1 Lecture 3

Goal: Aspects of 3-dimensional gauge theory on a point.

Recall: $\Gamma$, finite group, we have the identification

$$\text{Rep}_C \Gamma = C\Gamma - \text{mod}$$
$$\text{Rep}_k \Gamma = k\Gamma - \text{mod}$$

This gives a useful way of thinking about representations as modules over
an associative algebra. For infinite groups, one way to specify classes of represen-
tations it is often convenient to specify group algebra. E.g., $\mathbb{C}\mathcal{C}(G), L^1(G)$.

$G$, complex affine group (reductive: $GL_n\mathbb{C}, E_8(\mathbb{C})$).

Want to look at group actions on categories. In particular, dg categories or
infinity categories. If $G$ is algebraic we want the action to be somehow algebraic.

$$a_g : \mathcal{C} \rightarrow \mathcal{C}, \ g \in G$$

Want to say $a_g$ depend algebraically on $g$.

To get around this we think about group algebras.

$$\Gamma \rightarrow \text{Vect}\Gamma, * \ \Gamma \times \Gamma \rightarrow \Gamma$$

So we have a monoidal category.

$\Gamma$-action on $\mathcal{C} := \text{Vect}\Gamma$-module category.

Two natural candidates for a group algebra. These are the group algebras
for $G$ valued in quasi-coherent sheaves or $\mathcal{D}$-modules.

$$G \rightarrow Q(G) \ \text{quasi-coherent sheaves on } G$$
\[ G \rightarrow \mathcal{D}(G) \quad \mathcal{D} \text{ – modules on } G \]

These are both monoidal dg categories.

\[ M : G \times G \rightarrow G \]
\[ M(F \odot G) =: F \times G \]

We want to define topological field theories where to the point we can assign (four different possibilities):

- \( \bullet \mapsto \) Vect\( \Gamma \)-modules (3-dimensional TFT)
- \( Q(G) \)-categories \( \{QG \text{-modules}\} QG \otimes \mathcal{C} \rightarrow \mathcal{C} \) (2-dimensional TFT)
- \( \text{“smooth } G \text{-categories” } \{\mathcal{D} \text{-modules}\} \) (1-dimensional TFT)

There is a modified version to give 2-dimensional TFT’s.

For finite group \( \Gamma \):

We have an action of \( G \) on a variety \( X \), an action of \( Q(G) \) on \( Q(X) \), and \( \mathcal{D}(G) \) on \( \mathcal{D}(X) \).

This is where examples come from.

What kind of structures do these categories of \( G \)-categories have?

For \( H \subset G \), \( Q(G/H) \in Z^0_G(\cdot) \) is a natural example. So we look at the endomorphisms of the induced representation. This should be a Hecke algebra, i.e., functions on a double coset.

\[ \text{End}Q(G/H) = Q(H \backslash G / H) = \cdot / H \times /_G \cdot / H \]

E.g., (for \( H = G \)),

\[ \text{EndVect} = \cdot / G = Q(BG) = \text{Rep}_C G, \otimes \]

1.1 Morita theory

\( X \), finite set.

\( \text{Fun}(X \times X) \) algebra = \( \text{EndFun}(X) \) is Morita equivalent to \( \mathcal{C} = \text{Fun}(\cdot) \)

I.e., \( \text{Fun}(X \times X) - \text{mod} \cong \mathcal{C} - \text{mod} \)

For \( X \rightarrow Y \) surjective,

\[ \text{Fun}(X \times_Y X) - \text{mod} \cong \text{Fun}(Y) - \text{mod} = \text{Vect}(Y) \]
\[ Q(H \backslash G / H) = Q(\cdot / H \times /_G \cdot / H) \]
**Theorem 1.** (BZ-Francis-Nadler)

\[ \forall H, Q(H\backslash G/H) - \text{mod} \cong Q(G) - \text{mod} \cong (\text{Rep}G) - \text{mod} \]

**Primary Example:**

Let \( G \) be a reductive group, e.g., \( GL_n(\mathbb{C}) \), containing \( K = B \) Borel. 
\( G/B \) = flag variety = complete flags in \( \mathbb{C}^n \).

Beilinson-Bernstein says there is an isomorphism between:

- actions of \( DG \) on \( D(G/B) \)
- \( DG \)-actions on \( g - \text{mod}_0 \)

We have a left action by \( D(G) \) on \( D(G/B) \) and a right action by \( D(B \backslash B) = \mathcal{H} \) the finite Hecke category (roughly “Category O”)

\[ B \backslash (G/B) \leftrightarrow \text{Schubert cells} \leftrightarrow W \text{ Weyl group} \]

\[ K(\mathcal{H}) = ZW \]

“Bases” of \( \mathcal{H} \) labelled by \( w \in W \).

\[ \exists T_w \text{ standard } i_{w*}C_w \]

\( \mathcal{H} \rightarrow \text{version of } ZBG [B_{n-1} \text{ for } GL_n] \)

\[ T_{s_i} \times T_{s_j} \times T_{s_i} = T_{s_j} \times T_{s_i} \times T_{s_j} \]

\[ T_{s_i}^2 \neq Id \]

So if we give an action of \( \mathcal{H} \) on some category \( C \) gives an action of the braid group on \( C \). This is how many actions of braid group arise in Khovanov homology.

Studying modules for \( D(G) \) is not good.

We want

\[ \cdot \mapsto \mathcal{H} - \text{modules} \Rightarrow \text{Theorem (BZ-Nadler) 2d extended TFT’s} \]

We have the Harish Chandra modules \( D(K\backslash G/B) \) and we can take actions of \( \mathcal{H} \) on these.

\( Z(S^1) \) [TFT of dim \( \geq 2 \)]

**FIGURES**

For \( A \in Z(pt) \), we get two maps

\[ Z(S^1) \rightarrow \text{End}A \]

\( Z(S^1) = \text{“Hochschild cohomology/center”} \)

\[ \text{End}A \rightarrow Z(S^1) \]

\( Z(S^1) = \text{“Hochschild homology/dimension of } Z(pt)\text{”} \)

Center = end of \( Id_{Z(A)} = \text{hom}_{A-A^{op}}(A,A) \)
$Z(\cdot) = A\text{-mod}$

Dimension = FIGURES TO BE INCLUDED

$$Z(\cdot) = A - \text{mod} \Rightarrow \text{Dim} = A \otimes_{A \otimes A} A$$

$$\text{Vect} \Gamma - \text{mod} = Z(\cdot)$$

$$\text{Center}(\text{Vect} \Gamma) = Z(S^1)$$

Here the center is the Drinfel’d center.

**Theorem 2.** (BZ-Nadler-Francis) $QG$ has dimension and center equal to $Q(G_{\text{ad}}/G) = Z(S^1)$

Also for a Reimann surface $\Sigma$,

$$Z^Q_G(\Sigma) = R\Gamma(\mathcal{M}_G(\Sigma), \mathcal{O})$$

**Next time:**

Character theory

\[ \cdot \mapsto \mathcal{H} - \text{mod} \]

$S^1 \mapsto$ Luzstig’s character sheaves

\[ \begin{array}{c}
G/B \\
G_{\text{ad}}/G & \rightarrow & B \setminus G/B
\end{array} \]