1 Lecture 4

action : $Z(S^1) \leftrightarrow \text{End}A : \text{char}$

- $A \in Z(\cdot)$
- $\text{ch}(A) \in Z(S^1)$
- $\text{Id}_A \in \text{End}A$

- character of reps of $\Gamma$
- Chern character $\in (HH_*(Z\cdot))^{S^1}$ where $HH_*(Z\cdot) = \text{dim}Z(\cdot)$

Borel $B \subset G$ reductive $\mathbb{F}_\mathbb{C}$

\[
\begin{array}{c}
G/B \\
\pi \\
G/G \\
\downarrow \\
B\backslash G/B
\end{array}
\quad
\begin{array}{c}
\delta \\
\leftarrow \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
G/G \\
\downarrow \\
B\backslash G/B
\end{array}
\]

where $G/B = (G \times G/B)/G$ and $B\backslash G/B = G\backslash G/B \times G/B$.

\[ [g] \leftarrow (g, B) \rightarrow (B', g \cdot B') \]

Simultaneous Grothendieck-Springer resolution
Definition 1. (Lusztig) A character sheaf on $G$ is a $\mathcal{D}$-module on $G/G$ in the image of $\pi_*\delta$ on $\mathcal{H} = \mathcal{D}(B\backslash G/B)$.

The basic object is $\xi = \pi_*\delta^! = \pi_*\mathcal{C}_G\backslash G/G$.

$\bullet$ $\xi$ is Harish-Chandra’s system of differential equations satisfied by characters of $G$-representations.

Character sheaves are geometric avatars of representations (characters) of $G(\mathbb{F}_q)$.

Theorem 2. (BZ-Nadler) Character sheaves are characters of $\mathcal{H}$-modules. More precisely, $\mathcal{H}$-modules defines a $2$-dimensional (unoriented) TFT and $Z(S^1) = HH_*(\mathcal{H}) (\dim) = HH^*(\mathcal{H}) (\text{center})$

$act : Z(S^1) \leftrightarrow \mathcal{H} : tr$

E.g. $\xi = \text{char}(\mathcal{D}(B\backslash G/B))$ with action of Hecke category.

Local operators:

$$\int \mathcal{O}_{x}(\varphi) e^{-S(\varphi)} D\varphi$$

$E_n$ structures for $Z(S^{n-1})$ local operators

STUFF HERE

In 2-dimensional gauge theory, there are disorder operators

STUFF HERE

4d gauge theory I

The $B$-model $B_G$ (4D analog of $Z_G^Q$)

$$B_G(N^3) = R\Gamma(M + G(N^3), \mathcal{O})$$
\[ B_G(\Sigma) = \mathcal{O}_{cohsheaves}(\mathcal{M}_G(\Sigma)) \]

This is the main object of study. So we assign to the circle:

\[ B_G(S^1) = \mathcal{O}(G/G) - \text{modules} \]
\[ B_G(\cdot) = (\infty, 2) - \text{categories over } BG. \]

PICTURES
Wilson (loop) operators
TFT interpretation of Geometric Langland’s due to (Kapustin-Witten)(Belinson-Drinfeld)
\( A_G \) a 4-dimensional gauge theory (closer to \( Z_D^G \))

\( \Sigma \) now an algebraic curve/Riemann surface

\( Bundles_G \Sigma = \text{holomorphic } G\text{-bundles on } \Sigma. \)

\[ A_G(\Sigma) = D(Bun_G \Sigma) \]

Local operators”’ “t’Hooft/Hecke operators. These are examples of disorder operators.

PHYSICS PICTURE
Insert singularity, \( G \) is like a monopole alone line in 4-dimensions

\[ Bun_G(S^2) = \text{possible local singularities} \leftrightarrow \text{hom}(T^\vee, \mathbb{C}^*\!/W) \leftrightarrow \text{irreps of } G^\vee \]

\[ A_G(S^2) \leftrightarrow \mathcal{H}_{sph}, \text{spherical Hecke category} \]

A LITTLE MORE HERE...

**Theorem 3.** (Geometric Satake) \( A_G(S^2) = \mathcal{H}_{sph} \cong \mathcal{O}(\mathcal{M}_{G^\vee}(S^2)) = B_{G^\vee}(S^2) \)

**Conjecture:**
\[ D(Bun_G \Sigma) \cong \mathcal{O}(Loc_{G^\vee} \Sigma) \]

with actions of \( \mathcal{H}_{sph} \) and \( RepG^\vee \).

**Conjecture:** (Electric-Magnetic duality)
\[ A_G \cong B_{G^\vee} \]
as “TFT’s”.