1 Lecture 2: Overview of higher category theory

Definition 1. A category \( \mathcal{C} \) consists of

- a collection of objects \( X, Y, Z, \ldots \)
- for pairs of objects \( X, Y \in \mathcal{C} \), a set \( \text{hom}_\mathcal{C}(X,Y) \)
- composition maps
  \[
  \text{hom}_\mathcal{C}(X,Y) \times \text{hom}_\mathcal{C}(Y,Z) \to \text{hom}_\mathcal{C}(X,Z)
  \]
- associativity, units, etc.

Yesterday we saw the category of cobordisms. Here the objects and morphisms look a lot alike. This is a good candidate for what we will call a 2-category.

Bad definition:
A strict 2-category \( \mathcal{C} \) consists of

- a collection of objects \( X, Y, Z, \ldots \)
- for pairs of objects \( X, Y \in \mathcal{C} \), a category \( \text{hom}_\mathcal{C}(X,Y) \)
- composition functors
  \[
  \text{hom}_\mathcal{C}(X,Y) \times \text{hom}_\mathcal{C}(Y,Z) \to \text{hom}_\mathcal{C}(X,Z)
  \]
- associativity, units, etc.

Let \( X \) be a space, \( \pi_1(X,x) \) is a group whose elements are homotopy classes of loops. If we did not want to choose a basepoint, we could form a category: \( \pi_{\leq 1}X \) where the objects are points \( x \in X \) and morphisms are paths in \( X \) mod homotopy.
Here we also have 2-morphisms from \( p: x \to y \) to \( q: x \to y \) which are homotopies from \( p \) to \( q \) mod homotopy. This gives the 2-category \( \pi_{\leq 2}X \).

Now we look at composition of paths. Composition works because we can reparametrize. But when we are composing three paths reparametrization gets us into trouble regarding associativity. This is not a problem for the fundamental group or the fundamental groupoid since we only consider paths up to homotopy. So the fundamental 2-groupoid is a weak 2-category, which we will just call a 2-category.

We can get around the associativity problem, but uses an ad hoc tric which does not generalize to higher dimensions.

**Bad definition:**
A **strict \( n \)-category** consists of

- a collection of objects \( X, Y, Z, \ldots \)
- for pairs of objects \( X, Y \in C \), a strict \( (n - 1) \)-category \( \text{hom}_C(X, Y) \)
- composition functors
  \[ \text{hom}_C(X, Y) \times \text{hom}_C(Y, Z) \to \text{hom}_C(X, Z) \]
- strict associativity, units, etc.

**Example:**
Let \( X \) be a space, \( \pi_{\leq n}X \) is a possible example of an \( n \)-category where

- the objects are points \( x \in X \);
- morphisms are paths in \( X \);
- 2-morphisms homotopies between paths;
- 3-morphisms are homotopies of homotopies;
- ...
- \( n \)-morphisms are higher homotopies mod homotopy

So a new sketch of a definition will drop strictness everywhere:

An **\( n \)-category** consists of

- a collection of objects \( X, Y, Z, \ldots \)
- for pairs of objects \( X, Y \in C \), an \( (n - 1) \)-category \( \text{hom}_C(X, Y) \)
• composition functors

\[ \text{hom}_C(X,Y) \times \text{hom}_C(Y,Z) \to \text{hom}_C(X,Z) \]

• associativity up to coherent isomorphism, units, etc.

For any space \( X \), \( \pi_{\leq n} X \) is an \( n \)-groupoid.

**Thesis:**
Every \( n \)-groupoid should arise in this way.

**Better:**
There should be a theory of \( \infty \)-groupoids

**More precisely:**
For any space \( X \), \( \pi_{\leq \infty} X \) is an \( \infty \)-groupoid.

**Thesis:**
Every \( \infty \)-groupoid should arise in this way, for a space \( X \) which is unique up to weak homotopy equivalence.

\[ X \mapsto \pi_{\leq \infty} X \]

spaces mod homotopy go to \( \infty \)-groupoids mod equivalence.

One should regard this thesis as a criterion that a definition must pass.

**Definition 2.** An \( \infty \)-**groupoid** is a topological space (or maybe a simplicial set).

**Definition 3.** An \( n \)-**groupoid** is a space \( X \) such that its homotopy groups are trivial \( \pi_k X \cong * \) for \( k > n \).

So it is easier to define \( \infty \)-groupoids than \( n \)-groupoids.

**Sketch of definition**
An \((\infty,n)\)-category is a higher category in which all \( k \)-morphism are invertible for \( k > n \).

**Example:** \( (n = 0) \)
\((\infty,0)\)-category \( \cong \) \( \infty \)-groupoid \( \cong \) topological space

**Idea:**
An \((\infty,n)\)-category is

• a collection of objects \( X,Y,Z,\ldots \)

• for pairs of objects \( X,Y \in \mathcal{C} \), an \((\infty,n-1)\)-category \( \text{hom}_\mathcal{C}(X,Y) \)
• composition law
• associativity, units, etc....

**Example: (n=1)**

An \((\infty,1)\)-category is a topological category (or a simplicial category, an \(S\)-category).

Our definition is correct but often inconvenient. We want to think about strategies for making sense of associativity weakly.

In a Segal category, we have

\[
\begin{array}{ccc}
\text{hom}(X,Y,Z) & \sim & \hom(X,Y) \times \hom(Y,Z) \\
\uparrow & & \uparrow \\
\hom(X,Y) & & \hom(X,Z)
\end{array}
\]

So instead of a composition law, we have something that induces a composition law up to coherent homotopy.

\(\widetilde{\text{Cob}}(n)\) is an ordinary category.

\(n\text{Bord}\) is an \(n\)-category (a fancy version of \(\text{Cob}(n)\)).

\(\widetilde{\text{Cob}}(n)\) is an \((\infty,1)\)-category (a fancy version of \(\text{Cob}(n)\)).

**Definition 4.** \(\widetilde{\text{Cob}}(n)\) has

• objects which are \((n-1)\)-manifolds
• morphisms are bordisms of \((n-1)\)-manifolds
• 2-morphisms are diffeomorphisms of bordisms rel \(\partial\)
• 3-morphisms are isotopies of diffeomorphisms

This is an example of an \((\infty,1)\)-category.

We want \(\text{hom}(M,N)\), a classifying space for bordisms from \(M\) to \(N\).

This space exists and is uniquely determined up to homotopy.

By universality we get a map called composition which is determined up to homotopy, so we can only check associativity up to homotopy.
Describe: $\widehat{\text{Cob}(m)}$ as a Segal category.

We want a simplicial space $X$ with $X_0$ discrete.

$X_0 = \text{set of closed (m-1)-manifolds}$

$$X_n = \prod_{\text{objects } M_0, M_1, \ldots, M_n} \text{hom}(M_0, M_1, \ldots, M_n)$$

where

$$\text{hom}(M_0, M_1, \ldots, M_n) = \{(t_0 \leq t_1 \leq \cdots \leq t_n) \in \mathbb{R} - \text{dimensional proper submanifolds } B \subset [t_0, t_n] \times \mathbb{R}^\infty \text{ such that } B \text{ is transverse to } B \cap (t_i \times \mathbb{R}^\infty) \cong M_i\}$$

Similarly, we can describe more elaborate bordism categories using a more elaborate version of the theory of Segal categories/Segal spaces. Using these ideas, one can define an object $\widehat{n\text{Bord}}$.

Here the objects are 0-manifolds, morphisms are bordisms, $\ldots$, $n$-morphisms are $n$-manifolds with corners, and $(n + 1)$-morphisms are diffeomorphisms.