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Workshop on TFT’s at Northwestern  
Notes, errors, and incompleteness by Alex Hoffnung  
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DRAFT VERSION ONLY

1 Lecture 2

Reminders (and compliments):  
\( \mathfrak{k} \) commutative ring.  
\[
(H^0_{d\text{gCat}}/(d\text{gCat}/\mathfrak{k}), \otimes \mathfrak{k}) \subset (H^0_{\text{ct}}(d\text{gCat}/\mathfrak{k}), \otimes \mathfrak{k})
\]
(dualizable = saturated dgCat)  
same objects  
\[
[T, T'] = \{ E \in D(T \otimes T'^{\text{op}}), E_x \text{ is compact} \}
\]
(twisted \( \otimes \)-cats)

**Proposition 1.** \( T \in H^0_{d\text{g}}(d\text{gCat}/\mathfrak{k}) \) is saturated \( \iff T \sim B. \)

- \( B \) is a dg algebra/\( \mathfrak{k} \) such that
  - \( B \) is compact \( \in D(\mathfrak{k}) \) (“proper”)
  - \( B \) is compact \( \in D(B \otimes \mathfrak{k} B^{\text{op}}) \) (“smoothness”)

\( \alpha \in H^2_{\text{ct}}(X \mathbb{G}_m), X = \text{Spec} \mathfrak{k} \) then  
\( A_\alpha \) is Azumaya \( \mathfrak{k} \)-algebra, smooth and proper, then  
\( L_{\text{perf}}(A_\alpha) \) is saturated, which implies  
\( L_{\text{perf}}(A_\alpha) \in K_0^{(2)}(\mathfrak{k}) \), then  
\( H^2(X, \mathbb{G}_m) \to K_0^{(2)}(\mathfrak{k}) \)

More generally: \( A \) is a \( \mathfrak{k} \)-algebra  
- projective of finite type over \( \mathfrak{k} \)
- \( \forall L \text{ field}, \forall k \to L \)

\( A \otimes \mathfrak{k} L \) has finite homological dimension

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Then $L_{perf}(A)$ is saturated. (e.g., $A =$ quiver algebra)

Basic:

- $\mathfrak{k} \to K^{(2)}_0(\mathfrak{k})$ is a functor in $\mathfrak{k}$
  \[ \mathfrak{k} \to \mathfrak{k}' \otimes_{\mathfrak{k}} \mathfrak{k}' \colon H_0^{Mor}(dgCat/\mathfrak{k}) \to H_0^{Mor}(dgCat/\mathfrak{k}) \]

- $K^{(2)}_0(\mathfrak{k}) = \pi_0(K^{(2)}(\mathfrak{k}))$
  where $K^{(2)}(\mathfrak{k})$ is a commutative ring spectrum.

**Theorem 2.**

\[ \mathfrak{k} = \text{colim}_{\text{filtered}} \mathfrak{k}_x, \quad K^{(2)}(\mathfrak{k}) = \text{colim} K^{(2)}(\mathfrak{k}_x) \]

We want:

- extend $K^{(2)}$ to schemes and algebraic stacks
- $\text{ch}: K^{(2)}(X) \to ?$
  
  $X$ is a scheme stack

The above needs $\infty$-cagoy theory. Also uses cobordism hypothesis in dimension 1.

$\text{ch} := \text{“deRham realization of “N.C.Motives”} / X$

### 1.1 Really begin lecture 2: Segal categories

For ideas of what higher categories are see Jacob’s talk.

$(\infty, 1)$-categories: “$\left(1, \infty\right)$-category” = $\infty$-categories such that $n$-morphisms are invertible for all $n \geq 1 = \infty$-categories whose $\text{hom}$ are $\infty$-groupoids.

\[ | \cdot | : \infty \text{-groupoids} \leftrightarrow \text{homotopy types: } \Pi_\infty = \text{SSet} \]

“Naive definition”: A $(1, \infty)$-category is a $\text{SSet}$-enriched category = $\mathcal{S}$-categories:

$\mathcal{S}$-categories are badly behaved with respect to the category of functors.

**Definition 3.** A Segal category (or $\infty$-category) is

\[ A : \Delta^{op} \to \text{SSet} \]

such that

- $A_0$ is discrete (i.e., a set)
- $A_n \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ is a weak equivalence

with $[1] \to [n], \ 0 \to i, \ n \to i + 1.$
Remark 4. A is an $S$-category.

$$A: \Delta^{op} \to SSet$$

$$n \mapsto \prod_{(a_0, \ldots, a_n)} A(a_0, a_1) \times \ldots \times A(a_{n-1}, a_n)$$

Here $S-Cat \xhookrightarrow{\text{full}} \text{Segal Category} \subset \text{Fun}(\Delta^{op}, SSet)$.

A Segal category $A$ comes from an $S$-category iff $A_n \to A_1 \times_{A_0} \ldots \times_{A_0} A_1$ are isomorphisms.

Definition 5. A is a $\infty$-category, $[A] = \text{category with same objects as } A$ and $[A](x, y) := \Pi_0 A(x, y)$, where $[A] = 1$-truncation.

$$f: A \to B$$ a morphism between $\infty$-categories is:

- fully-faithful $A(x, y) \to B(fx, fy)$ weak equivalence
- essentially surjective if $[A] \to [B]$ is so.

Theorem 6. (Main theorem) (Hirschowitz-Simpson/Pellisier/Bergner) There exist a model category structure on \{ $F: \Delta^{op} \to SSet \mid F_0 \text{discrete}$ \} such that

- weak equivalences $\cap \{\text{Segal categories}\} = \text{fully faithful and essentially surjective}$
- cofibrations = morphisms
- fibrant objects = Segal category plus Reedy fibration condition
- It is a $\otimes$-monoidal category for the direct product. (most important property)

$$S-Cat \xhookrightarrow{\text{full}} \Pr SeCat$$

This is a Quillen equivalence.

So

$$H_0(S-Cat) \cong H_0(\infty - cat)$$

Consequence:

$A, B$ $\infty$-categories

$$\mathbb{R}\text{hom}(A, B) := \text{hom}(A, RB)$$

where $RB$ is a fibrant model for $B$.

$\mathbb{R}\text{hom}(A, B)$, an $\infty$-category, “is” the $\infty$-category of weak $\infty$-functors $A \to B$.

$SeCat^{fsh} + \text{hom} \Rightarrow \text{Cat enriched over } \infty$-categories = an example of a $(2, \infty)$-category.
Localization:

$A$ is an $\infty$-category

$S \subset [A]$ such that

$A \to L_SA$ of $\infty$-categories such that

\[
\begin{array}{ccc}
\mathbb{R} \text{hom}(L_SA, B) & \to & \mathbb{R} \text{hom}(A, B) \\
\downarrow & & \downarrow \\
\text{hom}(S^{-1}[A], [B]) & \to & \text{hom}([A], [B])
\end{array}
\]

Remark 7. $L_SA$ always exists.

We can start with $A$, a category (monoidal-$\infty$).

E.g., $A = M$, a model category $S = W$, then $L_WM$.

Theorem 8. $M$ is nice enough and simplicial, then $L_WM \cong M^{cofib}$

$L_WM(x, y) \cong Map_M(x, y)$

Under same assumptions as above, $\forall A$, a category

\[
\begin{array}{ccc}
\mathbb{R} \text{hom}(A, L_WM) & \to & L_WM(A) \\
\downarrow & & \downarrow \\
(M^A)_{fib, cofib}
\end{array}
\]

where all arrows are isomorphisms.

Next lecture:
- limits and colimits of $\infty$-categories
- $\infty$-categories of dg-categories
- $\otimes - \infty$-cat + rigity

Lecture 4:
- Derived algebraic geometry
- construct $ch$