1 Lecture 3

End of Lecture 2:
We talked about ∞-categories using the formalism of Segal categories. Here we will present some $\infty$-categories that we will present later. These have shown up in David’s lectures.

$$(C, W) \to L_w C \in \infty \text{- category}$$

where $C$ is a category.

- $L_{perf}(\mathfrak{t}) \in dgCat$

- $(\text{Perf}(\mathfrak{t}), q - \text{isom}) \to L\text{Perf}(\mathfrak{t}) \in \infty \text{- category}$

- $(dgCat/\mathfrak{t}, \text{Moritaeq}) \to D_g^{\text{Mor}}(\mathfrak{t}) = \infty \text{- category of dg Category}$

- $[L\text{Perf}(\mathfrak{t})] \cong D + \text{perf}(\mathfrak{t})$

- $[D_g^{\text{Mor}}(\mathfrak{t})] \cong H_0^{\text{Mor}}(dgCat/\mathfrak{t})$

- $E, F \in L\text{Perf}(\mathfrak{t})$

  $$\pi_i(L\text{Perf}(\mathfrak{t})(E, F), o) \cong \text{Ext}_{D(\mathfrak{t})}^{-i}(E, F)$$

  $$T, T' \in D_g^{\text{Mor}}(\mathfrak{t})$$

- $\pi_1(D_g^{\text{Mor}}(T, T'), E) \cong \text{aut}(E), \; E \in D(T, T')$
\[ \pi_i(D_{\mathcal{g}}^{Mor}(\mathfrak{t})(T, T'), E) \cong Ext^{1-i}_D(T \otimes T') \]

implied by

\[ \pi_i(D_{\mathcal{g}}^{Mor}(\mathfrak{t})(T, T)) \cong HH^{1-i}(T) = \text{Hochschild cohomology} =: Ext^{1-i}_{ \text{mathcal{D}}(T \otimes T)}(T, T) \]

**Adjunctions and limits**: \( f: A \to B \) in \( \infty \)-category.

**Definition 1.** \( f \) has a right adjoint if there exists:

\[ g: B \to RA \leftarrow \text{fibrant model} \]

and \( u \in \text{hom}(A, RA)(i, gf) \) such that \( \forall a \in A, \forall b \in B \)

\[ RA(i(a), g(b)) \]

This gives a notion of limit and colimit in a given \( \infty \)-category.

\[ A \in \infty - \text{cat}, I \in \infty - \text{category} \]

has right adjoint as limit and left adjoint as colimit.

\( M \) is a (nice) model category.

Then \( L_w M \) has all limits and colimits and they “are” the holim and hocolim defined in model category theory.

\[ L_w M \leftarrow \lim_{\text{holim}} \text{Hom}(I, L_w M) \]

**Lecture 3 begins now**: Want to talk about \( D_{\mathcal{g}}^{Mor}(X), D_{\mathcal{g}}^{ctg}(X), D_{\mathcal{g}}^{sat}(X) \), when \( X \) is now a scheme, an algebraic stack, or any \( \infty \)-stack.

\[ D_{\mathcal{g}}^{Mor}(X) := \lim_{\text{spec} \rightarrow X} D_{\mathcal{g}}^{Mor}(\mathfrak{t}) \in \infty - \text{cat} \]

This is equal to \( \Gamma(X, D_{\mathcal{g}}^{Mor}) \)

**Remark 2.**

\[ LPerf(X) := \lim_{\text{spec} \rightarrow X} LPerf(\mathfrak{t}), \infty - \text{category model for}[LPerf(X)] \cong \mathcal{D}_{Perf}(X) \]

\[ \text{End}_{\mathcal{D}^{ctg}(X)}(1) \cong LQCoh(X) \overset{\text{“}}{=} Q(X) \]
The Chern Character

The idea of the construction of $\text{ch}$:
Let $T \in D^c_g(X)$, $T = \text{sheaf of dgcat}/X$.

What is $\text{ch}(T) =$?

We consider the loops space

\[ \text{"hom}(S^1, X)" = \mathcal{L}X \xrightarrow{\pi} X \]

$\pi^*(T) \in D^c_g(\mathcal{L}X)$

$\pi^*(T)$ comes equipped with a natural auto-equivalence $m: \pi^*(T) \xrightarrow{\sim} \pi^*(T)$

$m = \text{monodromy along the loop}$

$\mathcal{L}X \times S^1 \xrightarrow{\mathcal{L}ev} X$

then self homotopy/equivalences of $|\pi|$ and self equivalence on $\pi^*$. $D^c_g(Y)$ is a rigid $\infty \otimes \infty$-category.

Then $\text{Tr}(M) \in D^c_g(\mathcal{L}X)(1, 1) \cong LQCoh(\mathcal{L}X) = \text{"Q}(\mathcal{L}X)"

\text{key result:}
$\text{Tr}(M)$ is $S^1$-invariant

This follows from Lurie’s theorem on $\widetilde{1Bord}$

Remark 3. • We can replace $D^c_g(-)$ by any functor “Schemes” $\xrightarrow{\mathcal{A}} \otimes - \infty$-category rigid.

$\mathcal{A}(X) \xrightarrow{\text{ch}} \text{End}_{\mathcal{A}(\mathcal{L}X)}(1)^{S^1}$

If $\mathcal{A} = \text{LPerf}$, then $\text{"L-hopf}(\mathcal{L}X)$ and $\text{ch}$ is usual Chern character.

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$\text{ch}(T) \in D_q\text{Coh}^{\mathcal{L}X}(\mathcal{L}X) \xrightarrow{\sim} D(\mathcal{D}_X - \text{mod})$

The LHS is deRham realization of $T$ as a noncommutative variety over $X$.

$\otimes \infty$-category is a “monoid object in $\infty$-category.”

commutative monoid in $\text{Sets} = \Gamma$-object in $\text{Set}$

$\Gamma$ = category of pointed finite sets

$M: \Gamma \to \text{Set}$

where $M_0 = *$ and $M_n \to M^n_1$ an iso.
A \otimes-\infty\text{-}category is

\[ A: \Gamma \tilde{\to} \infty -category \]
such that

\[ A_0 \tilde{\to} * \text{ and } A_n \tilde{\to} A_1 \times \cdots \times A_1 \]

These are equivalences of $\infty$-categories.

A $\otimes-\infty$-category from an $\infty$-category: $\infty - cat^\otimes$.

**Example:**
If $A$ is a $\times-\infty$-category, then $[A_1]$ is endowed with a natural $\otimes$-structures

**Remark:**
$\otimes-\infty$-category can be obtained by localization of $\otimes$-category with $A_0$ a set of maps

E.g., $dgCat/\mathfrak{t}, \otimes_{\mathfrak{t}},$ Morita