Lecture 4

End of Lecture 3:

We know about rigid $\otimes_{\infty}$-categories from Lurie’s talk.

Lecture 4: Derived algebraic geometry

Main problem: To construct a meaningful loop space $LX$ for $X$ a scheme or algebraic stack.

$$S^1 = BZ = \ast/\mathbb{Z}$$

is a stack. (here $\mathbb{Z}$ = discrete group scheme)

$X$ is a scheme or an algebraic stack ($[Y/G]$)

We want $LX := \text{hom}(S^1, X)$. If $X$ is a scheme, then $LX \cong X$ because $\ast = \text{coarse moduli space of } S^1$.

If $X$ is a stack, then $LX$ is the interia stack $I_X$. A point in $I_X$ is $(x, h)$, $x \in X$ and $h: x \sim x$.

E.g.,

$$I_{BG} \cong [G/G]$$

$$S^1 \cong \ast \amalg \ast \amalg \ast$$

$$\text{hom}(S^1, X) \cong X \times_X X \times X = \text{self intersection of the diagonal}$$

FIGURE $S^1 \sim \text{SQUARE}$
If $X = \text{Spec} A$:

$$A \otimes_{\text{Spec} k} A \longrightarrow A \otimes_{A \otimes_{k} A}^L A$$

The arrow means “replaced by” but the RHS is not a commutative ring anymore. But, it is a simplicial commutative $k$-algebra. (or $E_\infty$-ring, or commutative dg algebra)

$$A \otimes_{A \otimes_{k} A}^L A \in \infty\text{-category of simplicial commutative } k\text{-algebras.}$$

When $X$ is affine: we want $X := \text{Spec}(A \otimes_{A \otimes_{k} A}^L A)$ is a derived scheme.

**Definition 1.** A **derived scheme** is a topological space $Y$ and a sheaf of simplicial commutative ring $A$ and local conditions.

When $X$ is a scheme, these local models $A \otimes_{A \otimes_{k} A}^L A$ glue to give a derived scheme

$$\mathcal{L}X \rightarrow X$$

schemes $\subset$ derived schemes

Derived schemes form an $\infty$-category. Derived stacks obtained by taking quotients of derived schemes and there is a full sub-$\infty$-category of algebraic stacks.

**Definition 2.** $X$ is a stack.

$$\mathcal{L}X := \mathbb{R}\text{hom}(S^1, X)$$

where $\mathbb{R}$ means that hom is taken in the $\infty$-category of derived stacks.

The construction for $ch$ makes for this loop space $\mathcal{L}X$.

$X$ algebraic stack (or scheme)

$$ch : D^c(X) \rightarrow \text{End}_{D(\mathcal{L}X)}(k[S^1]) \cong L_{\text{geo.h}}(\mathcal{L}X)$$

a map of groupoids.

$$L_{\text{perf}}(X)[S^1] \rightarrow \text{End}_{L_{\text{perf}}(\mathcal{L}X)}(k[S^1]) \cong \mathbb{R}\Gamma(\mathcal{L}X, \mathcal{O})[S^1]$$

Yoneda gives us $T \in D^c_g(X) \Leftrightarrow (X \rightarrow D^c_g(-)[S^1]$ in the $\infty$-category of functors $\{\text{derived schemes}\} \rightarrow \infty$-groupoids, a rigid $\otimes_{\infty}$-category.

$$X \longrightarrow D^c_g(-)$$

$$1\text{Bord}_{\mathcal{O}}^\text{op}/X$$

$$1\text{Bord}_{\mathcal{O}}^\text{op}/X \rightarrow D_g(-)$$

is a 1-dimensional TFT$/X$ with values in a dg category.
moduli of “one” circle $X \subset \text{End}(\emptyset \to X) \to L_{\text{qcoh}}(-)^\sim \leftarrow [\mathcal{L}X/S^1]$

and the LHS and RHS are isomorphic.

Yoneda says $\text{ch}(T \in L_{\text{qcoh}}(\mathcal{L}X)^\sim_{S^1})$.

Relations between $\mathcal{L}X$ and deRham theory of $X$:

**Theorem 3.** (Folklore - BZ/Nadler) $X$ is a smooth scheme over $k$ with $\text{char} k = 0$.

1. $\pi + 0(\mathcal{R}\Gamma(\mathcal{L}X, O)^{S^1}) \cong H_{dR}^\ast(X/k)$ an iso of rings

2. $L_{\text{qcoh}}^S(\mathcal{L}(X)^u \cong L(X, \mathcal{R}x)$

$\mathcal{R}x$ is a sheaf of dg-algebras over $X$. $(= \oplus_i D_X^{|-2i|})$. THE RHS in (2) is the $\infty$-category of quasi-coherent $\mathcal{R}x$-dg modules.

“Pretheorem”: $E \in L_{\text{perf}}(X)$, $\text{ch}(E) =$ usual chern character in $H_{dR}^\ast(X)$ (using (1))

$L_{\text{qcoh}}(X, \mathcal{R}x)$ is some kind of $\infty$-category of “filtered $\infty$-modules”

$L_{\text{qcoh}}(X \mathcal{R}x) \xrightarrow{u=0} L_{\text{qcoh}}(\text{Sym}(T[-2])) \cong L_{\text{qcoh}}(\mathcal{L}X)$

**THIS DIAGRAM SHOULD HAVE MORE PIECES**

$T \to X$ a dgCat/$X$, then

$\text{ch}(T)$ is the “algebraic part” of the Hodge realization of $T$.

**Theorem 4.** $T \to X$ is saturated.

$\text{ch}(T) \in L_{\text{qcoh}}(X, \mathcal{R}x) \to L_{\text{qcoh}}(X)$

**A LITTLE HERE ABOUT HODGE FILTRATION**

Next steps:

$\text{ch} \mathcal{H}^{(2)}_0(X) \to L_{\text{perf}}^S(\mathcal{L}X) \to K_0^S(\mathcal{L}X) \xrightarrow{\text{ch}} \pi_0 \left( \mathcal{R}\Gamma(\mathcal{L}LX, O)^{S^1 \times S^1} \right)$

Want to understand the map from LHS to RHS.

What about a $\mathbb{Z}$-structure on the Hodge realization?

Candidate: topological $K$-theory for dgcat.