I Factorization algebras in perturbative quantum field theory
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I.1 Deformation quantization (work in progress with Owen Guillem).

Classical mechanics: described by a Poisson algebra \((A, \{ \cdot, \cdot \})\). In deformation quantization, want to replace \(A\) by an associative product \(A[[\hbar]]\), such that if \(f, g \in A\),
\[
\{ f, g \} = \lim_{\hbar \to 0} \frac{1}{\hbar}[f, g].
\]

Want an analogy in QFT:
1) Describe classical and quantum algebraic structures.
2) Show that we can get classical structure from classical field theory.
3) State a "quantization" theorem.

Let \(M\) be a manifold. A factorization algebra on \(M\) can be described as follows. Let \(B(M)\) be the space of balls in \(M\) and \(B_n(M)\) the collection of \(n\) disjoint unions of balls \(B_1, ..., B_n \in B(M)\) embedded into \(B_{n+1}\) (like in the definition of the little cubes operad).

A factorization algebra is a vector bundle \(V\) on \(B(M)\), together with maps
\[
B(M) \xrightarrow{p} B_n(M) \xrightarrow{q} B(M)^n, \quad q^*(V^\otimes n) \to p^*V
\]
satisfying some evident compatibility: \(V(B_1) \otimes V(B_2) \to V(B_3)\). The two maps
\[
V^\otimes \text{three balls} \to V^\otimes \text{two intermediate balls} \to V^\otimes \text{outer ball}
\]
and \(V^\otimes \text{three balls} \to V^\otimes \text{outer ball}\) commute.

\(\rightsquigarrow\) Close relation of \(E_n\)-algebras for \(\dim(M) = n\). Say a top. factorization algebra in \(M\) is this structure when \(V\) is a locally constant sheaf and the maps are morphisms of locally constant sheafs.

**Theorem I.1.** An \(E_n\)-algebra yields a top. factorization algebra on any framed manifold \(M\) with \(n = \dim(M)\).

A factorization algebra in \(\mathbb{C}\), where everything is holomorphic an invariant under \(\text{Aff}(\mathbb{C})\):
\[
V(\{ z \in \mathbb{C} : |z| < 1 \}) =: W \quad \text{a vector space}
\]
and the factorization algebra gives a map \(m_z : W \otimes W \to W\), depending holomorphically on \(z\) in an annulus. Thus \(m_z \sim \sum_{k \in \mathbb{Z}} \varphi_k z^k\), where \(\varphi_k\) in some completion of \(\text{Hom}(W, W)\) \(\rightsquigarrow\) reminiscent of the operator product expansion in vertex algebras.

**Claim:** Factorization algebras on \(M\) encode structure one expects from a quantum field theory on \(M\). This is motivated by the 2-dim. holomorphic setting, which fits with the known picture. In one dimension, this reduces to an associative algebra, the algebra of observables of quantum mechanics. In any dimension, one can construct a factorization algebra using perturbative quantum field theory.

Factorization algebras are symmetric monoidal algebras. A classical fact. algebra is a commutative algebra in this category. Suppose we have a classical field theory. For instance, take \(M\) compact riemannian, fields to be \(C^\infty(M, \mathbb{R})\) and the action
\[
S(\varphi) = \int_M \varphi \Delta \varphi + \varphi^3
\]
Let \(EL\) be the sheaf of solutions to the Euler-Lagrange equations \(2 \Delta \varphi + 3 \varphi^2 = 0\). If \(B \subseteq M\), let \(\mathcal{O}(EL(B))\) be functions on \(EL(B)\). This is a commutative fact. algebra \(EL(B_3) \to EL(B_1) \times EL(B_2)\) and applying \(\mathcal{O}\) to this gives a fact. algebra.

**Claim:** The \(E_\infty\)-algebra "wants" to become \(E_0\), i.e. just a factorization algebra.
Definition I.2. The $BV_0$-operad is the operad over $\mathbb{R}[[\hbar]]$, generated by a commutative product $\ast$ and a Poisson bracket of degree 1, s.th. $d\ast = \hbar \{\cdot, \cdot\}$.

Invert $\hbar$: $BV_0 \simeq E_0$. If $\hbar = 0$, then $BV_0/\hbar$ is the operad of commutative algebras with bracket of degree 1.

Let $X$ be a manifold, $f \in \mathcal{O}(X)$. Then $h(r, t(f))$, the derived critical scheme has a $\{\cdot, \cdot\}$ of degree 1 and "wants to become" $E_0$. $\mathcal{O}(h(r, t(f)))$ is a dga

$$\Lambda^2 TX \xrightarrow{\iota_\ast(df)} TX \xrightarrow{\iota_\ast(df)} \mathcal{O}(X)$$

The Schouten bracket gives $\mathcal{O}(h(r, t(f)))$ a Poisson bracket of degree 1. Apply this to the Euler-Lagrange situation and thus look at a "derived space of solutions". This is $\text{Crit}(S)$ and should acquire the Poisson bracket.

Example I.3. In free field theory, the derived critical locus of $\int \varphi \Delta \varphi$ is the 2-term complex

$$C^\infty(M) \xrightarrow{\Delta} C^\infty(M)$$

If we had an action with a cubic term, then differentiation acquires a non-linear term.

If $B \subseteq M$, the $\mathcal{O}(hEL(B))$ looks like

$$\prod \text{Hom}(\text{Sym}^n(C^\infty(B) \otimes \Lambda^k C^\infty(B)), \mathbb{R})$$

with a differential coming from the action $S$. If $X$ has a measure, i.e. a trivialization of the top-degree sheaf of differential forms, then we get a $BV$ operator $\Delta : \Lambda^\bullet TX \rightarrow \Lambda^\bullet TX$ such that $d + \hbar \Delta$ gives an algebra over $BV_0$.

Theorem I.4. The commutative factorization algebra, associated to the Euler-Lagrange equation of a classical field theory can be quantized into a factorization algebra in many interesting situations, e.g.

- scalar field theories
- Yang-Mills theories on $\mathbb{R}^4$.

Remark I.5. This is false over $\mathbb{Q}$!