

CONNECTIONS BETWEEN A CONJECTURE OF SCHIFFER'S AND INCOMPRESSIBLE FLUID MECHANICS

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ABSTRACT. We demonstrate connections that exists between a conjecture of Schiffer's (which is equivalent to a positive answer to the Pompeiu problem), stationary solutions to the Euler equations, and the convergence of solutions to the Navier-Stokes equations to that of the Euler equations in the limit as viscosity vanishes.

We say that a domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, has *the Pompeiu property* if, given that the integral of a continuous function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is zero for all translations and rotations of Ω , it follows that f is identically zero. The *Pompeiu problem* is to determine whether balls are the only simply connected domains with Lipschitz boundary not having the Pompeiu property.

From now on we assume that Ω is a nonempty simply connected domain with Lipschitz boundary Γ , having outward unit normal \mathbf{n} (defined almost everywhere on Γ).

Williams showed in [5] that an affirmative answer to the Pompeiu problem is equivalent to Conjecture 1 of Schiffer.

Conjecture 1 (Schiffer's conjecture). *Let α and λ be nonzero real numbers. Then there exists a non-identically vanishing solution ω to the overdetermined equation*

$$\begin{cases} \Delta\omega + \lambda\omega = 0 & \text{in } \Omega, \\ \omega = \alpha, \nabla\omega \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases} \quad (1)$$

if and only if Ω is a ball.

Observe that because ω is an eigenfunction of the Neumann Laplacian, necessarily $\lambda \geq 0$. The assumption that $\lambda \neq 0$ in Conjecture 1 simply rules out that possibility that $\omega \equiv \alpha$. Also, since ω is constant on Γ , it follows that $\nabla\omega = 0$, not just the normal component of it; that is, Equation (1) is equivalent to

$$\begin{cases} \Delta\omega + \lambda\omega = 0 & \text{in } \Omega, \\ \omega = \alpha, \nabla\omega = 0 & \text{on } \Gamma. \end{cases} \quad (2)$$

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(In fact, only the condition $\nabla\omega = 0$ on Γ is needed, since it then follows that ω is constant on Γ , but we will find it convenient to specifically state the value of ω on the boundary.)

We also observe that by Lemma 9, a solution to Equation (2), if it exists, is unique.

The following is a result of Williams in [6]:

Theorem 2. *The existence of a non-identically vanishing solution to Equation (2) implies that Γ is real analytic.*

We define the function space

$$V = \left\{ u \in (H^1(\Omega))^d : \operatorname{div} u = 0 \text{ in } \Omega, u = 0 \text{ on } \Gamma \right\}$$

and endow it with the H^1 -norm.

Given any u in $V \cap H^2(\Omega)$, the (classical) Stokes operator A_D applied to u is that unique element $A_D u$ of H such that $\Delta u + A_D u = \nabla p$ for some harmonic pressure field p . The operator A_D maps $V \cap H^2(\Omega)$ onto H (see, for instance, p. 49-50 of [2] for more details). This leads to the eigenvalue problem in Definition 3.

Definition 3. [Eigenvalue problem for A_D : strong form] An eigenfunction $u \in V \cap H^2(\Omega)$ of A_D with eigenvalue $\lambda > 0$ satisfies $A_D u = \lambda u$ or, equivalently,

$$\begin{cases} \Delta u + \lambda u = \nabla p, & \Delta p = 0, & \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & & & \text{on } \Gamma. \end{cases} \quad (3)$$

Theorem 4. *In two dimensions, the following are equivalent:*

- (1) *There exists a non-identically vanishing solution to Equation (2) on Ω .*
- (2) *There exists a pressureless eigenfunction of the Stokes operator on Ω —that is, Equation (3) has a solution with $\nabla p \equiv 0$.*
- (3) *There exists a steady state solution to the Euler equations that is also an eigenfunction of the Stokes operator.*

Remark 1. The equivalence of (2) and (3) shows that the steady state solution to the Euler equations in (3) is by necessity pressureless. Also, the nature of the steady state solution in (3) is very specific, as we describe following the proof of Theorem 4.

Proof of Theorem 4. Assume (1). Then Γ is real analytic by Theorem 2 so we can apply Lemma 8 giving a corresponding velocity field u satisfying

$$\begin{cases} \Delta u + \lambda u = 0, & \operatorname{div} u = 0 & \text{in } \Omega, \\ u \cdot \mathbf{n} = 0, & \omega = \alpha & \text{on } \Gamma. \end{cases}$$

But then from the identity $\Delta u = \nabla^\perp \omega$, it follows that $u = -(1/\lambda)\Delta u = 0$ on Γ , meaning that u satisfies Equation (3) with $\nabla p \equiv 0$.

Now assume (2), and let $\omega = \omega(u)$. Taking the vorticity of Equation (3), $\Delta\omega + \lambda\omega = 0$. It also follows from Equation (3) and the assumption $p \equiv 0$

that $\nabla\omega = -\Delta u^\perp = -\lambda u^\perp = 0$ on Γ from which Equation (2) follows for some value of α , and α cannot be zero by Lemma 9.

This shows the equivalence of (1) and (2).

If we assume that (2) holds, then $u \cdot \nabla\omega = -u \cdot \Delta u^\perp = -u \cdot (-\lambda u^\perp) \equiv 0$, which shows that u is a steady state solution to the Euler equations; that is, (3) holds.

Finally, assume (3), letting u be a steady state solution to the Euler equations that is also an eigenfunction of the Stokes operator with eigenvalue λ . Let ω be the vorticity of u and let ψ be the stream function for u , so that $u = \nabla^\perp\psi$.

Because u is a steady state solution to the Euler equations, it follows that $\omega = F(\psi)$ for some function F (see, for instance, Proposition 2.2 p. 46 of [4]) and because u is in $C^2(\Omega)$ it follows that F is in $C^1(\mathbb{R})$. Since ψ is constant on Γ (which gives the condition $u \cdot \mathbf{n} = 0$) so must ω be constant on Γ ; thus, $\nabla p \equiv 0$ by Lemma 8. This shows that (3) implies (2), completing the proof. \square

In the proof of Theorem 4 that (3) implies (2), we also have

$$\nabla\omega = F'(\psi)\nabla\psi \implies \Delta u = \nabla^\perp\omega = F'(\psi)\nabla^\perp\psi = F'(\psi)u.$$

But $\Delta u + \lambda u = 0$, since we showed that the solution to the Euler equations was pressure-free, so $F'(\psi) = -\lambda$ giving

$$\omega = F(\psi) = -\lambda\psi + C.$$

The stream function ψ is defined uniquely up to an additive constant. With the usual assumption that $\psi = 0$ on Γ , this becomes

$$\omega = -\lambda\psi + \alpha. \tag{4}$$

So if ψ is the stream function for a steady state solution to the Euler equations, then $\omega = \Delta\psi$ will satisfy Equation (2) as long as we also have $\nabla\psi = 0$ on Γ . We can also solve for ψ in terms of ω and use it to obtain the implications (1) implies (2) and (3) of Theorem 4.

These observations lead to Theorem 5, which is an alternate way of expressing Theorem 4.

Theorem 5. *Given the solution ω to Equation (2), setting*

$$\psi = (\alpha - \omega)/\lambda,$$

it follows that $u = \nabla^\perp\psi$ solves Equation (3) with $\nabla p \equiv 0$ and that u is a steady state solution to (E). Conversely, given that u is a steady state solution to the Euler equations that is also an eigenfunction of the Stokes operator, the pressure must vanish, and it follows that ω as given by Equation (4) solves Equation (2).

If any of the conditions in Theorem 4 are satisfied, then λ is an eigenvalue of the Neumann Laplacian. But u also satisfies Equation (3) with $\nabla p \equiv 0$, so it follows that both components of u are eigenfunctions of the Dirichlet

Laplacian having eigenvalue λ , or possibly one component is zero. So λ must be an eigenvalue of both the Dirichlet and Neumann Laplacian. In fact, we can say more:

Theorem 6. *If any of the conditions in Theorem 4 are satisfied then λ is an eigenvalue of the Neumann Laplacian and of the Stokes operator, and is an eigenvalue of the Dirichlet Laplacian with multiplicity at least two.*

Proof. Except for the multiplicity of λ , this all follows from the observations above. If as an eigenvalue of the Dirichlet Laplacian the multiplicity of λ were only one, then since both components of u are eigenfunctions of the Dirichlet Laplacian, or possibly one component is zero, it follows that $u = (f, Cf)$ or $u = (Cf, f)$ for some possibly zero constant C , where f is the only (up to normalization) eigenfunction of the Dirichlet Laplacian having eigenvalue λ . But $\operatorname{div} u = 0$, so if $u = (f, Cf)$ then $\partial_1 f + C\partial_2 f = 0$, so that $\nabla f = \partial_2 f(-C, 1)$. Thus, throughout Ω , ∇f points in the direction $-C\mathbf{i} + \mathbf{j}$, meaning that f is constant along lines in the direction $\mathbf{i} + C\mathbf{j}$. But u and so f is zero on Γ and so must be zero on the portions of all lines in the direction $\mathbf{i} + C\mathbf{j}$ lying in Ω ; that is, on all of Ω . This contradicts u being an eigenfunction of the Dirichlet Laplacian. A similar argument applies when $u = (Cf, f)$. We conclude that λ must be at least a double eigenvalue of the Dirichlet Laplacian. \square

We can also make the following observation:

Theorem 7. *Assume that u is a smooth solution of the Euler equations that vanishes on the boundary (in particular, this will be true if any of the conditions in Theorem 4 are satisfied). Then the solutions u_ν to the Navier-Stokes equations with the same initial velocity as u converge to u in $L^\infty([0, T]; L^2(\Omega))$ as $\nu \rightarrow 0$.*

Proof. We have,

$$\begin{aligned}\partial_t u_\nu + u_\nu \cdot \nabla u_\nu + \nabla p_\nu &= \nu \Delta u_\nu, \\ \partial_t u + u \cdot \nabla u + \nabla p &= 0,\end{aligned}$$

for some pressure p_ν , where $\operatorname{div} u_\nu = \operatorname{div} u = 0$ on Ω and $u_\nu = u = 0$ on Γ . Letting $W = u_\nu - u$, subtracting the two equations above, multiplying by W , and integrating over Ω gives

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\Omega)}^2 + \int_{\Omega} (u_\nu \cdot \nabla W) \cdot W + \int_{\Omega} (W \cdot \nabla u) \cdot W \\ + \int_{\Omega} \nabla(p_\nu - p) \cdot W = \nu \int_{\Omega} \Delta u_\nu \cdot W.\end{aligned}$$

The second and fourth terms above vanish after integrating by parts because $\operatorname{div} W = 0$ and $W \cdot \mathbf{n} = 0$ on Γ . Writing $\Delta u_\nu \cdot W = \Delta W \cdot W + \Delta u \cdot W$ and using $W = 0$ on Γ , integrating by parts once more, we have

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\Omega)}^2 + \nu \|\nabla W\|_{L^2(\Omega)}^2 = \nu \int_{\Omega} \Delta u \cdot W - \int_{\Omega} (W \cdot \nabla u) \cdot W.$$

But u is in $C^\infty(\Omega)$ so

$$\begin{aligned} \|W\|_{L^2(\Omega)} \frac{d}{dt} \|W\|_{L^2(\Omega)} &= \frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\Omega)}^2 + \nu \|\nabla W\|_{L^2(\Omega)}^2 \\ &\leq \nu \|\Delta u\|_{L^2(\Omega)} \|W\|_{L^2(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} \|W\|_{L^2(\Omega)}^2 \\ &\leq C \|W\|_{L^2(\Omega)} \left(\nu + \|W\|_{L^2(\Omega)} \right). \end{aligned}$$

It follows that

$$\frac{d}{dt} \|W\|_{L^2(\Omega)} \leq C \left(\nu + \|W\|_{L^2(\Omega)} \right)$$

and so by Gronwall's lemma that

$$\|u_\nu - u\|_{L^\infty([0,T];L^2(\Omega))} \leq C\nu T e^{CT},$$

which vanishes with the viscosity ν . \square

The following lemmas were used above.

Lemma 8. *Given ω in $H^2(\Omega)$ that satisfies*

$$\begin{cases} \Delta \omega + \lambda \omega = 0 & \text{on } \Omega, \\ \omega = g & \text{on } \Gamma \end{cases}$$

with g in $L^2(\Gamma)$ and $\lambda > 0$ there exists u and p in $H^2(\Omega)$ such that $\omega = \omega(u)$ and

$$\begin{cases} \Delta u + \lambda u = \nabla p, & \operatorname{div} u = 0, & \Delta p = 0 & \text{on } \Omega, \\ u \cdot \mathbf{n} = 0, & \omega = g & & \text{on } \Gamma. \end{cases}$$

The vector field u is unique and the pressure p is unique up to the addition of a constant. The pressure solves the Neumann problem,

$$\begin{cases} \Delta p = 0 & \text{on } \Omega, \\ \nabla p \cdot \mathbf{n} = -\nabla \omega \cdot \boldsymbol{\tau} & \text{on } \Gamma. \end{cases}$$

Furthermore, $\nabla p \equiv 0$ when $g \equiv C$.

Proof. See [3]. \square

Lemma 9 is the analog of the (only) lemma in [1] and, in fact, follows from it. For a proof see [3].

Lemma 9. *For all λ in \mathbb{R} , the only solution to Equation (2) with $\alpha = 0$ vanishes identically.*

REFERENCES

- [1] N. Filonov. On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator. *Algebra i Analiz*, 16(2):172–176, 2004. [5](#)
- [2] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001. [2](#)
- [3] James P. Kelliher. Eigenvalues of the Stokes operator versus the Dirichlet Laplacian for a bounded domain in the plane. *Submitted*. [5](#)
- [4] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002. [3](#)
- [5] Stephen A. Williams. A partial solution of the Pompeiu problem. *Math. Ann.*, 223(2):183–190, 1976. [1](#)
- [6] Stephen A. Williams. Analyticity of the boundary for Lipschitz domains without the Pompeiu property. *Indiana Univ. Math. J.*, 30(3):357–369, 1981. [2](#)

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