STREAM FUNCTIONS FOR DIVERGENCE-FREE VECTOR FIELDS
Version with Digressions

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Abstract. In 1990, Von Wahl and, independently, Borchers and Sohr showed that a divergence-free vector field u in a 3D bounded domain that is tangential to the boundary can be written as the curl of a vector field vanishing on the boundary of the domain. We extend this result to higher dimension and to Lipschitz boundaries in a form suitable for integration in flat space, showing that u can be written as the divergence of an antisymmetric matrix field. We also demonstrate how obtaining a kernel for such a matrix field is dual to obtaining a Biot-Savart kernel for the domain.

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Version with Digressions

This version includes Sections 3, 6, and 9 to 11, along with a few words in Section 1. These are digressions, not intended for publication (at least, not in their present form).

1. Overview

Let u be a divergence-free vector field on a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, that is tangential to the boundary. For a simply connected domain, it is well known that in two dimensions, $u = \nabla^\perp \psi := (-\partial_2 \psi, \partial_1 \psi)$ for a *stream function*, ψ, vanishing on the boundary. It is also well known that in three dimensions, we can write $u = \operatorname{curl} \psi$, where now the *vector potential* ψ is a divergence-free vector field tangential to the boundary. Perhaps somewhat less well-known is that ψ can also be chosen (non-uniquely) to vanish on the boundary, though sacrificing the divergence-free condition. This 3D form of the vector potential was developed in [8, 26], where it is studied in Sobolev, Hölder spaces, for $C^{1,1}$, C^∞ boundaries, respectively.

In higher dimension, we can no longer use a vector field as the potential; instead, we will use an antisymmetric matrix field A vanishing on the boundary, for which $u = \operatorname{div} A$, the divergence applied to A row-by-row. This was the manner it was utilized in [18], without, however, the key antisymmetric condition.

Our main result is Theorem 1.1.

Theorem 1.1. *Let H be the space of divergence-free vector fields on Ω that are tangential to the boundary and that have L^2 coefficients. Let H_c be the closed subspace of* curl*-free vector* *fields (see (4.1)) in H, let H_0 be its orthogonal complement in H, and let*

$$X_0 := \{A \in H_0^1(\Omega)^{d\times d} \colon A \text{ antisymmetric}\}.$$

Then $H_0 = \operatorname{div} X_0$, and there exists a bounded linear map $S \colon H_0 \to X_0$ with $\operatorname{div} Su = u$.

Specializing to $d = 2, 3$, we can write

$$H_0 = \begin{cases} \nabla^\perp H_0^1(\Omega), & d = 2, \\ \operatorname{curl}_3 H_0^1(\Omega)^3, & d = 3. \end{cases}$$

Because the term *matrix potential* is commonly used in the literature for other purposes, we will adopt the 2D terminology for all dimensions, calling A the *stream function* for u.

Closely connected to stream functions is the Hodge decomposition of L^2-vector fields on Ω. Indeed, one form of the Hodge decomposition in 3D is

$$H = H_c \oplus \operatorname{curl}(H \cap H^1(\Omega)^3).$$

That is, each element of $H_0 := H_c^\perp$ is the image of a classical, divergence-free vector potential tangential to the boundary. Moreover, for any $u \in H_0$, the boundary value problem

$$\begin{cases} \operatorname{curl} \psi = u & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

is (non-uniquely) solvable, and gives the 3D form of the stream function in Theorem 1.1.

In fact, solving the analog of (1.1) in any dimension in the more general setting of an oriented manifold with boundary was worked out by Schwarz in [22]. He shows that for such a manifold with $C^{1,1}$ boundary, given a 1-form α having L^2-regularity and vanishing normal component, the boundary value problem

$$\begin{cases} \delta\beta = \alpha & \text{on } M, \\ \beta|_{\partial M} = 0 & \text{on } \partial M \end{cases}$$

(δ is the codifferential) is solvable for a 2-form having H^1-regularity if and only if

$$\int_M \alpha \wedge *\lambda = 0 \text{ for all } \lambda \in \mathcal{H}_N^1(\Omega).$$

Here, $\mathcal{H}_N^1(\Omega)$ is the space of harmonic fields having vanishing normal component, the analog of H_c, and the integral condition on α defines the analog of H_0.

Schwarz's result is not restricted to 1-forms, but holds for k-forms and also allows non-zero boundary values. It is restricted, however, to $C^{1,1}$ boundaries. For manifolds embedded in $\mathbb{R}^d$, this restriction is loosened in [21], which applies to boundaries even less regular than Lipschitz. The authors show that, given an $(\ell - 1)$-form α for any $0 \le l \le d - 1$, there exists an ℓ-form β having prescribed boundary value for which $\delta\beta = \alpha$. They assume, however, that the $(\ell - 1)$-st Betti number vanishes. Since we need such a result for $\ell = 2$, this means that the first Betti number must vanish, which means that Ω must be simply connected, an assumption we wish to avoid.

We present our derivation of a stream function here, therefore, because it applies to non-simply connected domains having only a Lipschitz continuous boundary. Moreover, we obtain the stream function non-constructively, using simple functional analytic arguments, avoiding entirely the language of differential forms, making it more accessible and self-contained for our intended primary audience of analysts working in flat space.

Central to our approach is the fact that the divergence operator maps vector fields in $H_0^1(\Omega)^d$ *onto* $L_0^2(\Omega)$, the space of L^2 functions with mean zero. For arbitrary domains, this is a result of Bogovskiĭ [6, 7] (see Lemma 2.9, below). Bogovskiĭ produces an integral kernel for solving the problem $\operatorname{div} u = f$ in a star-shaped domain. This kernel and adaptations of it have been used in other approaches to Theorem 1.1 in 3D, such as [5] for star-shaped domains, but we use Bogovskiĭ's result as a "black box," for with it, we can easily obtain Theorem 1.1 except for the key antisymmetric condition on the stream function.

We assume that Ω is a bounded, connected, open subset of $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary, $\partial\Omega$. We define the L^2-based Sobolev spaces, $H^k(\Omega)$ and $H^k_0(\Omega)$, for nonnegative k in the usual way (the boundary is regular enough that all standard definitions are equivalent). Identifying L^2 with its own dual, we also define the dual spaces, $H^{-k}(\Omega) := H^k_0(\Omega)'$.

Defined this way, H^{-1} is what we will call an *abstract* dual space; that is, it is simply the space of all continuous linear functionals on a given Banach space (H^1_0, in this case). The usual realization of H^{-1} as what we will call a *concrete* dual space—by which me mean a specific, presumably useful space that is isometrically isomorphic to the abstract dual space—is as a subspace of distributions. This realization requires, however, making the identification of L^2 with its own (abstract) dual space $(L^2)'$, and leads to the continuous embeddings,

$$\mathcal{D}(\Omega) \subsetneq H^1_0(\Omega) \subsetneq L^2(\Omega) = L^2(\Omega)' \subsetneq H^{-1}(\Omega) \subsetneq \mathcal{D}(\Omega)'. \tag{1.2}$$

We then define weak derivatives of functions in L^2 in the usual way. So, for instance, given f in L^2, $\partial_i f$ is that element of H^{-1} for which

$$(\partial_i f, \varphi) = -(f, \partial_i \varphi) \text{ for all } \varphi \in \mathcal{D}(\Omega) := C^\infty_0(\Omega).$$

Defined this way, it is classical that any element of H^{-1} is a sum of an L^2 function and the divergence of a vector field in L^2. Another concrete manifestation of this definition of H^{-1} is given in Proposition 2.8, and there are many others.

We will work with the classical function spaces, H and V, of incompressible fluid mechanics:

$$\begin{aligned} H &:= \{u \in L^2(\Omega)^d \colon \operatorname{div} u = 0,\ u \cdot \boldsymbol{n} = 0\}, \\ V &:= \{u \in H^1_0(\Omega)^d \colon \operatorname{div} u = 0\}. \end{aligned} \tag{1.3}$$

The divergence here is defined in terms of weak derivatives, and $u \cdot \boldsymbol{n}$ is defined as an element of $H^{-\frac{1}{2}}(\partial\Omega)$ in terms of a trace (see Lemma 2.2), $\boldsymbol{n}$ being the outward unit normal vector. Both H and V are Hilbert spaces with norms and inner products as subspaces of L^2 and H^1_0. By virtue of the Poincaré inequality, we can use

$$\begin{aligned} (f, g)_{H^1_0} &:= (\nabla f, \nabla g)_{L^2}, \quad \|f\|_{H^1_0} := \|\nabla f\|_{L^2}, \\ (u, v)_V &:= (\nabla u, \nabla v)_{L^2}, \quad \|u\|_V := \|\nabla u\|_{L^2}. \end{aligned}$$

Because $\partial\Omega$ is Lipschitz, we know that $H^1_0(\Omega)$ is both the closure in the H^1 norm of $C^\infty_0(\Omega)$ and the subspace of all elements of $H^1(\Omega)$ whose trace on the boundary vanishes. It follows in the classical way that we can equivalently characterize H as

$$H = \text{closure of } V \text{ in the } L^2 \text{ norm}. \tag{1.4}$$

(Or we could use the closure of $\mathcal{V} = C^\infty_0(\Omega) \cap V$ in the L^2 norm.)

Now, H is, by its very definition, a subspace of $L^2(\Omega)^d$ and V is a subspace of $H^1_0(\Omega)^d$. Hence, any number of derivatives of functions lying in them will yield functions lying in some negative Sobolev space. That is, they are distribution spaces[1]: $H, V \subseteq \mathcal{D}'(\Omega)$.

With these very cursory definitions out of the way, we give in Section 2 some further necessary background material. Before moving on to our main result, however, we make a detour in Section 3 to explore a cautionary tale of J. Simon's [24] about how the dual space V' is not a distribution space. The tools we presented in Section 2 to construct our

[1] By a *distribution* space we mean any function space allowing for well-defined weak derivatives up to at least some finite order. Hence, we need not view the spaces as subspaces of distributions, which avoids the need to deal with their topology. For our purposes, a distribution space will always be a subspace of a Sobolev space.

stream function turn out to be well suited to describe, in a very concrete manner, the nature of these difficulties. In Section 4, we prove our main result, Theorem 1.1, extending it to the space V in Section 5. In Section 6, we prove that the adjoint of div as an operator on antisymmetric $d \times d$ matrices in H_0^1 is $-(1/2)\,\mathrm{curl}$. In Section 7 we show how the classical 3D vector potentials can be obtained from the stream function of Theorem 1.1.

In Section 8 we demonstrate that the Biot-Savart law, which recovers a vector field in H_0 from its vorticity (curl), is, in a precise way, dual to the problem of obtaining a stream function from a velocity field in H_0. We show that if there is an integral kernel associated with one of these problems it is also the kernel associated with the other problem.

In 3D, there is a further, useful, though somewhat non-standard, decomposition of H, that follows as a corollary of Theorem 1.1, and which we describe in Section 9. In Section 10, we give an alternate characterization of the space H and, for simply connected domains, a parallel characterization of ΔV as a subspace of $H^{-1}(\Omega)^d$. As an application of our main result, in Section 11 we use the stream function developed in Theorem 5.2 to prove Poincaré's lemma as a simple consequence of de Rham's lemma.

Throughout, we follow the convention that $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_H$.

We write (u, v) for the inner product in L^2 or H. We write v^i for the i-th coordinate of a vector v; A^i_j for the element in the i-th row, j-th column of a matrix A; A^i for the i-th row of A; A_j for the j-th column of A. We follow the convention that repeated indices are implicitly summed, even when both indices are superscripts or both are subscripts.

2. Background material

Here, we present a number of tools we will use in what follows. The results themselves are classical, but their form and proofs are based primarily upon Galdi's invaluable introductory chapters in [14] along with material from the equally invaluable [16]. Table 1 converts some of Galdi's notation to the notation we are using, which may be useful for the reader who wishes to examine our explicit references to Galdi's text.

Table 1. Some notation in Galdi's [14]

Galdi	Our notation
$\mathcal{D}(\Omega)$	$\mathcal{V} = V \cap C_0^\infty(\Omega)$: *divergence-free* test functions
H_2	the space H defined in (1.3)
H^1 or H_2^1	$H \cap H^1(\Omega)$, with H as defined in in (1.3)
D^m	$\dot{H}^m(\Omega)$, the homogeneous Sobolev space
D_0^m	$\dot{H}_0^m(\Omega)$, the homogeneous Sobolev space (for us, Ω is bounded, so $\dot{H}_0^m(\Omega) = H_0^m(\Omega)$)

Definition 2.1. *As in [25], we define the space*

$$E(\Omega) := \{u \in L^2(\Omega)^d \colon \operatorname{div} u \in L^2(\Omega)\},$$

endowed with the norm, $\|u\| + \|\operatorname{div} u\|$. We also define the space,

$$\widetilde{E}(\Omega) := \{u \in L^2(\Omega)^3 \colon \operatorname{curl} u \in L^2(\Omega)\},$$

endowed with the norm, $\|u\| + \|\operatorname{curl} u\|$. We use $\widetilde{E}(\Omega)$ only in 3D.

We frequently integrate by parts using Lemma 2.2 (see Theorem 2.5 and (2.17) of [16]):

Lemma 2.2. *There exists a normal trace operator from $E(\Omega)$ to $H^{-1/2}(\partial\Omega)$ that continuously extends $u \mapsto u \cdot \boldsymbol{n}|_{\partial\Omega}$ from $C(\overline{\Omega})$ to $E(\Omega)$. We will simply write $u \cdot \boldsymbol{n}$ rather than naming this trace operator. For all $u \in E(\Omega)$, $\varphi \in H^1(\Omega)$,*

$$(u, \nabla\varphi) = -(\operatorname{div} u, \varphi) + \int_{\partial\Omega} (u \cdot \boldsymbol{n})\varphi,$$

where we have written $(u \cdot \boldsymbol{n}, \varphi)_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$ in the form of a boundary integral.

In 3D, we also have the following (see Theorem 2.11 of [16]):

Lemma 2.3. *In 3D, there exists a tangential trace operator from $\widetilde{E}(\Omega)$ to $H^{-1/2}(\partial\Omega)$ that continuously extends $u \mapsto u \times \boldsymbol{n}|_{\partial\Omega}$ from $C(\overline{\Omega})$ to $\widetilde{E}(\Omega)$. We will simply write $u \times \boldsymbol{n}$ rather than naming this operator. For all $u \in \widetilde{E}(\Omega)$, $\varphi \in H^1(\Omega)$,*

$$(\operatorname{curl} u, \varphi) = (u, \operatorname{curl} \varphi) + \int_{\partial\Omega} (u \times \boldsymbol{n}) \cdot \varphi.$$

Poincaré's inequality holds not just for V, but for the larger space $H \cap H^1(\Omega)^d$:

Lemma 2.4. *There exists a constant $C = C(\Omega)$ such that for all $u \in H \cap H^1(\Omega)^d$,*

$$\|u\| \le C \|\nabla u\|.$$

Proof. For any $u \in H$,

$$\int_\Omega u^j = \int_\Omega u \cdot \nabla x^j = -\int_\Omega \operatorname{div} u \, x^j + \int_{\partial\Omega} (u \cdot \boldsymbol{n}) \, x^j = 0.$$

Hence, u has mean value zero, so Poincaré's inequality holds in the form stated. □

The well-posedness of solutions to the (stationary) Stokes problem is a classical, deep result, that lies at the heart of much of what we do. We will rely heavily upon the following version of it:

Proposition 2.5. *For any $f \in H^{-1}(\Omega)^d$, the (stationary) Stokes problem,*

$$\begin{cases} -\Delta v + \nabla q = f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Omega, \end{cases} \tag{2.1}$$

has a unique (up to an additive constant for q) weak solution, $(v, q) \in H^1(\Omega)^d \times L^2(\mathbb{R})$. Moreover,

$$\|v\|_{H^1} + \|q\| \le C \|f\|_{H^{-1}}.$$

Proof. See, for instance, Proposition 4.2 of [3] or Theorem IV.1.1 of [14]. □

The well-posedness of the Stokes problem quickly yields a proof of the version of de Rham's lemma in Proposition 2.6. (This makes de Rham's lemma appear quite simple, yet de Rham's lemma is generally used in the proof of the well-posedness of the Stokes problem, as it is in the proof in [14] that we referenced. This perceived simplicity, then, is merely a consequence of the presentation, and hardly a self-contained proof.)

Proposition 2.6 (de Rham's Lemma)**.** *A vector field $f \in H^{-1}(\Omega)$ is the gradient of an L^2 function if and only if*

$$(f, v) = 0 \text{ for all } v \in V.$$

Proof. The forward direction is immediate. For the converse, given $f \in H^{-1}(\Omega)^d$, let $(v, q) \in V \times L^2(\Omega)$ be the solution to (2.1) given by Proposition 2.5. Since $v \in V$, we then have

$$0 = (f, v) = (-\Delta v, v) + (\nabla q, v) = \|\nabla v\|^2.$$

Hence, $v = 0$, so $f = \nabla q$. □

Proposition 2.6 does not say that if f vanishes in V' then it is a gradient, for f must be an element in H^{-1}. Interpreting it that way is the origin of Simon's trap, which we explore in the next section.

Key tools for us will be the decomposition of vector fields in $H^1_0(\Omega)$ given in Proposition 2.7 and the surjectivity of the divergence operator in Lemma 2.9. These results employ the space

$$L^2_0(\Omega) := \{f \in L^2(\Omega) \colon \int_\Omega f = 0\}.$$

Proposition 2.7. *The orthogonal decomposition, $H^1_0(\Omega)^d = V \oplus V^\perp$, holds with*

$$V^\perp = \{z \in H^1_0(\Omega)^d \colon \Delta z = \nabla q \text{ for some } q \in L^2(\mathbb{R})\} \tag{2.2}$$

and $\|P_{V^\perp}\varphi\| \le C \|\operatorname{div} \varphi\|$. Moreover, the orthogonal projection $P_V \colon H^1_0(\Omega)^d \to V$ given by $\varphi = P_V \varphi + z$, where $(z, q) \in H^1_0(\Omega)^d \times L^2(\Omega)$ is a weak solution to

$$\begin{cases} -\Delta z + \nabla q = 0 & \text{in } \Omega, \\ \operatorname{div} z = \operatorname{div} \varphi & \text{in } \Omega, \\ z = 0 & \text{on } \Omega. \end{cases} \tag{2.3}$$

Proof. This decomposition is given in Corollary 2.3 p. 23 of [16] (also see Lemma 2.2 of [17]). We give a proof here for completeness.

Starting with $\varphi \in H^1_0(\Omega)^d$, set $g = \operatorname{div} \varphi \in L^2(\Omega)$ and solve (non-uniquely), $\operatorname{div} w = g$ for $w \in H^1_0(\Omega)$. That we can solve this is a matter we will return to in Section 4; specifically, see Lemma 2.9. We have, $\|w\|_{H^1(\Omega)} \le C \|g\|$, as shown, for instance, in Exercise III.3.8 of [14].

Next let $f = \Delta w \in H^{-1}(\Omega)^d$, and let (v, q) be the unique solution to (2.1). Set $z = v + w$ and observe that $-\Delta z + \nabla q = f - \Delta w = 0$, $\operatorname{div} z = g = \operatorname{div} \varphi$, and $z = 0$ on $\partial\Omega$. Hence, (z, q) is a solution to (2.3), and we see that $P_V \varphi = \varphi - z$. Moreover,

$$(P_V \varphi, z)_V = (\nabla P_V \varphi, \nabla z) = -(\Delta z, P_V \varphi)_{H^{-1}, H^1_0} = -(\nabla q, P_V \varphi)_{H^{-1}, H^1_0} = 0.$$

Hence, we see that $z \in V^\perp$, so $V^\perp$ contains the set on the righthand side of (2.2).

It remains to show that $V^\perp$ contains *only* the set on the righthand side of (2.2). To see this, suppose that $z \in V^\perp$. Let $u \in V$ be arbitrary. Then

$$(u, z)_V = (\nabla u, \nabla z) = (\Delta z, u)_{H^{-1}, H^1_0} = 0.$$

Thus, $\Delta z = \nabla q$ for some $q \in L^2(\Omega)$ by Proposition 2.6. The bound $\|P_{V^\perp}\varphi\| \le C \|\operatorname{div} \varphi\|$ follows, for instance, from the Stokes problem bound in Exercise IV.1.1 of [14]. □

We could have directly used the solution to (2.3) to obtain the decomposition of $H^1_0(\Omega)^d$, but we wished to reduce the problem to the classical Stokes problem and (non-unique) inversion of the divergence operator.

Note that the solution of (2.1) can be rephrased as follows:

Proposition 2.8.

$$H^{-1}(\Omega)^d = \Delta V \oplus \nabla L^2(\Omega) = \Delta V \oplus \Delta V^\perp = \Delta H^1_0(\Omega)^d.$$

Proof. Let $f \in H^{-1}(\Omega)^d$ and let (v, q) solve (2.1). This gives $H^{-1}(\Omega)^d = \Delta V + \nabla L^2(\Omega)$, and the uniqueness of the solution shows that the decomposition is a direct sum. Then (2.2) shows that $\Delta V^\perp = \nabla L^2(\Omega) = \nabla L^2(\Omega)$, hence also $H^{-1}(\Omega)^d = \Delta V \oplus \Delta V^\perp = \Delta(V + V^\perp) = \Delta H_0^1(\Omega)^d$, where we invoked Proposition 2.7. □

Lemma 2.9. *[Bogovskiĭ [6, 7]] For any $f \in L_0^2(\Omega)$ there exists $v \in H_0^1(\Omega)^d$ for which $\operatorname{div} v = f$. We can choose the (non-unique) solutions in such a way as to define a bounded linear operator $R\colon L_0^2(\Omega) \to H_0^1(\Omega)^d$ for which $\|\nabla Rf\| \le C\|f\|$. Moreover, we can assume that R maps into the space $V^\perp$.*

Proof. For the proof of all but the last sentence, see Bogovskiĭ [6, 7] or Theorem 2.4 of [8]. Then, for any $f \in L_0^2(\Omega)$, $\operatorname{div}(P_{V^\perp} Rf) = \operatorname{div} Rf = f$ and

$$\|\nabla(P_{V^\perp} Rf)\| = \|P_{V^\perp} Rf\|_{H_0^1(\Omega)^d} \le \|Rf\|_{H_0^1(\Omega)^d} = \|\nabla Rf\|.$$

So because $P_{V^\perp}$ is a continuous linear operator, we can replace R by $P_{V^\perp} R$. □

In fact, Bogovskiĭ in [6, 7] showed that the divergence is surjective for an arbitrary domain in $\mathbb{R}^d$. See, for instance, the historical comments on pages 208-209 of [2].

The difficult part of proving Lemma 2.9 is obtaining the surjectivity of the divergence as a map from $H_0^1(\Omega)^d$ to $L_0^2(\Omega)$: once that is obtained (or even just that the range of div is closed), the bounded linear (partial) inverse map R follows from basic functional analysis, by arguing much as we do in the proof of Theorem 1.1 in Section 4. (And see Remark 4.6.)

Much more can be said about the higher regularity of Rf when f is more regular. Moreover, it is shown in [8] also that $\|Rf\| \le C\|f\|_{H^{-1}}$, though for us the weaker bound $\|Rf\|_{H_0^1} \le C\|f\|$, which follows from Lemma 2.9 and Poincaré's inequality, will suffice.

Moreover, since $P_{V^\perp}$ does not change the divergence of a vector field, the constant in the inequality in Lemma 2.9 is at least as small as the constant in Proposition 2.7. (This is a little misleading, however, as Lemma 2.9 is generally used to prove the estimates on the Stokes problem that lead to the inequality in Proposition 2.7.)

From R of Lemma 2.9, we define a matrix-valued operator, which we continue to call R, by applying R on each component of any vector in $L_0^2(\Omega)^d$:

$$R\colon L_0^2(\Omega)^d \to H_0^1(\Omega)^{d\times d}, \quad (Ru)^i := Ru^i. \tag{2.4}$$

We have been somewhat formal in our proof of Proposition 2.7, as we never gave a definition of a weak solution to (2.3) or even to the special case in (2.1). For this purpose, we unwind the definitions and results in [14][2], leading to the following:

Definition 2.10. *The pair $(z, q) \in H_0^1(\Omega) \times L^2(\Omega)$ is a weak solution to (2.3) if $z = v + w$, where $v, w \in H_0^1(\Omega)$, $\operatorname{div} w = \operatorname{div} \varphi$, and*

$$(\nabla v, \nabla\psi) = \langle f, \psi\rangle \text{ for all } \psi \in \mathcal{V},$$
$$(\nabla v, \nabla\alpha) = \langle f, \alpha\rangle + (q, \operatorname{div}\alpha) \text{ for all } \alpha \in C_0^\infty(\Omega),$$

where $f = \Delta w$ and $\mathcal{V} := V \cap C_0^\infty(\Omega)$ (this is what Galdi, very confusingly, calls $\mathcal{D}(\Omega)$). Also, $\langle\cdot,\cdot\rangle$ is the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Now, since

$$(\nabla v, \nabla\psi) = (\nabla z, \nabla\psi) - (\nabla w, \nabla\psi) = (\nabla z, \nabla\psi) + (\Delta w, \psi) = (\nabla z, \nabla\psi) + (f, \psi)$$

and, similarly,

$$(\nabla v, \nabla\alpha) = (\nabla z, \nabla\alpha) + (f, \alpha),$$

[2]See Definition IV.1.1, Remark IV.1.1, (IV.1.3), Lemma IV.1.1 in [14], and note the sign change, since Galdi solves $\Delta v - \nabla q = f$.

we see that

$$\begin{aligned}&(\nabla z, \nabla \psi) = 0 \text{ for all } \psi \in V,\\&(\nabla z, \nabla \alpha) = (q, \operatorname{div} \alpha) \text{ for all } \alpha \in C_0^\infty(\Omega).\end{aligned} \tag{2.5}$$

In the first equality we used the density of $\mathcal{V}$ in V. Although f is eliminated in (2.5), q still appears, and q ultimately derives from f. Hence, we cannot use these identities together to define a weak solution.

We have the following simple proposition:

Proposition 2.11. *If $u \in V$ then for all $\varphi \in H_0^1(\Omega)$,*

$$(\Delta u, \varphi) = (\Delta u, P_V \varphi).$$

If $u \in V^\perp$ then for all $\varphi \in H_0^1(\Omega)$,

$$(\Delta u, \varphi) = (\Delta u, P_{V^\perp} \varphi).$$

Proof. Let $\varphi \in H_0^1(\Omega)$, which we can write as $\varphi = P_V\varphi + z$, as in Proposition 2.7. For $u \in V$,

$$(\Delta u, \varphi) = (\Delta u, P_V \varphi) + (\Delta u, z) = (\Delta u, P_V \varphi) - (\nabla u, \nabla z) = (\Delta u, P_V \varphi)$$

by $(2.5)_1$.

For $u \in V^\perp$, we know by Proposition 2.7 that $\Delta u = \nabla q$ for some $q \in L^2(\Omega)$. Hence,

$$(\Delta u, \varphi) = (\Delta u, P_V \varphi) + (\Delta u, z) = (\nabla q, P_V \varphi) + (\Delta u, P_{V^\perp} \varphi) = (\Delta u, P_{V^\perp} \varphi).$$ □

A simple and immediate consequence of Proposition 2.11 is the following:

Corollary 2.12. *Let $v \in \Delta V$. Then for all $\varphi \in H_0^1(\Omega)$,*

$$(v, \varphi) = (v, P_V \varphi).$$

Moreover,

$$(v, \psi) = 0 \text{ for all } \psi \in V \iff (v, \varphi) = 0 \text{ for all } \varphi \in H_0^1(\Omega).$$

Remark 2.13. *We might interpret Corollary 2.12 as saying that ΔV is a near proxy for V' as a distribution space, touching upon the subject of the next section.*

3. Dual spaces and Simon's trap

In the study of incompressible fluid mechanics, one sometimes makes the identification of H with H' rather than L^2 with $(L^2)'$, as we described in Section 1. The primary reason is that it allows for the use of some powerful functional analysis tools originating largely in the work of J. L. Lions in the 1950s and 1960s. These tools are used in the proof of the existence of solutions to PDEs, linear and nonlinear.

The identification of H with H', however, makes it impossible to treat V' as a distribution space. This is as detailed by J. Simon in [24], as a consequence of a general result. We will return to Simon's paper in a moment, but let us first look at the problem explicitly as it relates to V' to identify concretely the point of failure.

First, observe that for any linear functional $f \in V'$, there exists, by the Riesz representation theorem, a unique $u \in V \subseteq H_0^1(\Omega)^d$ for which

$$(f, \psi)_{V',V} = (u, \psi)_V \text{ for all } \psi \in V.$$

But,

$$(u, \psi)_V = (u, \psi)_{H_0^1} = (\nabla u, \nabla \psi) = -(\Delta u, \psi)_{H^{-1}, H_0^1}.$$

Hence, the mapping $\gamma \colon f \mapsto \Delta u$ is an isometric isomorphism between V' as an abstract dual space and a concrete manifestation of it as a subspace of H^{-1} (cf., Corollary 2.12). Or, we could view V' as composed of equivalence classes in $H^{-1}(\Omega)^d$ where $u_1 \sim u_2$ if $u_1 - u_2 = \nabla q$ for some $q \in L^2(\Omega)$.

Yet, we cannot employ either of these isomorphisms in an effective manner that allows for the usual, free "Calculus" operations of distribution or Sobolev spaces, such as "integration by parts." This is because the isomorphism is completely at odds with the identification of L^2 with its (abstract) dual space, which allows such operations in the duality between H^{-1} and H_0^1 (or as distributions).

To see this, by Proposition 2.8, any element of $H^{-1}(\Omega)^d$ can be written uniquely in the form $\Delta u + \nabla q$ for some $u \in V$, $q \in L_0^2(\Omega)$, where $L_0^2(\Omega)$ is the set of all functions in $L^2(\Omega)$ having mean zero. So fix $q \in L_0^2(\Omega)$. Since for any $\psi \in V$, $(\Delta u + \nabla q, \psi) = (\Delta u, \psi)$, we see that $\Delta V + \nabla q$, considered as a subspace of $H^{-1}(\Omega)^d$, is also isomorphic to V'. This simply reflects the observation in [24] that there is not a *unique* element of $H^{-1}(\Omega)^d$ corresponding to any given element of V'.

Now suppose we identify H with H'. If we expect to be able to operate on constructs such as Δu for $u \in V$, we are at a loss, for then $\Delta u = \Delta u + \nabla q$ as elements of some space that is to contain V', or else the rules applying to "integration by parts," as for distribution spaces of elements of H^{-1}, cannot apply. So, for instance, we could still define the Stokes operator as acting on V to produce an element of V', but then V' would be little more than an abstract dual space.

Returning to J. Simon's [24], at the center of the difficulties with V' is Proposition 2 of [24]. This proposition says that if two topological vector spaces, E and V, are subspaces of a common space, then to make sense of $V' \subseteq E'$ as a continuous embedding we must have both (i) $E \cap V$ dense in E and (ii) $E \cap V$ dense in V. The failure of (i) gives a non-unique representative of elements in V' as an element in E'. This is the point of failure when we apply Simon's Proposition with $E = H_0^1(\Omega)^d$, $V = V$, and $E' = H^{-1}(\Omega)^d$. Our observations above are merely an explicit unravelling of where this non-uniqueness occurs if one *tries* to make a concrete realization of V' as a distribution space.

Moreover, with the identification of $L^2(\Omega)$ with $L^2(\Omega)'$ as in (1.2), even the dual space H' becomes problematic to work with. Given a vector field $u \in H$, each of its components are in $L^2(\Omega)$, so each component is identified with an element of $L^2(\Omega)'$. But to any element v of $L^2(\Omega)^d$, $v + \nabla p$ for any $p \in H^1$ acts the same as v does on any element u of H; that is,

$$(v, u) = (v + \nabla p, u) \text{ for all } u \in H.$$

Hence, v and $v + \nabla p$ would need to be the same element of H', and we see that H', like V', cannot be treated concretely as a distribution space. The fundamental issue is the same as for V': in Simon's Proposition, while $H \cap L^2(\Omega)^d$ is dense in H it is not dense in $L^2(\Omega)^d$.

To summarize, if an element of H' or V' is to make sense as a distribution, it must be that $\nabla p = 0$ for any $p \in H^1(\Omega)$ or $L^2(\Omega)$, respectively. Yet if $p \in H_0^1(\Omega)$, say, then $\nabla p \in L^2(\Omega)^d$ is a regular-distribution that has a value pointwise almost everywhere, which is manifestly non-zero as long as $p \not\equiv 0$. So ∇p does not vanish as an element of $L^2(\Omega)^d$, yet it vanishes in the presumably containing distribution spaces H' and V'. Hence, H' and V' cannot be distribution spaces.

3.1. **Navier-Stokes Equations.** Let us turn now to how Simon's trap shows up in the classical theory of the existence of weak solutions to the Navier-Stokes equations, and try to understand the practical impact of the difficulties he points out.

A common formulation of what it means to be a weak solution to the Navier-Stokes equation on Ω with no-slip boundary conditions is that

$$\frac{d}{dt}(u, v) + \nu(\nabla u, \nabla v) - (u \otimes u, \nabla v) = \langle f, v \rangle \text{ for all } v \in V, \tag{3.1}$$

along with a condition for initial velocity in H. (Integrating this equation formally in time and using a time-varying test function, as well as imposing an energy inequality as a condition, yields another, closely related formulation.)

Now, suppose we want to assume, say, that $f \in L^2(0,T;V')$. Then the forcing term in (3.1) must be interpreted as the pairing of V with V'; that is, $\langle f, v\rangle = \langle f, v\rangle_{V',V}$. Authors then often add the parenthetical comment that (3.1) means equality in V'. If interpreted to mean that each side of (3.1) defines a continuous linear functional on V—the left-hand side through integration (with no need for distributions), the right-hand side in the sense of an element of the abstract dual space V' applied to a test function in V—this is perfectly legitimate.

As Simon points out in Proposition 3 of [24], however, we cannot write

$$\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = f$$

if we mean it as the equality of distributions, because the left-hand side is a distribution while the right-hand side is not. Yet, of course, we could apply a test function in V and interpret the left-hand side as a distribution, while we interpret the right-hand side as an element of the abstract dual space V' (this is (3.1)). Or, we could use the isomorphism described earlier and identify f with Δw for some $w \in V$, and treat both sides as distributions.[3]

All this could be, indeed has been, made to work, but the real problem, as Simon points out in Section 7 of [24], is that we cannot obtain a pressure if we assume that $f(t) \in V'$. Essentially, this is because, as we observed above, if ∇p is to have any meaning as an element of V' (which, being a distribution, it does not) it equals zero; that is, all pressure gradients are the same as elements of V'. The simple resolution to all this is to assume that $f(t) \in H^{-1}(\Omega)^d$ and avoid the use of V' entirely. Or, to allow a more direct physical interpretation and to avoid some other, minor technicalities, assume that $f(t) \in H$.

4. Proof of main result

In this section we prove our main result, Theorem 1.1. We present first some imporant existing results then establish a series of lemmas and propositions we will use in the (short) body of the proof of Theorem 1.1, with which we close the section.

Define the subspace

$$H_c := \{u \in H \colon \operatorname{curl} u = 0\}$$

of H. Here, we use the curl operator on $\mathbb{R}^d$ in the form,

$$\operatorname{curl} u := \nabla u - (\nabla u)^T. \tag{4.1}$$

That is, $\operatorname{curl} u$ is twice the antisymmetric gradient, the $d \times d$ matrix-valued function with $(\operatorname{curl} u)^i_j = \partial_j u^i - \partial_i u^j$. This form of the curl is convenient for integrating by parts (applying the divergence theorem) in flat space. In 2D, we can define $\operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1$, the scalar curl, and in 3D we can define it as a vector in the usual way, denoting it curl_3 for clarity.

H_c is clearly closed, so we can define

$$H_0 := H_c^\perp,$$

the orthogonal complement of H_c in H. Hence, $H = H_0 \oplus H_c$.

Remark 4.1. *H_c is finite-dimensional for a large class of domains for which $\partial\Omega$ has a finite number of components. For smooth boundaries, this follows, for instance, from the discussion in Section 4.1 of* [15]*. For special classes of 3D Lipschitz domains, Helmholtz domains of* [4]*, H_c (and H_0) can be characterized by making "cuts" in Ω that leave the remaining domain simply connected. This idea goes back to Helmholtz; see the historical comments in* [10].

[3] By doing this, we would be choosing a pressure arbitrarily; see the next paragraph in the text.

In [18] (Corollary 7.5), the simple tool in Lemma 4.2 was used to investigate conditions under which solutions to the Navier-Stokes equation for incompressible fluids converge to a solution to the Euler equations (the so-called *vanishing viscosity limit*).

Lemma 4.2. *For any $u \in H$ there exists (a non-unique) $A \in H_0^1(\Omega)^{d\times d}$ such that $u = \operatorname{div} A$; that is, such that $u^i = \partial_j A^i_j$.*

The idea of the proof is that a simple integration by parts as in the proof of Lemma 2.4 shows that each component of any $v \in H$ lies in $L^2_0(\Omega)$. But by Lemma 2.9, div maps $H^1_0(\Omega)^d$ onto $L^2_0(\Omega)$, so we can obtain each row of A independently. The proof of Lemma 4.2 is therefore quite simple, but it relies on the powerful and deep result in Lemma 2.9.

Left open in [18] was whether it could be assured that A in Lemma 4.2 is antisymmetric. In fact, such antisymmetry can be obtained, and was obtained in 3D by Borchers and Sohr in Theorem 2.1, Corollary 2.2 of [8], whose lowest regularity result can be stated as follows:

Lemma 4.3. *Assume that $d = 3$ and $\partial\Omega$ is $C^{1,1}$. For any $u \in H_0$ there exists $v \in H^1_0(\Omega)^3$ such that $u = \operatorname{curl}_3 v$ and $\Delta \operatorname{div} v = 0$. Moreover, one can choose the solutions in such a way as to define a bounded linear operator $S\colon H_0 \to H^1_0(\Omega)^3$ with $\|\nabla S u\| \le C \|u\|$.*

To see that Lemma 4.3 provides a 3D form of an extension of Lemma 4.2 to antisymmetric matrices, note that any 3×3 antisymmetric matrix can be written in the form,

$$A = \begin{pmatrix} 0 & \psi^3 & -\psi^2 \\ -\psi^3 & 0 & \psi^1 \\ \psi^2 & -\psi^1 & 0 \end{pmatrix}. \tag{4.2}$$

We can define a bijection Q from a vector in $\mathbb{R}^3$ to an antisymmetric $d \times d$ matrix, by setting $Q(\psi) = Q(\psi^1, \psi^2, \psi^3)$ to be the matrix in (4.2), and we can write that $\operatorname{div} Q\psi = \operatorname{curl}_3 \psi$. The claim in Theorem 1.1, then, is the natural extension of Lemma 4.3 to $d \ge 2$.

The simple argument in Proposition 4.4 shows that $\operatorname{div} X_0$ is at least dense in H_0:

Proposition 4.4. $H_0 = \overline{\operatorname{div} X_0}$.

Proof. First, we show that $\operatorname{div} X_0$ is a subspace of H. To see this, observe that if $u \in \operatorname{div} X_0$ then $u^i = \operatorname{div} A^i = \partial_j A^i_j$. Hence, $\operatorname{div} u = \partial_{ij} A^i_j = -\partial_{ij} A^j_i = -\partial_{ji} A^i_j = -\partial_{ij} A^i_j = -\operatorname{div} u$, so $\operatorname{div} u = 0$. (That $\operatorname{div} u = \operatorname{div}\operatorname{div} A = 0$ is a reflection of $\delta^2 = 0$ when A is expressed as a 2-form.)

Moreover, since A^i_j is constant along the boundary, ∇A^i_j is normal to the boundary, so we can write, $\nabla A^i_j = \alpha^i_j \boldsymbol{n}$, where

$$\alpha^i_j = \frac{\partial A^i_j}{\partial \boldsymbol{n}} = -\frac{\partial A^j_i}{\partial \boldsymbol{n}} = -\alpha^j_i.$$

Then,

$$\partial_j A^i_j = \nabla A^i_j \cdot \mathbf{e}^j = \alpha^i_j \boldsymbol{n} \cdot \mathbf{e}^j = \alpha^i_j n^j$$

so, using that $\alpha^i_j = -\alpha^j_i$,

$$u \cdot \boldsymbol{n} = \operatorname{div} A \cdot \boldsymbol{n} = \operatorname{div} A^i n^i = \partial_j A^i_j n^i = \alpha^i_j n^j n^i = -\alpha^j_i n^j n^i = -\alpha^i_j n^j n^i = -u \cdot \boldsymbol{n},$$

so $u \cdot \boldsymbol{n} = 0$. We conclude that $\operatorname{div} X_0 \subseteq H$.

We now show that $(\operatorname{div} X_0)^\perp = H_c$. Let $A \in X_0$ and $v \in H$ be arbitrary. Then $u := \operatorname{div} A$ is an arbitrary element of $\operatorname{div} X_0$. Applying Lemma 2.2 and using $A = 0$ on $\partial\Omega$,

$$\begin{aligned}(u, v) &= (\operatorname{div} A, v) = -(A, \nabla v) = -(A, \nabla v - (\nabla v)^T) - (A, (\nabla v)^T) \\ &= -(A, \operatorname{curl} v) - (A^T, \nabla v) = -(A, \operatorname{curl} v) + (A, \nabla v).\end{aligned}$$

Hence, $(A, \nabla v) = (1/2)(A, \operatorname{curl} v)$, and because both A and $\operatorname{curl} v$ are antisymmetric,

$$(u, v) = -(A, \nabla v) = -\frac{1}{2}(A, \operatorname{curl} v) = -\sum_{i<j} A^i_j (\operatorname{curl} v)^i_j.$$

We can choose the components A^i_j independently for $i < j$, and $H^1_0(\Omega)$ is dense in $L^2(\Omega)$, so we conclude that $(u, v) = 0$ for all $u \in \operatorname{div} X_0$ if and only if $\operatorname{curl} v = 0$; that is, if and only if $v \in H_c$. It then follows that $(\operatorname{div} X_0)^\perp = H_c$ so that, in fact, $\overline{\operatorname{div} X_0} = ((\operatorname{div} X_0)^\perp)^\perp = H_c^\perp = H_0$. $\square$

The operator R of (2.4) allows us to easily establish that $\operatorname{div} X_0$ actually yields all of H_0:

Proposition 4.5. *$H_0 = \operatorname{div} X_0$.*

Proof. We have, $\operatorname{div} X_0 = \operatorname{div}(R \operatorname{div} X_0) = \operatorname{div} Y$, where $Y = R \operatorname{div} X_0$. It follows from Proposition 4.4 that $\operatorname{div} Y$ is dense in H_0. If we can show that it is closed, then we are done.

Let (u_n) be a sequence in $\operatorname{div} Y$ converging to u in H_0. Then $u_n = \operatorname{div} B_n$ with $B_n = Ru_n$ in Y, and we have from Lemma 2.9 that $\|\nabla B_n\| \le C \|u_n\|$. Since (u_n) converges, it is Cauchy and hence (B_n) is Cauchy and so converges to some $B \in Y$ with $u = \operatorname{div} B$. This shows that $H_0 = \operatorname{div} Y = \operatorname{div} X_0$. $\square$

It remains only to obtain the bounded linear map S of Theorem 1.1. Examining the proof of Proposition 4.5, we see that $B_n = Ru_n$ in Y has some D_n in X_0 for which $R \operatorname{div} D_n = B_n$, but the convergence of (B_n) does not mean the convergence of (D_n). To surmount this difficulty, and obtain S, we restrict the domain of div to a subspace:

Proof of Theorem 1.1. Observe that $\operatorname{div} A = \operatorname{div} B$ for $A, B \in X_0$ if and only if $B = A + E$ for some E in $V^d \cap X_0$, a closed subspace of X_0. Letting $Y_0 = (V^d \cap X_0)^\perp$, the orthogonal complement of $V^d \cap X_0$ in X_0 *as a Hilbert space*, $\operatorname{div} \colon Y_0 \to H_0$ is a continuous bijection. It follows from a corollary of the open mapping theorem (see, for instance, Corollary 2.7 of [9]) that the inverse map, $S := \operatorname{div}|_{Y_0}^{-1}$, is also continuous. But this means that, $\|Su\|_{X_0} = \|Su\|_{Y_0} \le C \|u\|_{H_0}$, giving us the bounded linear map of Theorem 1.1. $\square$

The Baire category theorem appears through the proof of the corollary to the open mapping theorem we applied. Hence, the constant we obtain in $\|\nabla Su\| \le C \|\operatorname{div} u\|$ is not effectively computable, although we can see that C is no smaller than the constant in Lemma 2.9.

Remark 4.6. *Although the adjoints to the two forms of* div *appearing in Lemma 2.9 and Theorem 1.1 never appear explicitly, they are, in a sense, hiding in the proofs. We show in Section 6 that the adjoint of* $\operatorname{div} \colon X_0 \to H_0$ *is* $-(1/2) \operatorname{curl}$*, whose null space is* H_c*. Since* div *is a closed map,* $\operatorname{div} X_0$ *is closed if and only if it equals* $H_c^\perp =: H_0$*. Similarly, it can be shown that the adjoint of* $\operatorname{div} \colon H^1_0(\Omega)^d \to L^2_0(\Omega)$ *is* $-\nabla$*, whose null space is trivial. Hence,* $\operatorname{div} H^1_0(\Omega)^d$ *is closed if only if it equals all of* $L^2_0(\mathbb{R}^d)$*. Proving that the range of either version of* div *is closed is the hard part of each proof, but we were able to leverage the powerful result in Lemma 2.9 to obtain the hard part for Theorem 1.1 with minimal effort.*

We avoided characterizing the space $Y_0 = (V^d \cap X_0)^\perp$ *explicitly, but given that the adjoint of* $\operatorname{div} \colon X_0 \to H_0$ *is* $-(1/2) \operatorname{curl}$*, we show in Proposition 6.3 that* $Y_0 = \{z \in X_0 \colon \Delta z = \operatorname{curl} q \text{ for some } q \in L^2_0(\Omega)^d\}$*, in analogy with Proposition 2.7. In 3D, this is* $Y_0 = \{z \in H^1_0(\Omega)^3 \colon \Delta z = \operatorname{curl}_3 q, q \in L^2_0(\Omega)^d\}$*, which yields* $\Delta \operatorname{div} Su = 0$*, as in Lemma 4.3.*

5. Higher regularity

Bogovskiĭ in [6, 7] showed more than what we stated in Lemma 2.9 (see Theorem 2.4 of [8]):

Lemma 5.1. *[Bogovskiĭ [6, 7]] Let $p \in (1,\infty)$ and $m \geq 0$ be an integer. Define $H^{m,p}_{0,0}(\Omega)$ to be the functions in $H^{m,p}_0(\Omega)$ having mean zero. There exists a bounded linear operator $R = R_{m,p} \colon H^{m,p}_{0,0}(\Omega) \to H^{m+1,p}_0(\Omega)^d$ satisfying $\operatorname{div} Rf = f$ with $\left\|\nabla^{m+1} Rf\right\|_{L^p(\Omega)} \leq C \left\|\nabla^m f\right\|_{L^p(\Omega)}$.*

Restricting ourselves to $p = 2$, we define, as in (2.4), a matrix-valued operator $R_m = R_{m,2}$:

$$R_m \colon H^m_0(\Omega)^d \to H^{m+1}_0(\Omega)^{d\times d}, \quad (R_m u)^i := R_m u^i.$$

We will use Lemma 5.1 to study the stream function for an element of V.

Theorem 5.2. *The map S of Theorem 1.1 also maps $V \cap H_0$ continuously onto $Y_0 \cap H^2_0(\Omega)^{d\times d}$, where $Y_0 = (V^d \cap X_0)^\perp$.*

Proof. The space $Y^2_0 := Y_0 \cap H^2_0(\Omega)^{d\times d}$ is dense in Y_0 and $\operatorname{div}\colon Y_0 \to H_0$ is a continuous surjection, so $\operatorname{div} Y^2_0$ is dense in H_0. Moreover, $\operatorname{div} Y^2_0 \subseteq V \cap H_0$, so $\operatorname{div} Y^2_0$ is dense in $V \cap H_0$. Then, arguing as in the proof of Proposition 4.5, $\operatorname{div} Y^2_0 = \operatorname{div}(R_1 \operatorname{div} Y^2_0)$ is closed in $V \cap H_0$ and hence $\operatorname{div} Y^2_0 = V \cap H_0$. Because $\operatorname{div}|_{Y_0}$ is injective it also holds that $\operatorname{div}|_{Y^2_0}$ is injective. Finally, arguing as in the proof of Theorem 1.1, the inverse map, $\operatorname{div}|^{-1}_{Y^2_0}$, is continuous. But this is the same map S as in Theorem 1.1, restricted to $V \cap H_0$. □

Remark 5.3. *Using R_m, one can extend Theorem 5.2 to $S\colon H_0 \cap H^m_0(\Omega)^d \to Y_0 \cap H^{m+1}_0(\Omega)^{d\times d}$, though its utility is likely limited for $m \geq 2$. Similarly, one can employ Lemma 5.1 to develop L^p bounds in analog with Theorem 1.1.*

6. Of div and curl

We explore, now, the relation between div and curl, which we will see are (almost) adjoints.

Since $\operatorname{div}\colon X_0 \to H_0$, its adjoint is a map $\operatorname{div}^*\colon D(\operatorname{div}^*) \subseteq H'_0 \to X'_0$, where we must first determine $D(\operatorname{div}^*)$. (We do not reduce the domain of div to Y_0, as we did in the previous section, because the dual space of Y_0 is hard to characterize directly in a concrete form, so the adjoint, while it exists, would be hard to relate to the curl operator.)

Now, H_0 and X_0 are both Hilbert spaces, not just Banach spaces. To go further and obtain a concrete characterization of div^* we need to exploit this fact, but there is a delicate issue: we have identified L^2 with its dual space (as briefly mentioned in Section 1), but this is not compatible with identifying H_0 with H'_0 because of the divergence-free condition (Simon has an informative exposition on this issue in [24]). Nonetheless, this identification will be very useful to us, so, to be careful, we will only treat H'_0 as an abstract dual space, and use the identification explicitly: For any $v \in H'_0$ we will write Iv for that element of H_0 for which $\langle v, u\rangle_{H'_0, H_0} = (Iv, u)_{H_0}$ for all $u \in H_0$; that is, I gives the usual identification of the dual of a Hilbert space with itself.

The identification of $L^2(\Omega)$ with its dual is also not altogether compatible with identifying X'_0 with the space of antisymmetric matrices in $H^{-1}(\Omega)^{d\times d}$, as natural as that would be. This is because $X_0 \subseteq H^1_0(\Omega)^{d\times d}$, so we should have $H^{-1}(\Omega)^{d\times d} = (H^1_0(\Omega)^{d\times d})' \subseteq X'_0$.

However, X_0, is naturally isomorphic with $H^1_0(\Omega)^{d(d-1)/2}$, whose dual space we can identify with $H^{-1}(\Omega)^{d(d-1)/2}$ in a manner that is compatible with the identification of $L^2(\Omega)$ with its dual. Then, $H^{-1}(\Omega)^{d(d-1)/2}$ is naturally isomorphic with the space of antisymmetric matrices in $H^{-1}(\Omega)^{d\times d}$. This will allow us to treat X'_0 as the space,

$$X'_0 = \{A \in H^{-1}(\Omega)^{d\times d}\colon A \text{ antisymmetric}\} \subset H^{-1}(\Omega)^{d\times d}. \tag{6.1}$$

Remark 6.1. *More precisely, let J map antisymmetric $d\times d$ matrices into $\mathbb{R}^{d(d-1)/2}$ be given by $(JA)^k = A^i_j$, where $k = d(i-1) + j - 1$. Then define the operator $F\colon H^1_0(\Omega)^{d(d-1)/2} \to H_0$ by $FB = \operatorname{div}(J^{-1}B)$. Then F^* will map some subspace of H'_0 into $H^{-1}(\Omega)^{d(d-1)/2}$, and we will have $\operatorname{div}^* = J^{-1}F^*$. We will not, however, make this mapping explicit in what follows.*

With this concrete version of X_0', we can characterize div^* as in Proposition 6.2.

Proposition 6.2. *The following hold:*

(1) $\operatorname{div}\colon X_0 \to H_0$ *is a closed map;*
(2) $D(\operatorname{div}^*)$ *is all of* H_0'*;*
(3) $\operatorname{div}^*\colon H_0' \to X_0'$ *is given by* $\operatorname{div}^* = -(1/2)\operatorname{curl} I$*;*
(4) div *is surjective;*
(5) curl *is injective with* $\operatorname{curl} H_0$ *closed in* X_0'*;*
(6) $\|u\|_H \le C\,\|\operatorname{curl} u\|_{X_0'}$.

Proof. (1) We first show that $\operatorname{div}\colon X_0 \to H_0$ is a closed map (that is, its graph is closed in $X_0 \times H_0$). To see this, suppose that $A_n \to A$ in X_0 with $\operatorname{div} A_n \to u$ in H_0. But $A_n \to A$ in X_0 means that $\partial_k (A_n)^i_j \to \partial_k A^i_j$ in $L^2(\Omega)$ for all i, j, k so $\operatorname{div} A_n \to \operatorname{div} A$ in H_0. Hence, by the uniqueness of limits, $u = \operatorname{div} A$.

(2) By definition, the domain of div^* is

$$D(\operatorname{div}^*) = \{v \in H_0' \colon \exists\, C \ge 0 \text{ such that } |\langle v, \operatorname{div} A\rangle_{H_0', H_0}| \le C\,\|A\|_{X_0}\ \forall\, A \in X_0\}.$$

But, for any $v \in H_0'$, we have

$$|\langle v, \operatorname{div} A\rangle_{H_0', H_0}| \le \|v\|_{H_0'}\,\|\operatorname{div} A\|_{H_0} \le C\,\|\nabla A\|_{L^2} = C\,\|A\|_{X_0},$$

where $C = \|v\|_{H_0'}$. Hence, $D(\operatorname{div}^*)$ is all of H_0'.

(3) Also by the definition of the adjoint, we have

$$\langle v, \operatorname{div} A\rangle_{H_0', H_0} = \langle \operatorname{div}^* v, A\rangle_{X_0', X_0}$$

for all $v \in H_0'$, $A \in X_0$. For any $v \in H_0'$,

$$\langle v, \operatorname{div} A\rangle_{H_0', H_0} = (Iv, \operatorname{div} A) = -\langle \nabla I v, A\rangle_{H^{-1}, H^1_0},$$

where we applied Lemma 2.2 (to each component of v and row of A). But

$$\begin{aligned}
-\langle \nabla I v, A\rangle_{H^{-1}, H^1_0} &= -\langle \nabla I v - (\nabla I v)^T, A\rangle_{H^{-1}, H^1_0} - \langle (\nabla I v)^T, A\rangle_{H^{-1}, H^1_0} \\
&= -\langle \operatorname{curl} I v, A\rangle_{H^{-1}, H^1_0} - \langle \nabla I v, A^T\rangle_{H^{-1}, H^1_0} \\
&= -\langle \operatorname{curl} I v, A\rangle_{X_0', X_0} + \langle \nabla I v, A\rangle_{H^{-1}, H^1_0},
\end{aligned}$$

where we used that $\operatorname{curl} I v$ is antisymmetric and so lies in X_0'. It follows that

$$\langle \operatorname{div}^* v, A\rangle_{X_0', X_0} = \langle v, \operatorname{div} A\rangle_{H_0', H_0} = (Iv, \operatorname{div} A) = -\frac{1}{2}\langle \operatorname{curl} I v, A\rangle_{X_0', X_0}.$$

We conclude that $\operatorname{div}^* v = -(1/2)\operatorname{curl} I v$.

(4) div surjective follows from Proposition 4.5.

(5) and (6) follow, for instance, from Theorem 2.20 of [9]. □

We can now characterize the space $Y_0 = (V^d \cap X_0)^\perp$, which we used in the proof of Theorem 1.1:

Proposition 6.3. *Letting* $Y_0 = (V^d \cap X_0)^\perp$*, the orthogonal complement of* $V^d \cap X_0$ *in* X_0 *as a Hilbert space, we have*

$$Y_0 = \{z \in X_0 \colon \Delta z = \operatorname{curl} q \text{ for some } q \in L^2_0(\Omega)^d\}.$$

Proof. Fix $z \in X_0$. Then $z \in Y_0$ if and only if

$$(z, v)_{X_0} := (\nabla z, \nabla v) = -(\Delta z, v)_{X_0', X_0} = 0$$

for all $v \in V^d \cap X_0$. Thus, $z \in Y_0$ if and only if $\Delta z \in (V^d \cap X_0)^{\perp_B}$, where we have used $\perp_B$ here to refer to the subspace of X_0' that is orthogonal to $V^d \cap X_0$ in the duality between X_0' and X_0. But, $V^d \cap X_0 = \ker \operatorname{div}$, so $(V^d \cap X_0)^{\perp_B} = (\ker \operatorname{div})^{\perp_B} = \operatorname{range} \operatorname{curl}$. □

7. 3D VECTOR POTENTIALS

We can use Theorem 1.1 to obtain the more classical versions of 3D stream functions or vector potentials of Propositions 7.1 and 7.2 (cf., Theorems 3.5 and 3.6 Chapter I of [16] or Theorem 3.12 and 3.17 of [1]).

Proposition 7.1. *Let $u \in H_0$ for $d = 3$. There exists a vector potential $\overline{\psi} \in H$ for which* $\operatorname{curl}_3 \overline{\psi} = u$. *The vector potential is unique up to the addition of an arbitrary element in H_c; or, equivalently, the vector potential is unique if we require it to lie in H_0. If $\partial\Omega$ is $C^{1,1}$ then $\overline{\psi} \in H \cap H^1(\Omega)^3$.*

Proof. First, we show existence. Let ψ be the 3D stream function given by Theorem 1.1 and let p be the unique (up to an additive constant) solution to the Neumann problem,

$$\begin{cases} \Delta p = -\operatorname{div} \psi & \text{in } \Omega, \\ \nabla p \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

If $\partial\Omega$ is Lipschitz, we can only conclude that $p \in H^1(\Omega)$ so $\nabla p \in L^2(\Omega)^3$, but if $\partial\Omega$ is $C^{1,1}$ then $p \in H^2(\Omega)$ so $\nabla p \in H^1(\Omega)^3$. Letting $\overline{\psi} = \psi + \nabla p$, we see that

$$\begin{cases} \operatorname{curl}_3 \overline{\psi} = u & \text{in } \Omega, \\ \operatorname{div} \overline{\psi} = 0 & \text{in } \Omega, \\ \overline{\psi} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.2}$$

Hence, $\overline{\psi} \in H$ with $\operatorname{curl}_3 \overline{\psi} = u$, as required, with $\overline{\psi} \in H \cap H^1(\Omega)^3$ if $\partial\Omega$ is $C^{1,1}$.

Adding any element of H_c to $\overline{\psi}$ clearly yields another vector potential for u, and the difference of any two vector potentials for u lies in H and is curl-free; that is, it lies in H_c. This proves the uniqueness statement. □

Define the space,

$$\widetilde{H} := \{\psi \in L^2(\Omega)^3 \colon \operatorname{div} \psi = 0, \operatorname{curl} \psi \in L^2(\Omega)^3, \psi \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$$

with the norm $\|\psi\|_{\widetilde{H}} := \|\psi\| + \|\operatorname{curl} \psi\|$. That $\psi \times \boldsymbol{n}$ makes sense in terms of a trace is shown in Theorem 2.11 of [16]. Also let

$$\widetilde{H}_c := \{\psi \in \widetilde{H} \colon \operatorname{curl} \psi = 0\}.$$

Proposition 7.2. *Let $u \in H_0$ for $d = 3$. There exists a vector potential $\overline{\psi} \in \widetilde{H}$ for which* $\operatorname{curl}_3 \overline{\psi} = u$. *The vector potential is unique up to the addition of an arbitrary element in $\widetilde{H}_c$. If $\partial\Omega$ is $C^{1,1}$ then $\overline{\psi} \in \widetilde{H} \cap H^1(\Omega)^3$.*

Proof. The proof is the same as that of Proposition 7.1, but using the boundary condition $p = 0$ on $\partial\Omega$ in (7.1), noting that then $\nabla p \times \boldsymbol{n} = 0$. As in (7.2), this gives $\operatorname{curl}_3 \overline{\psi} = u$ and $\operatorname{div} \overline{\psi} = 0$ but with $\overline{\psi} \times \boldsymbol{n} = \psi \times \boldsymbol{n} + \nabla p \times \boldsymbol{n} = 0$ on $\partial\Omega$. Adding any element of $\widetilde{H}_c$ to $\overline{\psi}$ clearly yields another vector potential for u, and the difference of any two vector potentials for u lies in $\widetilde{H}$ and is curl-free; that is, it lies in $\widetilde{H}_c$. This proves the uniqueness statement. □

Suppose that $\Omega \subseteq \mathbb{R}^3$ has a finite number of boundary components $\Gamma_0, \cdots, \Gamma_N$. Then the vector potential $\overline{\psi}$ of Proposition 7.2 is unique if one imposes the condition $\int_{\Gamma_i} \overline{\psi} \cdot \boldsymbol{n} = 0$ for all i. This is shown in Theorem 3.6 Chapter I of [16] and 3.17 of [1]. The idea, in essence, is to use the boundary condition $p = c_i$ on Γ_i instead of $p = 0$ on $\partial\Omega$ in (7.1), and show that, fixing $c_0 = 0$, there exists a unique choice of the c_i such that $\int_{\Gamma_i} \nabla p \cdot \boldsymbol{n} = -\int_{\Gamma_i} \psi \cdot \boldsymbol{n}$ for all i. See, for instance, the argument on pages 49-50 of [16].

Remark 7.3. *The boundary condition $\psi \times \boldsymbol{n} = 0$ in the definition of $\widetilde{H}$ corresponds to $A\boldsymbol{n} = 0$ via the bijection given by (4.2). This suggests that Proposition 7.2 has a natural higher-dimensional formulation. Indeed for smooth boundaries it does, as follows from Theorem 3.1.1 of [23], in which $\overline{\psi}$ becomes a co-closed 2-form.*

8. A Biot-Savart kernel?

The Biot-Savart law is the classical method for obtaining a vector field in, say $H_0 \cap H^1(\Omega)^d$, having a given vorticity in $L^2(\Omega)$. But the existence of an integral representation for this law, that is, of a Biot-Savart kernel, for a bounded domain is a largely open question: the existence for all of $\mathbb{R}^d$ and for a bounded domain in $\mathbb{R}^2$ is quite classical, but only recently, in [13], has a kernel for a 3D bounded domain been obtained, and that was for domains with smooth boundary. In dimensions higher than 3 a kernel has not been obtained even for smooth domains. (Also, see the introductory comments in [13].)

To give a feeling for why obtaining a Biot-Savart kernel, even for smooth boundaries, is so difficult, let us examine an obvious approach that does not work. Start, following Section 1.3 of [11], with the Biot-Savart law for all of $\mathbb{R}^d$, which employs the fundamental solution E_d to the Laplacian in all of $\mathbb{R}^d$ (so $\Delta E_d * f = f$). We then define the vector-valued kernel $K_d = \nabla E_d$. Then if, say, $B \in (L^1 \cap L^\infty)(\mathbb{R}^d)^{d\times d}$ is antisymmetric, then defining the vector field u by $u^i := K_d^j * B_j^i$, we will have $u \in \dot{H}^1(\mathbb{R}^d)$, u divergence-free, with $\operatorname{curl} u = B$.

Now let $G(x,y) = E_d(x-y) + H(x,y)$ be the Green's function for the Dirichlet Laplacian on Ω. The obvious thing to try is to set $K_\Omega(x,y) := \nabla_x G(x,y) = K_d(x,y) + \nabla_x H(x,y)$. Then, operating formally, if $B \in L^2(\Omega)^{d\times d}$ is antisymmetric, let

$$\phi = \int_\Omega G(x,y)B(y)\,dy, \quad u^i = (\operatorname{div}\phi)^i = \int_\Omega K_\Omega^j(x,y)B_j^i(y)\,dy.$$

Since $G(x,\cdot) = 0$ for $x \in \partial\Omega$, we see that $\phi \in X_0$, so by Proposition 4.4, $u \in H_0$. And $\Delta\phi = B$, since G is the fundamental solution to the Laplacian. In 2D, $\Delta\phi = \operatorname{curl}\operatorname{div}\phi$, and one can verify that ϕ is the antisymmetric matrix form of the usual scalar 2D stream function, and in fact K_Ω this is the Biot-Savart kernel. In higher dimension, however, $\Delta \neq \operatorname{curl}\operatorname{div}$, so K_Ω is not the Biot-Savart kernel. Nor is there a clear way to correct this deficiency.

We can show, however, the conditional result in Theorem 8.1: a Biot-Savart kernel exists if and only if a kernel for the stream function exists, and there is a duality between them.

Theorem 8.1. *We say that $K \in L^1(\Omega^2)^d$ is a kernel for the Biot-Savart law on Ω if for all antisymmetric $B \in C(\overline{\Omega})^{d\times d}$,*

$$u^i(x) = \int_\Omega K^j(x,y)B_j^i(y)\,dy \tag{8.1}$$

lies in H_0 with $\operatorname{curl} u = B$. We say that $T \in L^1(\Omega^2)^d$ is a kernel for the stream function on Ω if for all $v \in H_0 \cap C^\infty(\overline{\Omega})^d$,

$$A_j^i(y) = \int_\Omega T_j(x,y)v^i(x)\,dx - \int_\Omega T_i(x,y)v^j(x)\,dx \tag{8.2}$$

lies in X_0 with $\operatorname{div} A = v$. A kernel K exists if and only if a kernel T exists, and in such a case, we can set $K = T$.

Proof. Assume that T exists. Let $v \in H_0 \cap C^\infty(\overline{\Omega})^d$ and let A be as given in (8.2). Let $u \in H_0 \cap C^\infty(\overline{\Omega})^d$ with $\operatorname{curl} u = B$. Then, applying Fubini's theorem,

$$\begin{aligned}(2u,v) &= 2(u,\operatorname{div} A) = -2(\nabla u, A) = -(\nabla u, A) - ((\nabla u)^T, A^T)\\ &= -(\nabla u, A) + ((\nabla u)^T, A) = -(\operatorname{curl} u, A) = -(B, A)\end{aligned}$$

$$= \int_\Omega \int_\Omega B^i_j(y) \left[T_i(x,y) v^j(x)\, dx - \int_\Omega T_j(x,y) v^i(x)\, dx \right] dy$$
$$= \int_\Omega \int_\Omega B^i_j(y) T_i(x,y) v^j(x)\, dx - \int_\Omega \int_\Omega B^i_j(y) T_j(x,y) v^i(x)\, dx\, dy$$
$$= \int_\Omega \int_\Omega B^i_j(y) T_i(x,y) v^j(x)\, dx - \int_\Omega \int_\Omega B^j_i(y) T_i(x,y) v^j(x)\, dx\, dy$$
$$= 2 \int_\Omega \int_\Omega B^i_j(y) T_i(x,y) v^j(x)\, dx\, dy = (2w, v),$$

where

$$w(x) = \int_\Omega T_i(x,y) B^i_j(y)\, dy.$$

Since $H_0 \cap C^\infty(\overline{\Omega})^d$ is dense in H_0 it follows that we must have $u = w$. Examining (8.1), then, we see that we can set $K = T$.

To show that the existence of K implies the existence of T, we reverse the order of the integrations by parts. □

9. A further decomposition of H in 3D

In 3D, we have two types of stream functions for any $u \in H_0$: that given by Theorem 1.1 and the more classical one given by Proposition 7.1. The former lacks the divergence-free condition, but, like 2D stream functions, vanishes entirely on the boundary, which eliminates many boundary terms when integrating by parts. The latter is only tangential to the boundary, but is divergence-free, a condition whose main usefulness is that $\operatorname{curl}_3^2 \psi = -\Delta\psi$ for such stream functions, so that $\operatorname{curl}_3 u = -\Delta\psi$, as for 2D stream functions.[4] Hence, each form has one and only one of these two key features of 2D stream functions.

Note that any element of V qualifies as a stream function of both types, as it is both divergence-free and vanishes on (and so is normal to) the boundary. Hence, it has both key features of 2D stream functions. Thus, it is natural to consider what elements of H are created from such a stream function; that is, to look at the space, $\operatorname{curl}_3 V \subseteq H_0$.

Another motivation for considering this space is that solutions to the Navier-Stokes equations with no-slip boundary conditions lie in V, and hence their curl lies in $\operatorname{curl} V$. In the vorticity formulation of the Navier-Stokes equations this is particularly important, since the velocity is recovered from the vorticity (curl of the velocity) via the Biot-Savart law. Hence, there may be utility in having some understanding of $\operatorname{curl} V$ as a subspace of H_0; this is the purpose of Proposition 9.1.

Proposition 9.1. *We have* $\operatorname{curl}_3 V^\perp = (\operatorname{curl}_3 V)^\perp$*, giving the orthogonal decomposition,*

$$H_0 = \operatorname{curl}_3(V \oplus V^\perp) = \operatorname{curl}_3 V \oplus \operatorname{curl}_3 V^\perp,$$

or, to be more explicit,

$$H_0 = \operatorname{curl}_3(V \oplus_{H_0^1(\Omega)^3} V^\perp) = \operatorname{curl}_3 V \oplus_H \operatorname{curl}_3 V^\perp.$$

Also,

$$\operatorname{curl}_3 V^\perp \subseteq \{u \in H_0 \colon u \text{ is harmonic}\}, \tag{9.1}$$

with equality if Ω *is simply connected.*

[4]If, in 2D, $\nabla^\perp$ is defined as ∇ rotated clockwise 90°; we have used the counterclockwise convention, which gives $\operatorname{curl} u = \Delta\psi$.

Proof. By Proposition 2.7, $H_0^1(\Omega)^3 = V \oplus V^\perp$, so by Theorem 1.1, $H_0 = \operatorname{curl}_3 V + \operatorname{curl}_3 V^\perp$. Hence, we need only show that $\operatorname{curl}_3 V^\perp \subseteq (\operatorname{curl}_3 V)^\perp$. To see this, write arbitrary elements in $\operatorname{curl}_3 V$, $\operatorname{curl}_3 V^\perp$ as $\operatorname{curl}_3 u$, $\operatorname{curl}_3 z$, where $u \in V$, $z \in V^\perp$. Then, applying Lemma 2.3,

$$(\operatorname{curl}_3 u, \operatorname{curl}_3 z) = (u, \operatorname{curl}_3^2 z) = -(u, \Delta z + \nabla \operatorname{div} z) = -(u, \nabla(q + \operatorname{div} z)) = 0,$$

where we applied Proposition 2.7 to know that $\Delta z = \nabla q$ for some $q \in L^2(\Omega)$.

Moreover, it follows that if $w \in (\operatorname{curl}_3 V)^\perp$ then $w = \operatorname{curl}_3 z$, where $\Delta z = \nabla q$ for some $q \in L^2(\Omega)$, since $z \in V^\perp$. But then $\Delta w = \operatorname{curl}_3 \Delta z = 0$, giving (9.1).

Now assume that Ω is simply connected and that $v \in H = H_0$ is harmonic. Then $v = \operatorname{curl}_3 \psi_v$ for some $v \in H_0^1(\Omega)^3$, so $\Delta v = \operatorname{curl}_3 \Delta \psi_v = 0$ in $H^{-2}(\Omega)$, so we know by Poincaré's lemma that $\Delta \psi_v$ is a gradient, and hence $\psi_v \in V^\perp$. That is, $v \in H$ harmonic implies that $v = \operatorname{curl}_3 \psi_v \in \operatorname{curl}_3 V^\perp$, giving equality in (9.1). □

10. An alternate characterization of H and ΔV

Proposition 10.1 shows that H_0 is the space of minimizers of the L^2 norm over all vector fields in L^2 having a given H^{-1} vorticity. This gives a characterization of H_0 without a priori assuming either the divergence-free condition or the no-penetration condition.

Proposition 10.1. *Let u be a vector field in $L^2(\Omega)$. There exists a unique minimizer $\overline{u} \in L^2(\Omega)$ to*

$$\min\{\|w\|_{L^2} : w \in L^2(\Omega)^d, \operatorname{curl} w = \operatorname{curl} u\}.$$

Moreover, $\overline{u} = P_{H_0} u$, where P_{H_0} is orthogonal projection onto the space H_0 defined in Section 4. When Ω is simply connected, $\overline{u} = P_H u$, where P_H is the classical Leray projector of vector fields in $L^2(\Omega)$ onto H. (The equalities $\operatorname{curl} w = \operatorname{curl} u$ and $\operatorname{div} \overline{u} = 0$ are as elements of H^{-1}.)

Proof. This is an immediate consequence of the decomposition, $L^2(\Omega)^d = H_0 \oplus H_c \oplus G$, where G is the space of gradients in $L^2(\Omega)^d$, since elements of H_c and G both have vanishing curl. □

More interesting is the analogous statement for ΔV as a subspace of $H^{-1}(\Omega)^d$:

Proposition 10.2. *Add the assumption that Ω is simply connected. Let $u \in H^{-1}(\Omega)^d$. There exists a unique minimizer $\overline{u} \in H^{-1}(\Omega)^d$ to*

$$\min\{\|w\|_{H^{-1}} : w \in H^{-1}(\Omega)^d, \operatorname{curl} w = \operatorname{curl} u\}.$$

Moreover, $\overline{u}$ is in the image of Δ applied to V; in particular, $\operatorname{div} \overline{u} = 0$. (The equalities $\operatorname{curl} w = \operatorname{curl} u$ and $\operatorname{div} \overline{u} = 0$ are as elements of H^{-2}.)

Proof. By Proposition 2.8, we can uniquely write $u = \Delta v + \nabla q$ for some $v \in V$, $q \in L^2/\mathbb{R}$, where $L^2/\mathbb{R}$ is the set of all functions in $L^2(\Omega)$ having mean zero. We will directly show that $\overline{u}$ exists and that, in fact, $\overline{u} = \Delta v$.

First, let us characterize all possible candidates for our desired minimizer. So let

$$w \in S := \{w \in H^{-1}(\Omega)^d \colon \operatorname{curl} w = \operatorname{curl} u\}$$

be arbitrary. Applying Poincaré's lemma to $u - w$, we see that u and w differ by a gradient. Hence, we seek a minimizer $\overline{u}$ of the form

$$\overline{u} = \Delta v + \nabla \overline{q}$$

for some $\overline{q} \in L^2/\mathbb{R}$. It is only in establishing this form for the minimizer that we use Ω being simply connected. (Note that this would not follow simply from Proposition 2.8, which would only give $\overline{u} = \Delta \overline{v} + \nabla \overline{q}$ with $\operatorname{curl} \Delta \overline{v} = \operatorname{curl} \Delta v$.)

What we must show is that choosing $\overline{q} = 0$ produces the minimizer. Toward this end, first let us determine $\|\Delta v\|_{H^{-1}}$ by pairing it with an arbitrary $\varphi \in H_0^1(\Omega)^d$. By Proposition 2.11,

$$(\Delta v, \varphi) = (\Delta v, P_V \varphi) = -(\nabla v, \nabla P_V \varphi).$$

Hence,

$$(\Delta v, \varphi) = -(\nabla v, \nabla P_V \varphi) \le \|\nabla v\| \, \|\nabla P_V \varphi\| = \|v\|_V \, \|P_V \varphi\|_V \le \|v\|_V \, \|\varphi\|_V \, .$$

It follows that $\|\Delta v\|_{H^{-1}} \le \|v\|_V$. Choosing $\varphi = -v$, we see that equality is achieved. Hence,

$$\|\Delta v\|_{H^{-1}} = \|v\|_V = \|v\|_{H_0^1} \, .$$

Now assume that $\nabla \overline{q} \ne 0$. We will show that there exists $\varphi \in H_0^1(\Omega)^d$ for which

$$\frac{|(\overline{u}, \varphi)|}{\|\varphi\|_{H_0^1}} > \|v\|_V \, ,$$

from which it will follow that $\overline{u} = \Delta v$ is the desired unique minimizer.

By Proposition 2.7, we can write any $\varphi \in H_0^1(\Omega)^d$ in the form $\varphi = P_V \varphi + z$. We will choose φ so that $P_V \varphi = -v$, giving $\varphi = -v + z$, leaving $z \in V^\perp$ and z alone to be freely chosen.

Then,

$$\begin{aligned}
(\overline{u}, \varphi) &= (\Delta v + \nabla \overline{q}, -v + z) = -(\Delta v, v) + (\Delta v, z) - (\nabla \overline{q}, -v) + (\nabla \overline{q}, z) \\
&= (\nabla v, \nabla v) - (\nabla v, \nabla z) - 0 - (\overline{q}, \operatorname{div} z) = (\nabla v, \nabla v) + 0 - 0 - (\overline{q}, \operatorname{div} z) \\
&= \|v\|_V^2 - (\overline{q}, \operatorname{div} z),
\end{aligned}$$

where (2.5) gave us $(\nabla v, \nabla z) = 0$. Hence,

$$\|\overline{u}\|_{H^{-1}} \ge \sup_{z \in V^\perp} \frac{\|v\|_V^2 - (\overline{q}, \operatorname{div} z)}{\|-v + z\|_V} = \sup_{z \in V^\perp} \frac{\|v\|_V^2 - (\overline{q}, \operatorname{div} z)}{\sqrt{\|v\|_V^2 + \|z\|_V^2}},$$

where we used the orthogonality of the projection operator, P_V.

Now, given any $a > 0$, we can choose (z_a, r_a) so that it is a weak solution to

$$\begin{cases} -\Delta z_a + \nabla r_a = 0 & \text{in } \Omega, \\ \operatorname{div} z_a = -a \overline{q} & \text{in } \Omega, \\ z_a = 0 & \text{on } \Omega. \end{cases}$$

This is uniquely solvable for $z_a \in H^{-1}$, $r_a \in L^2/\mathbb{R}$ by Exercise IV.1.1 of [14], the same result we reference in the proof of Proposition 2.7, because the compatibility condition,

$$\int_\Omega (-a \overline{q}) = -a \int_\Omega \overline{q} = 0 = \int_{\partial\Omega} z_a$$

is satisfied. Noting that $z_a = a z_1$, we have $\|z_a\|_V = a \|z_1\|_V$. Thus, setting $z = z_a$ in our estimate on $\|\overline{u}\|_{H^{-1}}$, it follows that

$$\|\overline{u}\|_{H^{-1}} \ge \sup_{a > 0} \frac{\|v\|_V^2 + a \|\overline{q}\|^2}{\sqrt{\|v\|_V^2 + a^2 \|z_1\|_V^2}}.$$

At this point, $\overline{q}$ and hence z_1 are fixed, but we are free to choose any $a > 0$ so that

$$a \|\overline{q}\|^2 > a^2 \|z_1\|_V^2 \, ;$$

that is, so that

$$a < \frac{\|\overline{q}\|^2}{\|z_1\|_V^2}.$$

This allows us to conclude that $\|\overline{u}\|_{H^{-1}} > \|v\|_V$. Or, more explicitly,

$$\|\overline{u}\|_{H^{-1}} \geq \frac{\|v\|_V^2 + a\|\overline{q}\|^2}{\sqrt{\|v\|_V^2 + a^2\|z_1\|_V^2}} > \|v\|_V = \|\Delta v\|_{H^{-1}}.$$

□

To be more explicit in the bound from below of $\|\overline{u}\|_{H^{-1}}$, we can use the classical estimate on solutions to the Stokes problem, which for z_1 gives

$$\|z_1\|_V \leq C_0 \|\overline{q}\|.$$

(See, for instance, Exercise IV.1.1 of [14].) Then,

$$\|\overline{u}\|_{H^{-1}} \geq \frac{\|v\|_V^2 + a\|\overline{q}\|^2}{\sqrt{\|v\|_V^2 + C_0^2 a^2\|\overline{q}\|^2}}$$

for any $a < C_0^{-1}$. Using elementary Calculus, the resulting maximal lower bound occurs when $a = C_0^{-2}$, giving

$$\|\overline{u}\|_{H^{-1}} \geq \frac{\|v\|_V^2 + C_0^{-2}\|\overline{q}\|^2}{\sqrt{\|v\|_V^2 + C_0^{-2}\|\overline{q}\|^2}} = \sqrt{\|v\|_V^2 + C_0^{-2}\|\overline{q}\|^2}.$$

11. Application: A simple proof of Poincaré's Lemma

Proposition 11.1 is a version of Poincaré's Lemma, which we prove as a corollary of de Rham's lemma, Proposition 2.6, along with Theorem 5.2.

Proposition 11.1. *Adding the assumption that Ω is simply connected, let f be a vector field in $H^{-1}(\Omega)^d$. Then $\operatorname{curl} f = 0$ in $H^{-2}(\Omega)$ if and only if $f = \nabla q$ for some unique $q \in L^2_0(\Omega)$.*

Proof. The reverse implication is immediate. For the forward implication, fix $f \in (H^{-1})^d$ and let $v \in V \subseteq H = H_0$ be arbitrary. By Theorem 5.2, $v = \operatorname{div} A$ for some $A \in Y_0 \cap H^2_0(\Omega)^{d \times d}$. Then,

$$(f, v)_{H^{-1}, H^1_0} = (f, \operatorname{div} A)_{H^{-1}, H^1_0} = -(\nabla f, A)_{H^{-2}, H^2_0} = (\nabla f, A^T)_{H^{-2}, H^2_0},$$

since A is antisymmetric. But also,

$$\begin{aligned} -(\nabla f, A)_{H^{-2}, H^2_0} &= -(\nabla f - (\nabla f)^T, A)_{H^{-2}, H^2_0} - ((\nabla f)^T, A)_{H^{-2}, H^2_0} \\ &= -((\nabla f)^T, A)_{H^{-2}, H^2_0} = -(\nabla f, A^T)_{H^{-2}, H^2_0}, \end{aligned}$$

since $\operatorname{curl} f = 0$. We conclude that $(f, v)_{H^{-1}, H^1_0} = 0$ and hence from Proposition 2.6 that $f = \nabla q$ for some $q \in L^2(\Omega)$. If $\overline{q}$ is another such element of $L^2(\Omega)$ then $\nabla(q - \overline{q}) = 0$ in $H^{-1}(\Omega)$ so they must differ by a constant. This gives the uniqueness of $q \in L^2_0(\Omega)$.[5] □

Remark 11.2. *For other relatively simple proofs of Proposition 11.1, see Theorem 2.1 of [12] (also see Theorem 3.1 of [20]). There is a short, clear, and simple proof of the result in [12] given by Kesavan in [19]. He uses a solution to the stationary Stokes problem and, most important, uses a Lemma of Lions, which states that if $q \in \mathcal{D}'(\Omega)$ with $\nabla q \in H^{-1}(\Omega)$ then $q \in L^2(\Omega)$, whose proof for Lipschitz domains is due to Amrouche and Girault [3]. Also, see the historical comments in [2].*

[5]Ultimately, this relies upon the divergence operator mapping $H^1_0(\Omega)^d$ *onto* $L^2(\Omega)$, itself a non-trivial result.

Remark 11.3. *The decomposition in Proposition 2.8 is not sufficient to prove Proposition 11.1. To see this, observe that by Proposition 2.8, we have $f = \Delta v + \Delta z$ for some $v \in V$, $z \in V^\perp$. Then*

$$\operatorname{curl} f = \Delta \operatorname{curl} v + \Delta \operatorname{curl} z = \Delta \operatorname{curl} v,$$

since $\Delta \operatorname{curl} z = \operatorname{curl} \Delta z = -\operatorname{curl} \nabla q = 0$ as elements of $H^{-2}(\Omega)$. Assuming $\operatorname{curl} f = 0$ it follows that $\operatorname{curl}(\Delta v) = \Delta \operatorname{curl} v = 0$. But this only shows that it is sufficient to establish Proposition 11.1 for $f \in \Delta V$, our near proxy for V' (see Remark 2.13).

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