VANISHING VISCOSITY FOR VORTICITY ZERO IN A LAYER NEAR THE BOUNDARY: IS THERE AN EASY WAY OUT?

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ABSTRACT. Assuming that the initial vorticity is zero in a boundary layer and the initial velocity vanishes on the boundary allows us to conclude that the Euler velocity is a gradient in a boundary layer. We attempt to exploit this to obtain the vanishing viscosity limit using a straightforward energy argument, but with a corrector that is a gradient rather than being divergence-free. The approach fails, but it in a somewhat instructive way.

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1. Introduction

We assume that Ω is a bounded domain in $\mathbb{R}^2$ with, for simplicity, a C^∞ boundary Γ.[1] Define the classical functions spaces,

$$\begin{aligned} H &:= \left\{ v \in L^2(\Omega)^2 \colon \operatorname{div} v = 0, v \cdot \boldsymbol{n} = 0 \right\}, \\ V &:= \left\{ v \in H^1(\Omega)^2 \colon \operatorname{div} v = 0, v = 0 \right\}. \end{aligned}$$

Adopting the notation of Kato [4], we let u be the solution to the Navier-Stokes equations with no-slip boundary conditions on

$$Q := [0, T] \times \Omega$$

for some fixed $T > 0$ with initial velocity $u^0 \in H \cap C^\infty(\Omega)^2$. That is,

$$(NS) \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u & \text{in } Q, \\ \operatorname{div} u = 0 & \text{in } Q, \\ u(0) = u^0 & \text{in } \Omega, \\ u = 0 & \text{on } [0, T] \times \partial\Omega. \end{cases}$$

Note that u depends upon ν, though, following Kato, we suppress ν in our notation.

[1] A $C^{2,\varepsilon}$ boundary for some $\varepsilon \in (0, 1)$ would suffice.

When $\nu = 0$, (NS) formally reduces to the Euler equations with no-penetration boundary conditions:

$$(E) \begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = 0 & \text{in } Q, \\ \operatorname{div} \overline{u} = 0 & \text{in } Q, \\ \overline{u}(0) = u^0 & \text{in } \Omega, \\ \overline{u} \cdot \boldsymbol{n} = 0 & \text{on } [0, T] \times \partial\Omega. \end{cases}$$

Here, $\boldsymbol{n}$ is the outward unit normal to Γ. We also define the unit tangent vector field $\boldsymbol{\tau}$ with $(\boldsymbol{n}, \boldsymbol{\tau})$ in the standard orientation of $(\boldsymbol{n}, \boldsymbol{\tau})$.

For both (NS) and (E) we have assumed zero external forcing.

Also as in [4], we define the boundary layer

$$\Gamma_\delta := \{x \in \Omega \colon \operatorname{dist}(x, \Gamma) < \delta\}.$$

We say that the *(classical) vanishing viscosity limit* holds if

$$u \to \overline{u} \text{ in } L^\infty(0, T; H) \text{ as } \nu \to 0. \tag{1.1}$$

Whether or not (1.1) holds in general is a long open problem. In 1983, Tosi Kato [4] gave a simple, elegant energy argument that gave a set of necessary and sufficient conditions on the solution to the Navier-Stokes equations for the limit to hold. The focus was on the Navier-Stokes solutions, but a corrector to the Euler equations was employed in such a way as to give hope that a more refined corrector might allow the limit to be established, at least in special circumstances, with less detailed analysis of the Navier-Stokes solutions. Indeed, this turned out to be the case for initial data and geometry with special symmetries. We mention only [3], which contains further references (many of them by one or more of the authors of [3]) to this branch of the literature.

In 2014, Yasunori Maekawa [8] published a startling result. Working in a half-plane, he showed that if the initial vorticity is supported away from the boundary and the initial velocity vanishes on the boundary ($u_0 \in V$) then (1.1) holds. The result is startling in part because it has a direct and meaningful physical interpretation that would be broadly applicable. Since then, multiple proofs and refinements have been published, and the connections to [1, 9, 10] and the role that partial analyticity of solutions to the Navier-Stokes equations became progressively clearer. We mention only [2], which also contains references to this branch of the literature

All of these analyses following on from Maekawa's paper are technically complex, so it is natural to wonder whether or not Kato's original, simple energy-based approach might be able to yield the result more easily. The reason to think that perhaps Kato's argument might be adapted in Maekawa's case is that for the Euler solutions, the vorticity remains zero over time in a (albeit, shrinking) layer near the boundary, and the velocity field in that layer is a gradient. One reason to think that Kato's argument should not be adaptable is that it can take no account of partial analyticity of a solution, and hence it would seem that if the approach works, it must work in the broader setting of a domain in which the boundary itself need not be analytic.

One further motivation for trying the Kato-like approach is that, should it work, it would not require the boundary to be analytic, which would make the result much more robust and physically meaningful. All existing results that apply to Maekawa-like initial data require an analytic boundary for convergence to hold,[2] so that even the smallest (and physically meaningful) change in the boundary might yield to a failure of convergence in (1.1), a kind of instability. Should the Kato approach work,

It what follows we explore a Kato-like approach, doomed to failure though it may be.

2. Vorticity vanishing near the boundary

From now on, we add assumptions on the initial data like those in [8]:

$$u^0 \in V \cap C^\infty(\Omega)^2 \text{ with } \omega_0 := \operatorname{curl} u_0 \equiv 0 \text{ in } \Gamma_{\delta_0} \text{ for some } \delta_0 > 0. \tag{2.1}$$

As a consequence of these assumptions, we know that $\omega(t) := \operatorname{curl} u(t)$ is zero on $\Gamma_{\delta(t)}$ with

$$\delta(t) \geqslant Ce^{-Ct}.$$

The velocity field $\overline{u}$ does not, however, continue to vanish on the boundary (in general).

To each component of Γ there exists one component of Γ_δ. Let Γ_i be one boundary component and U_i the corresponding component of Γ_δ. Remove a cut made perpendicularly to the boundary from U_i to create the simply connected domain $\dot{U}_i$. On $\dot{U}_i$, $\overline{u}$ is curl-free and hence is a gradient: $\overline{u} = \nabla q_i$ on $\dot{U}_i$. Moreover, the circulation of u_0 along each boundary component is zero, since $u_0 \in V$, and circulation is conserved for force-free solutions to the Euler equations, so $\overline{u}(t)$ has circulation zero on Γ_i. The change in the value of q_i on Γ_i from one side of the cut to the other is equal to the total circulation along Γ_i, and is therefore zero. From this, we conclude that, in fact, $\overline{u} = \nabla q_i$ on Γ_i, and hence there is some $q_{\overline{u}}$ on $\overline{\Omega}$ for which

$$\overline{u} = \nabla q_{\overline{u}} \text{ in } \Gamma_\delta.$$

Now let $\varphi \in C^\infty([0,\infty))$ with $\varphi \equiv 1$ on $[0, 1/2]$ and $\varphi \equiv 0$ on $[1,\infty)$, and define $\varphi_\delta(\cdot) := \varphi(\cdot/\delta)$. Then define

$$q := \varphi_\delta q_{\overline{u}}, \quad z := \nabla q = \varphi_\delta \nabla q_{\overline{u}} + q_{\overline{u}} \nabla\varphi_\delta = \varphi_\delta \overline{u} + q_{\overline{u}} \nabla \varphi_\delta.$$

This is the analog of the divergence-free corrector of Kato, as in [7]. Unlike the Kato corrector, however, z is not small even in the L^2 norm. This is because $q_{\overline{u}}$ will not, in general, equal zero on the boundary, so $q_{\overline{u}} \nabla \varphi_\delta$ will scale like δ^{-1}. Hence, $\|z\| \sim C\delta^{-\frac{1}{2}}$. By contrast, this term is $\psi \nabla^\perp \varphi_\delta$ for the divergence-free corrector of Kato, where ψ is the stream function for $\overline{u}$. Since we can assume that ψ is zero near any given boundary component, $\psi \nabla^\perp \varphi_\delta$ is bounded by a constant, so $\|z\| \sim C\delta^{\frac{1}{2}}$.

In short, we cannot assume that z is small or even of unit size. And, in fact, we can even used a fixed $\delta(t)$ for all $t \in [0, T]$, so we will assume that

$$\delta(t) = \delta(T) \tag{2.2}$$

for all $t \in [0, T]$. Then z and all of its derivatives are bounded uniformly over $t \in [0, T]$. We will still refer to z as a *corrector*, as it does "correct" the boundary value of the Euler velocity $\overline{u}$ to make it zero, but we should keep in mind that it is only of unit size in all pertinent norms.

[2]We note that in one of these results, [11], required assumptions on the boundary are never stated. We can see from the beginning of Section 8 of [11], however, that some degree of analyticity is assumed indirectly by assuming that the corresponding Riemann map lies in a specific analytic class.

3. Quick analysis

As a warm up, we make a quick energy argument.

Letting

$$w := u - \overline{u} \in H,$$

we see that

$$\partial_t w + P_H(u \cdot \nabla w + w \cdot \nabla \overline{u}) = \nu P_H \Delta u.$$

Pairing with w and using that $(P_H v, w) = (v, P_H w) = (v, w)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 &= -(w \cdot \nabla \overline{u}, w) + \nu(\Delta u, w) \leqslant \|\nabla \overline{u}\|_{L^\infty}\|w\|^2 + \nu(\Delta u, w - z) + \nu(\Delta u, z) \\ &\leqslant C\|w\|^2 - \nu(\nabla u, \nabla(w - z)) + \nu(\Delta u, z).\end{aligned}$$

But,

$$\begin{aligned}-(\nabla u, \nabla(w - z)) &= -(\nabla u, \nabla w) + (\nabla u, \nabla z), \\ -(\nabla u, \nabla w) &= -(\nabla w, \nabla w) - (\nabla \overline{u}, \nabla w) \leqslant -\|\nabla w\|^2 + \frac{1}{2}\|\nabla w\|^2 + \frac{1}{2}\|\overline{u}\|^2 \\ &\leqslant C - \frac{1}{2}\|\nabla w\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{\nu}{2}\|\nabla w\|^2 &\leqslant C\nu + C\|w\|^2 + \nu(\Delta u, z) + \nu(\nabla u, \nabla z) \\ &= C\nu + C\|w\|^2 - \nu(\nabla u, \nabla z) + \nu\int_\Gamma (\nabla u \cdot \boldsymbol{n}) \cdot z + \nu(\nabla u, \nabla z) \\ &= C\nu + C\|w\|^2 + \nu\int_\Gamma (\nabla u \cdot \boldsymbol{n}) \cdot \overline{u}.\end{aligned}$$

If we could control the boundary integral sufficiently, then we could apply Grönwall's lemma to obtain (1.1).

Notice that although we used the corrector, the inequality we obtained does not involve the corrector. If we had used (NS) to replace the term $\nu(\Delta u, z)$ with $(\partial_t u, v) + (u \cdot \nabla u, v) + (\nabla p, v)$, we would have been led to Kato's classical conditions in [4] for the vanishing viscosity limit to hold, giving an alternate, perhaps shorter but more roundabout, derivation of Kato's conditions. Regularity issues and having only an energy inequality in dimensions three and higher would make this approach more difficult, however.

Moreover, we never used our special assumptions in (2.1). To do so, as in Section 8 of [6], we can use (4.2) of [5], to write,

$$(\nabla u \cdot \boldsymbol{n}) \cdot \overline{u} = ((\nabla u \cdot \boldsymbol{n}) \cdot \boldsymbol{\tau})\, \overline{u} \cdot \boldsymbol{\tau} = \omega\, \overline{u} \cdot \boldsymbol{\tau} = \omega\, \nabla q \cdot \boldsymbol{\tau} \tag{3.1}$$

and

$$\int_\Gamma \omega\, \nabla q \cdot \boldsymbol{\tau} = -\int_\Gamma (\nabla \omega \cdot \boldsymbol{\tau})\, q = \int_\Gamma (\nabla^\perp \omega \cdot \boldsymbol{n})\, q = \int_\Gamma (\Delta u \cdot \boldsymbol{n})\, q,$$

so

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 + \frac{\nu}{2}\|\nabla w\|^2 &\leqslant C\nu + C\|w\|^2 + \nu\int_\Gamma \omega\, \overline{u} \cdot \boldsymbol{\tau} \\ &= C\nu + C\|w\|^2 + \int_\Gamma (\Delta u \cdot \boldsymbol{n})\, q.\end{aligned}$$

This inequality, however, would seem to be of no advantage.

Perhaps in employing the corrector only this late stage, we have missed something that would allow us to make productive use of it. We explore this in the next section, by making a longer argument in which we modify Kato's energy argument in [4] to use our special assumptions in (2.1).

4. Kato's basic energy argument

Observe that Kato's basic energy argument uses that the corrector z is divergence-free in only one (and a very important one) way: to eliminate the pressure.[3] Let us try to parallel that energy argument, as it appears in the proof of Proposition 4.1 of [7], taking care to avoid any use that appears in that proof of bounds on z.

As in the proof of Proposition 4.1 of [7], we define

$$\begin{aligned} &w := u - \overline{u} \in H, \\ &\widetilde{w} := w - z = u - \overline{u} - z \in H_0^1(\Omega)^2. \end{aligned}$$

Examining the proof of Proposition 4.1 of [7], we see that after pairing (4.3) of [7] with $\widetilde{w}$, we have

$$(\partial_t w, \widetilde{w}) = \nu(\Delta u, \widetilde{w}) - (u \cdot \nabla w, \widetilde{w}) - (w \cdot \nabla \overline{u}, \widetilde{w}) + (\nabla(\overline{p} - p), \widetilde{w}).$$

To bound these terms, we depart somewhat from [7]. We start with

$$\begin{aligned} (\partial_t w, \widetilde{w}) &= \frac{1}{2}\frac{d}{dt}\|w\|^2 - (\partial_t w, z) = \frac{1}{2}\frac{d}{dt}\|w\|^2 - (\partial_t w, \nabla q) = \frac{1}{2}\frac{d}{dt}\|w\|^2, \\ \nu(\Delta u, \widetilde{w}) &= -\nu(\nabla u, \nabla \widetilde{w}) = -\nu(\nabla u, \nabla w) + \nu(\nabla u, \nabla z) \\ &= -\nu(\nabla w, \nabla w) - \nu(\nabla \overline{u}, \nabla w) + \nu(\nabla u, \nabla z) \\ &\leqslant -\nu\|\nabla w\|^2 + \frac{\nu}{2}\|\nabla \overline{u}\|^2 + \frac{\nu}{2}\|\nabla w\|^2 + \nu(\nabla u, \nabla z) \\ &\leqslant C\nu - \frac{\nu}{2}\|\nabla w\|^2 + \nu(\nabla u, \nabla z), \\ -(u \cdot \nabla w, \widetilde{w}) &= -(u \cdot \nabla w, w) + (u \cdot \nabla w, z) = (u \cdot \nabla w, z) \\ &= (u \cdot \nabla u, z) - (u \cdot \nabla \overline{u}, z) = -(u \cdot \nabla z, u) - (u \cdot \nabla \overline{u}, z), \\ -(w \cdot \nabla \overline{u}, \widetilde{w}) &= -(w \cdot \nabla \overline{u}, w) + (w \cdot \nabla \overline{u}, z) \\ &= -(w \cdot \nabla \overline{u}, w) + (u \cdot \nabla \overline{u}, z) - (\overline{u} \cdot \nabla \overline{u}, z). \end{aligned}$$

Hence,

$$\begin{aligned} -(u \cdot \nabla w, \widetilde{w}) - (w \cdot \nabla \overline{u}, \widetilde{w}) &= -(u \cdot \nabla z, u) - (w \cdot \nabla \overline{u}, w) - (\overline{u} \cdot \nabla \overline{u}, z) \\ &\leqslant \|\nabla \overline{u}\|_{L^\infty}\|w\|^2 - (u \cdot \nabla z, u) - (\overline{u} \cdot \nabla \overline{u}, z) \\ &\leqslant C\|w\|^2 - (u \cdot \nabla z, u) + ((\partial_t \overline{u} + \nabla \overline{p}), z) \\ &= C\|w\|^2 - (u \cdot \nabla z, u) + \nabla \overline{p}, z), \end{aligned}$$

since $\partial_t \overline{u} \in H$, so $(\partial_t \overline{u}, z) = (\partial_t \overline{u}, \nabla q) = 0$.

[3]Although in the estimates on the divergence-free corrector, $\operatorname{div} z = 0$ is also used to trade a normal derivative for a benign tangential derivative, yielding tighter bounds on z.

This leads to the bound,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{\nu}{2}\|\nabla w\|^2 \leqslant & C\nu+C\|w\|^2\\ &-(u\cdot\nabla z,u)+\nu(\nabla u,\nabla z)+(\nabla\overline{p},z)+(\nabla(\overline{p}-p),\widetilde{w}).\end{aligned}$$

In [7], $(\nabla(\overline{p}-p),\widetilde{w})$ vanishes because z, and so $\widetilde{w}$, is divergence-free, but now we have

$$(\nabla(\overline{p}-p),\widetilde{w})=(\nabla(\overline{p}-p),w-z)=-(\nabla(\overline{p}-p),z)=(\nabla p,z)-(\nabla\overline{p},z).$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{\nu}{2}\|\nabla w\|^2 \leqslant C\nu+C\|w\|^2-(u\cdot\nabla z,u)+\nu(\nabla u,\nabla z)+(\nabla p,z). \tag{4.1}$$

Now,

$$(\nabla p,z)=(\nabla p,\nabla q)=-(\Delta p,q)+\int_\Gamma(\nabla p\cdot\boldsymbol{n})q.$$

But,

$$-(\Delta p,q)=(\operatorname{div}(u\cdot\nabla u),q)=-(u\cdot\nabla u,\nabla q)=-(u\cdot\nabla u,z)=(u\cdot\nabla z,u)$$

and

$$\int_\Gamma(\nabla p\cdot\boldsymbol{n})q=\nu\int_\Gamma(\Delta u\cdot\boldsymbol{n})q=\nu(\Delta u,\nabla q)+(\operatorname{div}\Delta u,q)=\nu(\Delta u,z).$$

We have,

$$\nu(\Delta u,z)=-\nu(\nabla u,\nabla z)+\nu\int_\Gamma(\nabla u\cdot\boldsymbol{n})\cdot z=-\nu(\nabla u,\nabla z)+\nu\int_\Gamma(\nabla u\cdot\boldsymbol{n})\cdot\overline{u}.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{\nu}{2}\|\nabla w\|^2 \leqslant C\nu+C\|w\|^2+\nu\int_\Gamma(\nabla u\cdot\boldsymbol{n})\cdot\overline{u}.$$

Using (3.1), this becomes

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{\nu}{2}\|\nabla w\|^2 \leqslant C\nu+C\|w\|^2+\nu\int_\Gamma\omega\,\overline{u}\cdot\boldsymbol{\tau}. \tag{4.2}$$

Or we could write,

$$\begin{aligned}\int_\Gamma(\nabla p\cdot\boldsymbol{n})q&=\nu\int_\Gamma(\Delta u\cdot\boldsymbol{n})q=\nu\int_\Gamma(\nabla^\perp\omega\cdot\boldsymbol{n})q=-\nu\int_\Gamma(\nabla\omega\cdot\boldsymbol{\tau})q\\ &=\nu\int_\Gamma\omega\,\nabla q\cdot\boldsymbol{\tau}=\nu\int_\Gamma\omega\,(\overline{u}\cdot\boldsymbol{\tau}),\end{aligned} \tag{4.3}$$

where $\omega:=\operatorname{curl}u$, so that

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\frac{\nu}{2}\|\nabla w\|^2 \leqslant C\nu+C\|w\|^2+\nu(\nabla u,\nabla z)+\nu\int_\Gamma\omega\,\overline{u}\cdot\boldsymbol{\tau}.$$

So this option does not eliminate from (4.1) the term $\nu(\nabla u,\nabla z)$, but that was never necessary anyway, since

$$\nu(\nabla u,\nabla z)=-\nu(u,\Delta z)=-\nu(u,\nabla\Delta q)=0,$$

and we recover (4.2).

5. Summary

To summarize, although with our special assumptions in (2.1), the two difficult-to-bound terms, $(\nabla u, \nabla z)$ and $(u \cdot \nabla u, z)$, in Kato's energy inequality cancel, the original, impossible-to-bound boundary integral of $\omega\, \overline{u} \cdot \boldsymbol{\tau}$ reappears. Hence, these special assumptions would seem to be of no use if we focus only on the corrector to the Euler equations. The focus has to be on the solution to the Navier-Stokes equations and its partial analyticity, or the solution has to be exploited using much more involved, and sophisticated, techniques. There seems to be no easy way out.

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