LACK OF HÖLDER REGULARITY OF THE FLOW FOR 2D EULER EQUATIONS WITH UNBOUNDED VORTICITY

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Abstract. We construct a class of examples of initial vorticities for which the solution to the Euler equations in the plane has an associated flow that lies in no Hölder space of positive exponent for any positive time. Our initial vorticities have L^p-norms that do not grow much faster than $\log p$, which Yudovich showed ensures the uniqueness of solutions to the Euler equations ([8]). Our class of examples extends an example of Bahouri's and Chemin's ([1]) with bounded initial vorticity, for which the flow lies in no Hölder space of exponent greater than e^{-t}.

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1. Introduction

Yudovich showed in [7] that there exists a unique solution to the Euler equations in a bounded domain of the plane with bounded initial vorticity. Also, there exists a unique continuous flow, and this flow lies in the Hölder space of exponent e^{-Ct} for all positive time t. Yudovich's result is easily modified to apply to solutions in the whole plane. Bahouri and Chemin in [1] showed that this regularity of the flow was in a sense optimal by constructing an example for which the flow lies in no Hölder space of exponent higher than e^{-t}.

In [8] Yudovich extended his uniqueness result to a certain class $\mathbb{Y}$ of unbounded vorticities and showed that there exists as well a unique flow. There is an upper bound on the modulus of continuity of this flow that depends on how unbounded the initial vorticity is. It would be useful to know whether this upper bound is achieved for certain initial vorticities; that is, whether one can find examples for which the upper bound is also a lower bound (to within a constant factor). The hope is that this might cast some light on how near to the "edge of uniqueness" the class $\mathbb{Y}$ has brought the Euler equations.

We give some partial information by extending the example of Bahouri and Chemin to a class of initial vorticities in $\mathbb{Y}$ having a point singularity. We show that for some such initial vorticities the flow lies in no Hölder space of positive exponent for any positive time.

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2. Yudovich's theorem for unbounded vorticity

We first define the function spaces in which our initial velocity will lie. We let E_m be as in [2] (although we could also work in the larger spaces of [3]). For any real number m, a vector v belongs to E_m if it is divergence-free and can be written in the form $v = \sigma + v'$, where v' is in $L^2(\mathbb{R}^2)$ and where σ is a *stationary vector field*, meaning that σ is of the form

$$\sigma = \left(-\frac{x_2}{r^2} \int_0^r \rho g(\rho)\, d\rho,\ \frac{x_1}{r^2} \int_0^r \rho g(\rho)\, d\rho \right) \tag{2.1}$$

for some g in $C_0^\infty(\mathbb{R})$. E_m is an affine space; fixing an origin, σ, in E_m we can define a norm by $\|\sigma + v'\|_{E_m} = \|v'\|_{L^2(\Omega)}$. Convergence in E_m is equivalent to convergence in the L^2–norm to a vector in E_m.

We will further restrict our initial velocities to have vorticities that are "only slightly unbounded" in a sense we now make precise.

Definition 2.1. Let $\theta : [p_0, \infty) \to \mathbb{R}^+$ for some p_0 in $[1, 2)$. We say that θ is *admissible* if the function $\beta : (0, \infty) \to [0, \infty)$ defined, for some $M > 0$, by[1]

$$\beta_M(x) := C \inf \left\{ (M^\epsilon x^{1-\epsilon}/\epsilon)\theta(1/\epsilon) : \epsilon \text{ in } (0, 1/p_0] \right\}, \tag{2.2}$$

where C is a fixed absolute constant, satisfies

$$\int_0^1 \frac{dx}{\beta(x)} = \infty. \tag{2.3}$$

Because

$$\begin{aligned}\beta_M(x) &= CM\frac{x}{M} \inf \left\{ ((x/M)^{-\epsilon}/\epsilon)\theta(1/\epsilon) : \epsilon \text{ in } (0, 1/p_0] \right\} \\ &= M\beta_1(x/M),\end{aligned}$$

this definition is independent of the value of M. Also, β_M is a monotonically increasing continuous function, with $\lim_{x \to 0^+} \beta_M(x) = 0$.

Definition 2.2. We say that a vector field v has *Yudovich vorticity* if for some admissible function $\theta : [p_0, \infty) \to \mathbb{R}^+$ with p_0 in $[1, 2)$, $\|\omega(v)\|_{L^p} \le \theta(p)$ for all p in $[p_0, \infty)$.

Examples of admissible bounds on vorticity are

$$\theta_0(p) = 1, \theta_1(p) = \log p, \ldots, \theta_m(p) = \log p \cdot \log^2 p \cdots \log^m p, \tag{2.4}$$

where $\log^m$ is log composed with itself m times. These admissible bounds are described in [8] (see also [4].) Roughly speaking, the L^p–norm of a Yudovich vorticity can grow in p only slightly faster than $\log p$ and still be admissible. Such growth in the L^p–norms arises, for example, from a point singularity of the type $\log\log(1/|x|)$ (see Lemma A.1).

[1] The definition of β in Equation (2.2) differs from that in [4] in that it directly incorporates the factor of p that appears in the Calderón-Zygmund inequality; in [4] this factor is included in the equivalent of Equation (2.3).

Given an admissible function θ and fixing m in $\mathbb{R}$, we define the space

$$\mathbb{Y}_\theta = \{v \in E_m : \|\omega(v)\|_{L^p} \le C\theta(p) \text{ for all } p \text{ in } [p_0, \infty)\},$$

for some constant C. We define the norm on $\mathbb{Y}_\theta$ to be

$$\|v\|_{\mathbb{Y}_\theta} = \|v\|_{E_m} + \sup_{p\in[p_0,\infty)} \|\omega(v)\|_{L^p} / \theta(p). \tag{2.5}$$

This space is not separable, because $L^{p_0} \cap L^\infty$, which is not separable, is compactly embedded in it.

Finally, we define the space

$$\mathbb{Y} = \{v \in Y_\theta : \theta \text{ is admissible}\},$$

but place no norm on this space.

The final thing we must do before stating Yudovich's theorem is to define what we mean by a weak solution to the Euler equations.

Definition 2.3 (Weak Euler Solutions)**.** Given an initial velocity v^0 in $\mathbb{Y}_\theta$, v in $L^\infty([0,T]; \mathbb{Y}_\theta)$ is a weak solution to the Euler equations (without forcing) if $v(0) = v^0$ and

$$(\mathbf{E}) \qquad \frac{d}{dt}\int_\Omega v \cdot \varphi + \int_\Omega (v \cdot \nabla v)\cdot \varphi = 0$$

for all divergence-free φ in $(H^1(\mathbb{R}^2))^2$.

Our form of the statement of Yudovich's theorem is a generalization of the statement of Theorem 5.1.1 of [2] from bounded to unbounded vorticity.

Theorem 2.4 (Yudovich's Theorem for Unbounded Vorticity)**.** ***First part****:* *For any v^0 in $\mathbb{Y}$ there exists a unique weak solution v of (E). Moreover, v is in $C(\mathbb{R}; E_m) \cap L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$. Also,*

$$\|\omega(t)\|_{L^p(\mathbb{R}^2)} = \|\omega^0\|_{L^p} \text{ for all } 1 \le p \le \infty. \tag{2.6}$$

Second part*: The vector field has a unique continuous flow. More precisely, if v^0 is in $\mathbb{Y}_\theta$ then there exists a unique mapping ψ, continuous from $\mathbb{R}\times\mathbb{R}^2$ to $\mathbb{R}^2$, such that*

$$\psi(t,x) = x + \int_0^t v(s, \psi(s,x))\, ds.$$

Let

$$\mu(r) = (C/r)\beta_1(r^2/4), \tag{2.7}$$

where β_1 is the function of Definition 2.1 associated with θ and let $\Gamma_t \colon [0,\infty) \to [0,\infty)$ be defined by $\Gamma_t(0) = 0$ and for $s > 0$ by

$$\int_s^{\Gamma_t(s)} \frac{dr}{\mu(r)} = t \text{ or equivalently } \int_{s^2/4}^{\Gamma_t(s)^2/4} \frac{dr}{\beta_1(r)} = t. \tag{2.8}$$

Then $\delta \mapsto \Gamma_t(\delta)$ *is an upper bound on the modulus of continuity of the flow at time* $t > 0$*; that is, for all* x *and* y *in* $\mathbb{R}^2$

$$|\psi(t,x) - \psi(t,y)| \le \Gamma_t(|x-y|).$$

Also, for all x *and* y *in* $\mathbb{R}^2$

$$|v(t,x) - v(t,y)| \le \mu(|x-y|), \tag{2.9}$$

Existence in the first part of Yudovich's theorem can be established, for instance, by modifying the approach on p. 311-319 of [6], which establishes existence under the assumption of bounded vorticity; the uniqueness argument is given by Yudovich in [8]. The second part is Theorem 2 of [8], the bound on the modulus of continuity of the flow following from working out the details of Yudovich's proof (see Sections 5.2 through 5.4 of [5]).

The function μ of Equation (2.7) will have all the important properties of β_1: $\mu(0) = 0$, μ is nondecreasing, and $\int_0^1 (\mu(r))^{-1}\, dr = \infty$.

The factors of 1/4 in Equation (2.8) are inconsequential, since if we define $\Gamma_t(s)$ by

$$\int_{s^2}^{\Gamma_t(s)^2} \frac{dr}{\beta_1(r)} = t \tag{2.10}$$

we will simply obtain a slightly weaker upper bound on the modulus of continuity of the flow.

3. Square-symmetric vorticities

Ignoring for the moment the Euler equations, we will assume that the vorticity has certain symmetries, and from these symmetries deduce some useful properties of the divergence-free velocity having the given vorticity. In Section 4, we will then consider what happens to a solution to the Euler equations whose initial vorticity possesses these symmetries.

For convenience, we number the quadrants in the plane Q_1 through Q_4, starting with

$$Q_1 = \{(x_1, x_2) : x_1 \ge 0, x_2 \ge 0\},$$

and moving counterclockwise through the quadrants.

Definition 3.1. We say that a Yudovich vorticity (vorticity as in Definition 2.2) is *symmetric by quadrant*, or SBQ, if ω is compactly supported and $\omega(x) = \omega(x_1, x_2)$ is odd in x_1 and x_2; that is, $\omega(-x_1, x_2) = -\omega(x_1, x_2)$ and $\omega(x_1, -x_2) = -\omega(x_1, x_2)$—so also $\omega(-x) = \omega(x)$.

Lemma 3.2. *Let* ω *be SBQ. Then there exists a unique vector field* v *in* $E_0 \cap \mathbb{Y}$ *with* $\omega(v) = \omega$*, and* v *satisfies the following:*

(1) $v_2(x_1, 0) = 0$ *for all* x_1 *in* $\mathbb{R}$*;*
(2) $v_1(0, x_2) = 0$ *for all* x_2 *in* $\mathbb{R}$*;*
(3) $v(0,0) = 0$.

If, in addition, $\omega \ge 0$ *in* Q_1*, then*

(4) $v_1(x_1, 0) \geq 0$ *for all* $x_1 \geq 0$.

Proof. Let p be in $[1, 2)$ and let $q > 2p/(2-p)$. By Proposition 3.1.1 p. 44 of [2], for any vorticity ω in L^p there exists a unique divergence-free vector field v in $L^p + L^q$ whose curl is ω, with v being given by the Biot-Savart law,

$$v = K * \omega. \tag{3.1}$$

Here, K is the Biot-Savart kernel, $K(x) = (1/2\pi)x^\perp / |x|^2$, which decays like $1/|x|$ with a singularity of order $1/|x|$ at the origin.

Because ω is compactly supported and lies in $L^2(\mathbb{R}^2)$, ω is in $L^p(\mathbb{R}^2)$, and Equation (3.1) gives our velocity v, unique in all the spaces $L^p + L^q$. Also, because $\int_{\mathbb{R}^2} \omega = 0$, v is in $(L^2)^2 = E_0$ (see the comment following Definition 1.3.3 of [2], for instance).

Then

$$\begin{aligned} v_1(x_1, 0) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{|x-y|^2} \omega(y)\, dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y)\, dy \\ &= \frac{1}{2\pi} \sum_{j=1}^{4} \int_{Q_j} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y)\, dy. \end{aligned}$$

Making the changes of variables, $u = (-y_1, y_2)$, $u = -y$, and $u = (y_1, -y_2)$ on Q_2, Q_3, and Q_4, respectively, in all cases the determinant of the Jacobian is ± 1, and we obtain

$$\begin{aligned} v_1(x_1, 0) = \frac{1}{2\pi} \Bigg[&\int_{Q_1} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(y)\, dy - \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega(u)\, du \\ &+ \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega(u)\, du - \int_{Q_1} \frac{u_2}{(x_1 - u_1)^2 + u_2^2} \omega(u)\, du \Bigg] \end{aligned}$$

or

$$v_1(x_1, 0) = \frac{1}{\pi} \int_{Q_1} (f_1(x_1, y) - f_2(x_1, y))\, \omega(y)\, dy, \tag{3.2}$$

where

$$f_1(x_1, y) = \frac{y_2}{(x_1 - y_1)^2 + y_2^2}, \quad f_2(x_1, y) = \frac{y_2}{(x_1 + y_1)^2 + y_2^2}. \tag{3.3}$$

It follows from $(x_1 - y_1)^2 + y_2^2 \leq (x_1 + y_1)^2 + y_2^2$ on Q_1 that $f_1(x_1, y) > f_2(x_1, y)$ for all $x_1, y_1 > 0$. Conclusion (4) then follows from Equation (3.2).

By the Biot-Savart law of Equation (3.1),

$$\begin{aligned} v_2(x_1, -x_2) &= (K_2 * \omega)(x_1, -x_2) \\ &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, -x_2 - y_2)\omega(y_1, y_2)\, dy \\ &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, x_2 + y_2)\omega(y_1, -y_2)\, dy \\ &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, x_2 - (-y_2))\omega(y_1, -y_2)\, dy \\ &= -v_2(x_1, x_2). \end{aligned}$$

Here we used $K_2(x_1, -x_2) = -K_2(x_1, x_2)$ and the symmetry of ω. A similar calculation shows that $v_1(-x_1, x_2) = -v_1(x_1, x_2)$. Thus, the velocity along the x-axis is directed along the x-axis and the velocity along the y-axis is directed along the y-axis, so the axes are preserved by the flow. In particular, the origin is fixed. This gives conclusions (1)-(3). □

Lemma 3.3 is Proposition 2.1 of [1] (see also Proposition 5.3.1 of [2]).

Lemma 3.3. *Let ω be SBQ with*

$$\omega = 2\pi \mathbf{1}_{[0,1]\times[0,1]} \tag{3.4}$$

on Q_1. Then there exists a constant $C > 0$ such that

$$v_1(x_1, 0) \geq 2x_1 \log(1/x_1) \tag{3.5}$$

for all x_1 in $(0, C]$.

The following lemma is a slight generalization of Lemma 3.3 that will give us our key inequality.

Lemma 3.4. *Let ω be SBQ with*

$$\omega = 2\pi \mathbf{1}_{[0,r]\times[0,r]} \tag{3.6}$$

on Q_1 for some r in $(0, 1)$. Then for any λ in $(0, 1)$ there exists a right neighborhood of the origin, $\mathcal{N}$, such that

$$v_1(x_1, 0) \geq 2(1 - \lambda)x_1 \log(1/x_1) \tag{3.7}$$

for all x_1 in $(0, r^{1/\lambda}] \cap \mathcal{N}$.

Remark 3.1. The neighborhood $\mathcal{N}$ depends only upon λ; in particular, it is independent of r.

Proof. The result follows from scaling the result in Lemma 3.3. Indeed, if we write $\omega^r(x)$ for the function ω defined by Equation (3.6) then ω^1 is the function defined by Equation (3.4) and $\omega^r(\cdot) = \omega^1(\cdot/r)$. Letting $v^r = K * \omega^r$ we see that $v^r(x) = rv^1(x/r)$, since then $\omega(v^r(x)) = r(1/r)\omega(v^1)(x/r) =$

$\omega^1(x/r) = \omega^r(x)$ and $v^r(x)$ is divergence-free. It follows from Lemma 3.3 that for all x_1 such that x_1/r lies in $[0, C]$,

$$\begin{aligned} v_1^r(x_1, 0) &= r v_1^1(x_1/r, 0) > r 2(x_1/r)\log(1/(x_1/r)) \\ &= 2x_1 \left[\log(1/x_1) + \log r\right]. \end{aligned}$$

Thus, if $x_1^\lambda \le r$ then $\log r \ge \lambda \log x_1 = -\lambda \log(1/x_1)$ so

$$v_1^r(x_1, 0) > 2x_1(1 - \lambda)\log(1/x_1).$$

Thus, Equation (3.7) holds for all x_1 in $[0, r^{1/\lambda}] \cap [0, rC]$. But $r^{1/\lambda} \le rC$ if and only if $r \le C^{\lambda/(1-\lambda)}$, which gives us the right neighborhood, $\mathcal{N} = (0, C^{\lambda/(1-\lambda)})$. □

Definition 3.5. We say that ω is square-symmetric if ω is SBQ and $\omega(x_1, x_2) = \omega(\max\{x_1, x_2\}, 0)$ on Q_1.

Being square-symmetric means that a vorticity is SBQ and is constant in absolute value along the boundary of any square centered at the origin.

Lemma 3.6. *Assume that ω is square-symmetric, finite except possibly at the origin, and $\omega(x_1, 0)$ is non-increasing for $x_1 > 0$. Then for any λ in $(0, 1)$*

$$v_1(x_1, 0) \ge C\omega(x_1^\lambda, 0)x_1 \log(1/x_1) \tag{3.8}$$

for all x_1 in the neighborhood $\mathcal{N}$ of Lemma 3.4, where $C = 1/\pi$.

Proof. We can write ω on Q_1 as

$$\omega(x) = 2\pi \int_0^1 \alpha(s) \mathbf{1}_{[0,s]\times[0,s]}(x)\, ds, \tag{3.9}$$

for some measurable, nonnegative function $\alpha\colon (0, 1) \to [0, \infty)$. This means that

$$\omega(x_1, 0) = 2\pi \int_{x_1}^1 \alpha(s)\, ds. \tag{3.10}$$

Let $V(s)$ be the value of $v_1(x_1, 0)$ that results from assuming that ω is given by Equation (3.6). By Lemma 3.2, $V(s) > 0$. Then because the Biot-Savart law of Equation (3.1) is linear, and using Lemma 3.4, for all x_1 in the neighborhood $\mathcal{N}$,

$$\begin{aligned} v_1(x_1, 0) &= \int_0^1 \alpha(s) V(s)\, ds \\ &= \int_0^{x_1^\lambda} \alpha(s) V(s)\, ds + \int_{x_1^\lambda}^1 \alpha(s) V(s)\, ds \\ &\geq \int_{x_1^\lambda}^1 \alpha(s) V(s)\, ds \\ &\geq 2\pi \left(\int_{x_1^\lambda}^1 \alpha(s)\, ds \right) \frac{2}{2\pi} x_1 \log(1/x_1) \\ &= \frac{1}{\pi} \omega(x_1^\lambda, 0) x_1 \log(1/x_1). \end{aligned}$$

In the final inequality, $V(s)$ is bounded as in Lemma 3.4 because $x_1^\lambda \leq s$ in the integrand. □

Remark 3.2. Properly speaking, we must allow the function α of Equation (3.9) to be a distribution since, for instance, to obtain ω of Lemma 3.4, we would need $\alpha = \delta_r$. We could avoid this complication, however, by assuming that ω is strictly decreasing and that $\omega(x_1, 0)$ is sufficiently smooth as a function of $x_1 > 0$.

4. Square-symmetric initial vorticities

We now assume that our initial vorticity is square-symmetric, and consider what happens to the solution to (E) over time.

Theorem 4.1. *Assume that ω^0 is square-symmetric, finite except possibly at the origin, and $\omega^0(x_1, 0)$ is nonnegative and non-increasing for $x_1 > 0$. Then for any λ in $(0, 1)$,*

$$v_1(t, x_1, 0) \geq C\omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1) \tag{4.1}$$

for all x_1 in the neighborhood $\mathcal{N}$ of Lemma 3.4 and all time $t \geq 0$, where Γ_t is defined as in Theorem 2.4. The constant $C = 1/\pi$.

Further, let $L(t, x_1)$ be any continuous lower bound on $v_1(t, x_1, 0)$, Equation (4.1) being one possibility. Then if $x_1(t)$ is the solution to

$$\frac{dx_1(t)}{dt} = L(t, x_1)$$

with $x_1(0) = a > 0$ in $\mathcal{N}$, then $\psi^1(t, a, 0) \geq x_1(t)$ for all $t \geq 0$.

Proof. Since $\omega^0(x_1, x_2) = -\omega^0(x_1, -x_2)$, if $\omega(t, x_1, x_2)$ is a solution to (E) then $-\omega(t, x_1, -x_2)$ is also a solution. But the solution to (E) is unique by Theorem 2.4, so we conclude that $\omega(t, x_1, x_2) = -\omega(t, x_1, -x_2)$. Similarly, $\omega(t, x_1, x_2) = -\omega(t, -x_1, x_2)$, and we see that ω is SBQ. By Lemma 3.2, then, it follows that the flow transports vorticity in Q_k, $k = 1, \ldots, 4$, only within

Q_k, because of the direction of v along the axes for all $t \geq 0$. Therefore, $\omega(t)$ is also nonnegative in Q_1 for all time.

Our approach then will be to produce a point-by-point lower bound $\overline{\omega}(t)$ on $\omega(t)$ that satisfies all the requirements of Lemma 3.6. In particular, it is SBQ, so $\omega(t) - \overline{\omega}(t)$ is SBQ and nonnegative in Q_1. It follows from Lemma 3.2 that $v_1(t, x_1, 0) - \overline{v}_1(t, x_1, 0) \geq 0$ for all $t \geq 0$, where $\omega(\overline{v}(t)) = \overline{\omega}(t)$. Thus, the lower bound on $\overline{v}_1(t, x_1, 0)$ coming from Lemma 3.6 will also be a lower bound on $v_1(t, x_1, 0)$. We now determine $\overline{\omega}(t)$.

Because conclusion (3) of Lemma 3.2 holds for all time, ω being SBQ for all time, the origin is fixed by the flow; that is $\psi(t, 0) = \psi^{-1}(t, 0) = 0$ for all t. Also, the Euler equations are time reversible, and the function Γ_t of Equation (2.8) depends only upon the Lebesgue norms of the vorticity, which are preserved by the flow; therefore, Γ_t is a bound on the modulus of continuity of $\psi^{-1}(t, \cdot)$ as well. Thus,

$$\left|\psi^{-1}(t, x)\right| = \left|\psi^{-1}(t, x) - \psi^{-1}(t, 0)\right| \leq \Gamma_t(|x|).$$

In Q_1, the value of $\omega(t, x)$, then, is bounded below by using the minimum value of ω^0 on the circle of radius $\Gamma_t(|x|)$ centered at the origin, since this is the furthest away from the origin that $\psi^{-1}(t, x)$ can lie, and ω decreases with the distance from the origin. That is,

$$\omega(t, x) = \omega^0(\psi^{-1}(t, x)) \geq \omega^0(\Gamma_t(|x|), 0)$$

because ω^0 is square-symmetric.

Since $\sqrt{2} \max\{x_1, x_2\} \geq |x|$, Γ_t is nondecreasing, and ω_0 is nonincreasing on Q_1, $\omega^0(\Gamma_t(\sqrt{2} \max\{x_1, x_2\}), 0) \leq \omega^0(\Gamma_t(|x|), 0)$ on Q_1. Letting

$$\overline{\omega}(t, x_1, x_2) = \omega^0(\Gamma_t(\sqrt{2} \max\{x_1, x_2\}), 0)$$

we see that $\overline{\omega}$ is square-symmetric, and on Q_1, $\overline{\omega}(t, x) \leq \omega(t, x)$, so $\overline{\omega}$ is our desired lower bound on ω.

Then from Equation (3.8),

$$\begin{aligned} v^1(x_1, 0) &\geq C\overline{\omega}(t, x_1^\lambda, 0) x_1 \log(1/x_1) \\ &= C\omega^0(\Gamma_t((\sqrt{2} \max\{x_1, 0\})^\lambda), 0) x_1 \log(1/x_1) \\ &= C\omega^0(\Gamma_t(2^{\lambda/2} x_1^\lambda), 0) x_1 \log(1/x_1). \end{aligned}$$

The lower bound on the flow, $\psi^1(t, a, 0) \geq x_1(t)$, follows from using the minimum possible value of $v^1(t, x_1, 0)$ in Equation (4.1), setting it equal to $dx_1(t)/dt$, and integrating over time. □

5. Bounded vorticity

We now apply Theorem 4.1 to the first in the sequence of Yudovich's vorticity bounds in Equation (2.4) in which we have bounded vorticity. We assume that ω is square-symmetric with $\omega^0 = \mathbf{1}_{[0,1/2] \times [0,1/2]}$ in Q_1 so that

$\|\omega^0\|_{L^1\cap L^\infty} = 1$. We have,

$$\beta_1(r) = C\inf\left\{r^{1-\epsilon}/\epsilon : \epsilon \text{ in } (0,1]\right\} = C\inf\left\{g(\epsilon) : \epsilon \text{ in } (0,1]\right\},$$

where $g(\epsilon) = r^{1-\epsilon}/\epsilon$. Then

$$g'(\epsilon) = C\frac{r^{1-\epsilon}(\epsilon\log(1/r) - 1)}{\epsilon^2},$$

which is zero when $\epsilon = \epsilon_0 := 1/(\log(1/r))$ if $r < 1$ and $\epsilon_0 < 1$, and never zero otherwise. But

$$\begin{aligned}\epsilon_0 < 1 &\iff \frac{1}{\log(1/r)} < 1 \iff \log(1/r) > 1\\ &\iff \frac{1}{r} > e \iff r < e^{-1},\end{aligned}$$

so the condition $r < 1$ is redundant.

Assume that $r < e^{-1}$. Then $g(\epsilon)$ approaches infinity as ϵ approaches either zero or infinity; hence, ϵ_0 minimizes g. Thus,

$$\begin{aligned}\beta_1(r) &= Cr^{1-\epsilon_0}/\epsilon_0 = Cr(1/r)^{\epsilon_0}\log(1/r)\\ &= Cre^{\log(1/r)\epsilon_0}\log(1/r) = -Cer\log(r).\end{aligned}$$

Then from Equation (2.8),

$$\begin{aligned}&\int_{x_1^2/4}^{\Gamma_t(x_1)^2/4}\frac{dr}{\beta_1(r)} = -C\left[\log(-\log r)\right]_{x_1^2/4}^{\Gamma_t(x_1)^2/4} = t\\ &\qquad\implies \log(-\log(x_1^2/4)) - \log(-\log(\Gamma_t(x_1)^2/4)) = Ct\\ &\qquad\implies \Gamma_t(x_1)^2 = 4(x_1^2/4)^{e^{-Ct}} \implies \Gamma_t(x_1) = 2(x_1/2)^{e^{-Ct}}\end{aligned}$$

as long as $\Gamma_t(x_1)^2/4 < e^{-1}$.

Thus, Theorem 4.1 gives

$$\begin{aligned}v^1(t,x_1,0) &\geq C\omega^0(2(2^{\lambda/2}x_1^\lambda/2)^{e^{-Ct}},0)x_1\log(1/x_1)\\ &\geq Cx_1\log(1/x_1)\end{aligned}$$

as long as $2(2^{\lambda/2}x_1^\lambda/2)^{e^{-Ct}} < 1/2$.

Solving $dx_1(t)/dt = Cx_1\log(1/x_1)$ with $x_1(0) = a$ gives

$$\psi^1(t,a,0) \geq x_1(t) = a^{\exp(-Ct)},$$

which applies for sufficiently small a.

Since $\psi(t,0,0) = 0$,

$$\frac{|\psi(t,a,0) - \psi(t,0,0)|}{a^\alpha} = \frac{\left|\psi^1(t,a,0)\right|}{a^\alpha} \geq a^{\exp(-Ct)-\alpha},$$

which is infinite for any $\alpha > e^{-Ct}$. This shows that the flow can be in no Hölder space C^α for $\alpha > e^{-Ct}$, reproducing, up to a constant, the result of [1] (or see Theorem 5.3.1 of [2].)

6. Yudovich's higher examples

Assume that $m \geq 2$ and let ω^0 have the symmetry described in Theorem 4.1 with

$$\omega^0(x_1, 0) = \log^2(1/x_1) \cdots \log^m(1/x_1) = \theta_m(1/x_1)/\log(1/x_1), \tag{6.1}$$

for x_1 in $(0, \exp^{-m}(0))$, and ω^0 equal to zero elsewhere in the first quadrant. Then by Lemma A.1

$$\theta(p) = \|\omega^0\|_{L^p} \leq C \log p \cdots \log^{m-1} p = \theta_{m-1}(p)$$

for all p larger than some p^*, with θ_{m-1} given by Equation (2.4).

Using an observation of Yudovich's in [8], if β_1 is the function of Definition 2.1 associated with the admissible function θ, then letting $\epsilon_0 = 1/\log(1/r)$ for $r < e^{-p^*}$,

$$\begin{aligned}\beta_1(r) &\leq C(r^{1-\epsilon_0}/\epsilon_0)\theta(1/\epsilon_0) = -Crr^{1/\log r}\log r\theta(\log(1/r)) \\ &= Cr\log(1/r)\log^2(1/r)\cdots\log^m(1/r) \\ &= Cr\theta_m(1/r).\end{aligned}$$

Then, if we define the upper bound on the modulus of continuity of the flow by Equation (2.10) instead of Equation (2.8), we have

$$\begin{aligned}-C\left[\log^{m+1}(1/r)\right]_{s^2}^{\Gamma_t(s)^2} &= \int_{s^2}^{\Gamma_t(s)^2} \frac{dr}{Cr\theta_m(1/r)} \\ &\leq \int_{s^2}^{\Gamma_t(s)^2} \frac{dr}{\beta_1(r)} = t \\ &\implies -C\log^{m+1}(1/\Gamma_t(s)^2) \leq -C\log^{m+1}(1/s^2) + t \\ &\implies \log^{m+1}(1/\Gamma_t(s)^2) \geq \log^{m+1}(1/s^2) - Ct \\ &\implies \log^m(1/\Gamma_t(s)^2) \geq e^{-Ct}\log^m(1/s^2) \\ &\implies \log^m(1/\Gamma_t(s)) \geq (1/2)e^{-Ct}\log^m(1/s^2) \geq (1/2)e^{-Ct}\log^m(1/s)\end{aligned}$$

for sufficiently small s by Lemma 6.2.

Using this bound, we have, from Equation (4.1),

$$\begin{aligned}v^1(t,x_1,0) &\geq C\omega^0(\Gamma_t(2^{\lambda/2}x_1^\lambda), 0)x_1\log(1/x_1) \\ &\geq C\log^2(1/\Gamma_t(2^{\lambda/2}x_1^\lambda))\cdots\log^m(1/\Gamma_t(2^{\lambda/2}x_1^\lambda))x_1\log(1/x_1) \\ &\geq Ce^{-Ct}\log^2(1/\Gamma_t(2^{\lambda/2}x_1^\lambda))\cdots\log^m(1/2^{\lambda/2}x_1^\lambda)x_1\log(1/x_1) \\ &\geq Ce^{-Ct}\log^2(1/\Gamma_t(2^{\lambda/2}x_1^\lambda))\cdots\log^m(1/x_1)x_1\log(1/x_1)\end{aligned} \tag{6.2}$$

as long as $x_1 > 0$ is sufficiently small. (The argument $1/\Gamma_t(2^{\lambda/2}x_1^\lambda)$ appears in each of the $\log^2, \dots, \log^{m-1}$ factors above, but not in the log factor or, unless $m = 2$, the $\log^m$ factor.)

Specializing to the case $m = 2$ and combining the previous two inequalities, the explicit dependence of the bound in Equation (6.2) on Γ_t disappears, and we have

$$v^1(t, x_1, 0) \geq Ce^{-Ct} \log^2(1/x_1) x_1 \log(1/x_1) = Ce^{-Ct} x_1 \theta_2(1/x_1).$$

Solving for

$$\frac{dx_1(t)}{dt} = Ce^{-Ct} x_1 \theta_2(1/x_1) \tag{6.3}$$

with $x_1(0) = a$, we get

$$\log^3(1/x_1(t)) = \log^3(1/a) + C\left(e^{-Ct} - 1\right),$$

so

$$\begin{aligned}\psi^1(t, a, 0) \geq x_1(t) &= \exp\left(-(-\log a)^{\exp\left(C(e^{-Ct}-1)\right)}\right) \\ &= e^{-(-\log a)^\gamma},\end{aligned}$$

where $\gamma = \exp\left(C(e^{-Ct} - 1)\right)$.

Observe that $\gamma < 1$ for all $t > 0$. Thus, for any α in $(0, 1)$ and all $t > 0$,

$$\begin{aligned}\|\psi\|_{C^\alpha} &\geq \lim_{a \to 0^+} \frac{\psi^1(t, a, 0) - \psi^1(t, 0, 0)}{a^\alpha} \geq \lim_{a \to 0^+} \frac{x_1(t)}{a^\alpha} \\ &= \lim_{a \to 0^+} \frac{e^{-(-\log a)^\gamma}}{e^{-(-\log a)\alpha}} = \lim_{u \to \infty} \frac{e^{-u^\gamma}}{e^{-\alpha u}} = \lim_{u \to \infty} e^{\alpha u - u^\gamma} = \infty.\end{aligned} \tag{6.4}$$

We conclude that the flow lies in no Hölder space of positive exponent for all positive time, a result that we state explicitly as a corollary of Theorem 4.1.

Corollary 6.1. *There exists initial velocities satisfying the conditions of Theorem 2.4 for which the unique solution to (E) has an associated flow lying, for all positive time, in no Hölder space of positive exponent.*

We used the following lemma above:

Lemma 6.2. *Let m be a positive integer. Then for sufficiently small positive x,*

$$\log^m(1/x) \geq (1/2) \log^m(1/x^2).$$

Proof. The proof is by induction. For $x < 1$,

$$\log(1/x^2) = 2 \log(1/x),$$

which establishes the inequality (in fact, equality) for $m = 1$. So assume the inequality holds for all positive integers up to m. Then

$$\begin{aligned}\log^{m+1}(1/x^2) &= \log \log^{m-1}(1/x^2) \leq \log(2 \log^{m-1}(1/x)) \\ &= \log 2 + \log^m(1/x) \leq 2 \log^m(1/x)\end{aligned}$$

as long as x is sufficiently small that $\log 2 \leq \log^m(1/x)$. $\square$

7. Final remarks

It is natural to try to extend the analysis of Section 6 to Yudovich initial vorticities for $m > 2$. But this is, in fact, quite difficult, because when $m > 2$ the explicit dependence of the bound in Equation (6.2) on Γ_t remains, so we must also bound $\log^k(1/\Gamma_t(s))$ for $k = 2, \dots, m-1$. Doing so makes the analog of Equation (6.3) no longer exactly integrable. It is clear that one obtains a worse bound on the modulus of continuity than for $m = 2$, but it is not at all clear what happens as we take m to infinity.

We chose to give the initial vorticity SBQ symmetry because such symmetry works well with the symmetry of the Biot-Savart law to produce a lower bound on the velocity along the x- or y-axes. Having made this choice, the rest of our choices concerning the vorticity were inevitable, up to interchanging the roles of x and y or changing the sign of the vorticity. Because the initial vorticity is SBQ, the function $f = f_1 - f_2$, where f_1 and f_2 are defined in Equation (3.3), controls the bound on the velocity, and f is continuous except for a singularity at $y = (x_1, 0)$, where it goes to positive infinity (for $x_1 > 0$). Thus, whatever lower bound we derive on $v_1(x_1, 0)$, it will increase the fastest at a singularity of $|\omega(t)|$ that lies along the x-axis and this effect is most pronounced when ω is of a single sign in $Q1$ (this follows from Equation (3.2)). The lower bound on the modulus of continuity of the flow then follows from allowing a point $a = (a_1, 0)$ to approach the singularity and looking at how large the appropriate difference quotient becomes, as in Equation (6.4). But to do this, we need control on the position of the singularity of $|\omega(t)|$, and, when assuming SBQ, the origin is the one point at which we have the most control—the singularity doesn't move at all.

Thus, the assumption of SBQ naturally leads us to assume a point singularity at the origin. Then, because it appears that we can only bound from below the effect on $v_1(x_1, 0)$ of the vorticity outside of the square on which a point lies (actually, an even larger square because of the exponent λ in Lemma 3.6), we are naturally led to the assumption that $|\omega^0|$ decreases with the distance from the origin, which leads to Lemma 3.6. Perhaps we could improve our lower bound on $v_1(x_1, 0)$ by accounting for the vorticity all the way to the origin, but this may simply be impossible (and in any case would not change our conclusion for $m = 2$).

Appendix A. Logarithmic singularities

We show how to bound the L^p-norms of a function with a singularity like those we use in Section 6.

Lemma A.1. *Let $m \geq 2$ and let ω^0 have the symmetry described in Theorem 4.1 with*

$$\omega^0(x_1, 0) = \log^2(1/x_1) \cdots \log^m(1/x_1) = \theta_m(1/x_1)/\log(1/x_1),$$

for x_1 *in* $(0, \exp^{-m}(0))$, *and* ω^0 *equal to zero elsewhere in the first quadrant. Then*

$$\|\omega^0\|_{L^p} \sim \log p \cdots \log^{m-1} p = \theta_{m-1}(p)$$

for large p.

Proof. Because of the symmetry of ω^0,

$$\begin{aligned} \|\omega^0\|_{L^p}^p &= 4 \int_0^{\exp^{-m}(0)} 2 \int_0^{x_1} (\omega^0(x_1, 0))^p \, dx_2 \, dx_1 \\ &= 8 \int_0^{\exp^{-m}(0)} x_1 \left[\log^2(1/x_1) \cdots \log^m(1/x_1)\right]^p \, dx_1. \end{aligned} \tag{A.1}$$

Making the change of variables, $u = \log(1/x_1) = -\log x_1$, it follows that $x_1 = e^{-u}$ and $du = -(1/x_1)\, dx_1$ so $dx_1 = -e^{-u}\, du$. Thus,

$$\begin{aligned} \|\omega^0\|_{L^p}^p &= 8 \int_\infty^{\exp^{m-1}(0)} e^{-u} \left[\log u \cdots \log^{m-1} u\right]^p (-e^{-u})\, du \\ &= 8 \int_{\exp^{m-1}(0)}^\infty e^{-2u} \left[\log u \cdots \log^{m-1} u\right]^p \, du. \end{aligned}$$

Making the further change of variables $x = u/p$, so that $u = px$ and $du = p\, dx$, we have

$$\|\omega^0\|_{L^p}^p = 8p \int_{\exp^{m-1}(0)/p}^\infty e^{-2xp} \left[\log(xp) \cdots \log^{m-1}(xp)\right]^p \, dx. \tag{A.2}$$

Obtaining a lower bound on $\left\|\omega^0\right\|_{L^p(\mathbb{R}^2)}$ is easy. For $x \ge 1$,

$$\log(xp) \cdots \log^{m-1}(xp) \ge \log p \cdots \log^{m-1} p,$$

so

$$\begin{aligned} \|\omega^0\|_{L^p}^p &\ge 8p \int_{\exp^{m-1}(0)/p}^\infty e^{-2xp} \left[\log p \cdots \log^{m-1} p\right]^p \, dx \\ &= 8p \left[\log p \cdots \log^{m-1} p\right]^p \int_{\exp^{m-1}(0)/p}^\infty e^{-2xp}\, dx \\ &= 8p \left[\log p \cdots \log^{m-1} p\right]^p \left(-\frac{1}{2p}\right) \left[e^{-2xp}\right]_{\exp^{m-1}(0)/p}^\infty \\ &= 4\exp^{-2(m-1)}(0) \left[\log p \cdots \log^{m-1} p\right]^p. \end{aligned}$$

Thus, asymptotically, $\|\omega^0\|_{L^p} \ge \log p \cdots \log^{m-1} p$.

We now obtain an upper bound on $\|\omega^0\|_{L^p}$. For $x \le 1$,

$$\left|\log(xp) \cdots \log^{m-1}(xp)\right| \le \left|\log p \cdots \log^{m-1} p\right|,$$

while for $x \geq 1$ and sufficiently large p, Equation (A.3) holds. Thus,

$$\begin{aligned}
\|\omega^0\|_{L^p}^p &\leq 8p \left(\int_0^1 + \int_1^\infty \right) e^{-2xp} \left| \log(xp) \cdots \log^{m-1}(xp) \right|^p \, dx \\
&\leq 8p \int_0^1 e^{-2xp} \left| \log p \cdots \log^{m-1} p \right|^p \, dx \\
&\qquad + 8p \int_1^\infty e^{-2xp} \left[\left| \log p \cdots \log^{m-1} p \right| e^{x-1} \right]^p \, dx \\
&\leq 8p \left| \log p \cdots \log^{m-1} p \right|^p \left[\int_0^1 e^{-2xp} \, dx + e^{-p} \int_1^\infty e^{-xp} \, dx \right] \\
&= 8p \left| \log p \cdots \log^{m-1} p \right|^p \left[\frac{1}{2p} \left(1 - e^{-2p} \right) + e^{-p} \frac{e^{-p}}{p} \right] \\
&\leq 8 \left| \log p \cdots \log^{m-1} p \right|^p .
\end{aligned}$$

It follows that for sufficiently large p, $\|\omega^0\|_{L^p} \leq 8^{1/p} \log^{m-1} p$, so asymptotically, $\|\omega^0\|_{L^p} \leq \log p \cdots \log^{m-1} p$, which completes the proof. □

Lemma A.2. *Let m be a positive integer. Then for sufficiently large p,*

$$\log(xp) \cdots \log^{m-1}(xp) \leq (\log p \cdots \log^{m-1} p) e^{x-1} \tag{A.3}$$

for all $x \geq 1$.

Proof. We prove this for $m = 3$, the proof for other values of m being entirely analogous. First, by taking the logarithm of both sides of Equation (A.3), that equation holds if and only if

$$\begin{aligned}
f(x) &:= \log \log(xp) + \log \log \log(xp) \\
&\leq g(x) := \log(\log p \log \log p) + x - 1.
\end{aligned}$$

Because equality holds for $x = 1$, our result will follow if we can show that $f' \leq g'$ for all $x \geq 1$ and sufficiently large p. This is, in fact, true, since

$$f' = \frac{1}{x \log(xp)} + \frac{1}{x \log(xp) \log \log(xp)} \leq 1 = g'$$

for all $x \geq 1$ and $p \geq e^e$. □

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