SOLUTIONS TO THE AUGUST 2001 ANALYSIS PRELIM AT THE UNIVERSITY OF TEXAS, AUSTIN

JIM KELLIHER

Contents

1. August 2001 Analysis Prelim, UT Austin

Time allowed is four hours [they gave us more like 4-1/2 hours]. Work on as many problems as possible, but do at least three problems from each section.

Real Analysis

Here m denotes Lebesgue measure on $\mathbb{R}^n$.

1. Find a sequence of real-valued nonnegative functions $\{f_k\}$ on $[0,1]$ so that

$$\text{(a) } \limsup f_k = +\infty \ \forall x, \text{ (b) } \int_{[0,1]} f_k \, dx \to 0.$$

Adapt this argument to the case where the domain is $\mathbb{R}^n$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function such that

$$m\{x : |f(x)| > \lambda\} \le C\lambda^{-2}, \lambda > 0.$$

Prove that there is a constant C_1 such that for any Borel set $E \subset \mathbb{R}^n$ of finite and positive measure

$$\int_E |f(x)| \, dx \le C_1 \sqrt{m(E)}.$$

3. For $g \in L^1(\mathbb{R})$ and any $f \in C^1(\mathbb{R})$, suppose that

$$g * f'(x) = f(x+h) - f(x-h).$$

Show that $g \equiv \chi_{[-h,h]}$.

4. Let $B_{m,1}$ denote the ball of radius one centered at the origin in $\mathbb{R}^m$.

a) Show that there exists a function $f : \mathbb{R} \to [0,1]$ such that

$$m\{B_{n+1,1}\} = m\{B_{n,1}\} \int [f(t)]^n \, dt.$$

b) Show that $\int [f(t)]^n \, dt \to 0$ as $n \to \infty$.

c) Show that for any positive number A, $A^n m\{B_{n,1}\} \to 0$ as $n \to \infty$.

5. Let $\{f_n\}$ be a sequence of real-valued functions in L^2 such that $\|f_n\|_{L^2} = 1$, and for $n < m$

$$\left| \int f_n f_m \, dx \right| \leq 2^{-m} \to 0, \quad m \to \infty.$$

Let $\{e_n\}$ be a Gram-Schmidt orthogonalization for this sequence with $\|e_n\| = 1$.

a) Show that

$$\|f_m - e_m\|^2 \leq 2^{-m}.$$

b) For any $g \in L^2$, show that as $m \to \infty$, $\int g f_m \, dx \to 0$.

6. Consider a real-valued nonnegative function $g \in L^1[0,1]$. Suppose that $\int gf \, dx < A$ whenever $\int e^f \, dx \leq 1$.

a) What can you say about the measure of the set $\{g > \lambda\}$ for large λ? (Try $f = c\chi_E$.)[1]

b) Can you say that $g \in L^2$? Why or why not?

Complex Analysis

Here $\mathbb{C}$ denotes the complex plane, Ω denotes a region in $\mathbb{C}$, $dx\, dy$ denotes Lebesgue measure on $\mathbb{C}$.

1. Suppose that $\{f_n\}$ is a sequence of analytic functions on a region Ω, and F a continuous function on Ω such that for any disk $D \subset \Omega$,

$$\lim_{n \to \infty} \int_D |f_n(z) - F(z)| \, dx\, dy = 0.$$

Show that F is analytic on Ω, and that $f_n \to F$ and $f_n' \to F'$ uniformly on compact subsets of Ω.

2. let $p(z)$ be a polynomial of degree $N \geq 1$ and define

$$L = \{z \in \mathbb{C} : |p(z)| = 1\}.$$

Prove that the open set $\mathbb{C} - L$ has at most $N + 1$ components.

3. Suppose f is a complex-valued function defined on $[0, \infty)$ that belongs to $L^p([0,\infty))$ for some p, $1 < p < \infty$. Define G on the right half-plane by

$$G(z) = \int_0^\infty f(t) e^{-tz} \, dt.$$

Prove that $G(z)$ is analytic for $\Re Z > 0$. What additional assumption on f would ensure that G is entire?

[1] This hint can most charitably be described as highly misleading. I would describe it as wrong. A reasonable hint would have been "try $f = c_1\chi_{E_1} + c_2\chi_{E_2}$"

4. Use a contour integral to compute the value of one of the following integrals:

a) The improper Riemann integral,

$$\int_0^\infty e^{ix^2}\,dx.$$

b) The Fourier transform of a Gaussian function on $\mathbb{R}$,

$$\int_{-\infty}^\infty e^{ixy}e^{-x^2}\,dx.$$

5. Consider a compact set $K \subset \mathbb{R}$ with positive Lebesgue measure. On $\mathbb{C} - K$ define the function

$$g(z) = \int_K \frac{1}{t-z}\,dt.$$

a) Prove that g is analytic on $\mathbb{C} - K$;

b) Prove that g cannot be extended by analytic continuation to an entire function;

c) Show that $\lim_{z\to\infty}[zf(z)]$ exists and determine its value.

6. a) Construct a one-to-one conformal mapping of the upper half-plane $H = \{z \in \mathbb{C} : \Im x > 0\}$ onto the angle region with an interval removed,

$$A = \{z \in \mathbb{C} : z \neq 0, \arg z \in (-\pi/4, \pi/4)\} - \{z : \Im z = 0, 0 \le \Re z \le 1\}.$$

b) Describe all the one-to-one conformal mappings of H onto A.

2. Hints and Comments

Real Analysis

1. Hint: I can't think of a good one.

2. Hint: Use the "distribution" theorem–Theorem 8.16 of Rudin, and split the integral into two parts.

Comment: Even with this hint, if you haven't seen the trick before then it's hard to kill the problem off. This problem tested whether or not you had seen this particular trick before.

3. Hint: Show that $\chi_{[-h,h]}$ itself satisfies the given property, so that $\varphi = g - \chi_{[-h,h]}$ satisfies $\varphi * f'(x) = 0$. Use Wheeden and Zygmund Theorem 9.3 and apply the Fourier transform.

4. Hint: Don't be intimidated by how scary this problem looks (I was!): treat it as a M408C problem, then justify your calculations.

5. Hint: Don't even try to solve part (a). Just do part (b), which follows easily from (a).

Comment: After a lot of hard work, Jason Deblois came up with a nice induction argument for part (a). But it required manually verifying the result for $n = 1, 2$, and 3, and the manual verification for $n = 3$ was already harder than most prelim problems. If you find an easier way to do this, please let me know.

6. Hint: Ignore the hint they gave for part (a), because it is a bogus hint which will lead you down a time-wasting blind alley. The hint should probably have read "try $f = c_1\chi_{E_1} + c_2\chi_{E_2}$," which is my hint to you. For part (b), show that any nonnegative function in $L^p([0,1])$ for $1 < p \leq \infty$ satisfies the property in part (a).

Comment: This is a seriously hard problem, made even harder by being phrased open-endedly. It would have been tough enough if they had just told you the answer to both parts and then asked you to verify the answer. It is possible that there is an easier way to do part (b) than the approach I indicated in my hint, because showing this (at least the way I went about it) is harder than almost any other analysis prelim problem I've ever seen.

Complex Analysis

1. Hint: To show that F is analytic, write $\int_D$ in polar coordinates, then try to write the integral over θ as a contour integral; the result will be an inequality. For the convergence, repeat almost the same argument.

Comment: This problem was a more complicated version of the first question on Dr. Beckner's Spring 2001 complex analysis final, which is worth working first. There's no way to perform all the steps in this problem properly in a reasonable amount of time. They should just have asked us to show that F is analytic, since doing that demonstrated facility with all the techniques required to solve the problem.

2. Hint: Apply min/max mod theorem to each component.

Comment: This is a repeat of problem 1 on the January 2000 prelim, which may already have been a repeat of an earlier prelim.

3. Hint: First show G is finite, then form the difference quotient (this is why G must be finite–two infinities could appear to subtract to a finite value) and take the limit. Use the mean value theorem or, with considerably more difficulty, an argument using Morera's.

Comment: This is a repeat of problem 4 on the August 1999 prelim.

4. Hint: No hint. I haven't yet worked this problem.

5. Hint: Start off in a manner similar to problem 3.

Comment: This is a perennial favorite. It appeared in slightly varying forms in at least the Jan 1998 and Jan 2000, and I believe on older, prelims (I don't have those copies with me right now).

6. Hint: No hint. I have seen Jason Deblois's solution, but don't remember it. I believe this exact problem is worked in Palka.

3. Real Analysis 1 Solution

Define

$$\varphi : \left\{[a,b] \subseteq \mathbb{R}^+\right\} \to \mathcal{M}([0,1]),$$

by

$$\varphi([a,b]) = \{x \bmod 1 : x \in [a,b]\}.$$

So for $a, b > 0$, $\varphi([a,b])$ consists of the fractional part of each point in $[a,b]$, and $\varphi([a,b])$ is always the union of one or two intervals. Also, if $b - a \leq 1$, then $|\varphi([a,b])| = |[a,b]|$, where $|E|$ is the Lebesgue measure of the measurable set E.

Let

$$\begin{aligned}
\psi(m) &= \sum_{k=1}^{m} \frac{1}{k}, \\
E_n &= \varphi([\psi(n), \psi(n+1)], \\
f_n &= \sqrt{n+1}\chi_{E_n}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{[0,1]} f_n \, dm =& \sqrt{n+1}\,(\psi(n+1) - \psi(n)) \\
=& \frac{\sqrt{n+1}}{n+1} = \frac{1}{\sqrt{n+1}} \to 0,
\end{aligned}$$

so property (b) is satisfied.

Since $\psi(n) \to \infty$, the E_n's cycle through $[0,1]$ an infinite number of times, missing no points on each pass. This is enough to show that $\limsup f_n = +\infty$, and so property (a) is satisfied as well.

To extend this approach to $\mathbb{R}^n$ we first extend to $\mathbb{R}^1$. We can do this by leaving all our definitions above the same except for changing the underlying function φ, which we redefine so that

$$\varphi : \left\{[a,b] \subseteq \mathbb{R}^+\right\} \to \mathcal{M}(\mathbb{R}),$$

as follows: Let m be the largest positive integer such that $1+2+\cdots+m < a$. Then

$$\varphi([a,b]) = \{(x \bmod m) - m/2 : x \in [a,b]\}.$$

The same argument as before shows that properties (a) and (b) hold, where now the successive sets E_n cycle through $[-1/2, 1/2]$, $[-1, 1]$, $[-3/2, 3/2]$,... in succession, so $\limsup f_n = +\infty$ for all real x, not just $x \in [0,1]$.

Extending this solution to $\mathbb{R}^n$ is a little messy. What we want to do is repeatedly spiral outward from the origin using the φ function along each axis, spiraling farther and farther each time before returning to the origin. But I have to admit that I have never managed to work out a clean way of expressing this, and I don't want to bother you with my ugly way of doing it.

Comment: There are several ways to solve this problem, and some of them probably generalize from the $[0, 1]$ case to the $\mathbb{R}^n$ case a lot more easily. A nice approach might be to use the functions defined in the solution to problem 4 on the real analysis portion of the August 1998 prelim, which involves an enumeration of the rationals, but I can't quite get this to work on $[0, 1]$. But if one could, it would probably extend painlessly.

$$\begin{aligned}
\int_E |f|\ dm =& \int_0^\infty m\left(\{|f| > t\} \cap E\right)\ dt \qquad \text{(Theorem 8.16 Rudin)} \\
=& \int_0^{\sqrt{\frac{C}{|E|}}} m\left(\{|f| > t\} \cap E\right)\ dt + \int_{\sqrt{\frac{C}{|E|}}}^\infty m\left(\{|f| > t\} \cap E\right)\ dt \\
\leq& |E|\sqrt{\frac{C}{|E|}} + \int_{\sqrt{\frac{C}{|E|}}}^\infty \frac{C}{t^2}\ dt \\
=& \sqrt{C}\sqrt{|E|} + \left[\frac{-C}{t}\right]_{\sqrt{\frac{C}{|E|}}}^\infty \\
=& \sqrt{C}\sqrt{|E|} + \frac{C}{\sqrt{\frac{C}{|E|}}} \\
=& 2\sqrt{C}\sqrt{|E|} \\
=& C_1\sqrt{|E|},
\end{aligned}$$

where $C_1 = 2\sqrt{C}$.

Comment: Dr. Beckner worked this problem in his real analysis class the semester following the prelim (so it was probably his problem). He used the approach above (that's where it came from), except that he used a real parameter α in place of $\sqrt{\frac{C}{|E|}}$. He then determined what value of α minimizes the resulting integral a la M408C: $\alpha = \sqrt{\frac{C}{|E|}}$ results. I chose $\sqrt{\frac{C}{|E|}}$ up front because it appeared in Jason Deblois's almost miraculous, from-the-basics solution, which I have a copy of if you want to see it.

$$\begin{aligned}\chi_{[-h,h]} * f'(x) &= \int \chi_{[-h,h]}(t) f'(x-t)\,dt \\ &= \int_{-h}^{h} f'(x-t)\,dt \\ &= -\left[f(x-h) - f(x+h)\right] = f(x+h) - f(x-h) \\ &= g * f'(x),\end{aligned}$$

so letting $\varphi = g - \chi_{[-h,h]}$, it follows that $\varphi * f'(x) = 0$.

Now let $f \in C_C^1$ (so f is compactly supported and at least one-time differentiable). Then

$$\begin{aligned}&\varphi * f'(x) = (\varphi * f)'(x) \qquad \text{(Wheeden and Zygmund Thm 9.3)} \\ &\Rightarrow (\varphi * f)' \equiv 0 \Rightarrow \varphi * f = c,\end{aligned}$$

where c is a constant that may depend on f.

Now, both g and $\chi_{[-h,h]}$ are in $L^1(\mathbb{R})$, so φ is in $L^1(\mathbb{R})$ and hence $\widehat{\varphi}$ exists and is (pointwise) finite, and the same is true of $\widehat{f}$. Hence,

$$\widehat{\varphi * f} = \widehat{\varphi}\widehat{f} = \widehat{c}$$

exists and is finite, which it can only be if $c = 0$. Therefore, $\widehat{\varphi}\widehat{f} \equiv 0$ for all $f \in C_C^1$ and thus $\widehat{\varphi} \equiv 0$ since we can choose an $f \in C_C^1$ so that $\widehat{f}$ is nonzero at any desired point. But then $\widehat{\varphi} = 0$ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and so $\varphi = \widehat{\widehat{\varphi}}$ in $L^1(\mathbb{R})$–that is, almost everywhere– and so $g = \chi_{-h,h]}$ almost everywhere.

Comment: When they wrote "$g \equiv \chi_{-h,h]}$" in the statement of the problem, they must have meant "$g = \chi_{-h,h]}$ almost everywhere." But by my use of the symbol $\equiv$ I mean "identically equal to", and I could just as well have used "=," except that I wanted to stress the point-by-point equality.

(a) Let $B_{n+1,r} = \left\{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n+1}^2 = r\right\}$, and integrate over $t = x_{n+1}$ to determine the volume of $B_{n+1,1}$:

$$\begin{aligned} m\left\{B_{n+1,1}\right\} &= \int_{-1}^{1} m\left\{B_{n,\sqrt{1-t^2}}\right\}\,dt \\ &= 2\int_{-1}^{1} m\left\{B_{n,1}\right\}\left(\sqrt{1-t^2}\right)^n\,dt \\ &= m\left\{B_{n,1}\right\}\int_{-\infty}^{\infty} f(t)\,dt, \end{aligned}$$

where $f(t) = \chi_{[-1,1]}\sqrt{1-t^2}$.

We need to justify the first two integrals above beyond the justification one might give in M408C. The first integral follows from Fubini's theorem, where we are integrating $n+1$ times over the characteristic function of $B_{n+1,1}$, once in each dimension; $m\left\{B_{n,\sqrt{1-t^2}}\right\}$ we recognize as the first n of these integrals. To get the second integral, we use the fact that $m\left\{B_{n,r}\right\} = r^n m\left\{B_{n,1}\right\}$, which follows from a linear change of variables.

(b) A quick way to prove this is to use Maple, to calculate that

$$\int [f(t)]^n\,dt = \sqrt{\pi}\frac{\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2}n+\frac{3}{2})},$$

which approaches 0 inverse-linearly with n. This would not have been a practical approach during the prelim, however.

So instead, note that for all $t \in \mathbb{R} - \{0\}$, $[f(t)]^n \to 0$ pointwise and if we define $f_n = f^n$, then $f = f_1 \geq f_2 \geq \ldots \geq 0$ and $f_n \to 0$ almost everywhere, so all the criteria of Exercise 7 of Chapter 1 of Rudin (which is a quick consequence of the monotone convergence theorem) are satisfied, so $\int [f(t)]^n\,dt \to 0$.

(c)

$$\begin{aligned} A^n m\left\{B_{n,1}\right\} =& A^n m\left\{B_{n-1,1}\right\}\int [f(t)]^{n-1}\,dt \\ =& A^n m\left\{B_{n-2,1}\right\}\int [f(t)]^{n-2}\,dt\int [f(t)]^{n-1}\,dt \\ =& \cdots = A^n m\left\{B_{0,1}\right\}\int [f(t)]^0\,dt \cdots \int [f(t)]^{n-1}\,dt \\ =& \left(A\int [f(t)]^0\,dt\right)\cdots\left(A\int [f(t)]^{n-1}\,dt\right), \end{aligned}$$

since $m\left\{B_{0,1}\right\} = 1$. By part (b), for k sufficiently large, $A\int [f(t)]^k\,dt < r < 1$ for some fixed r with $0 \leq r < 1$. But then $A^n m\left\{B_{n,1}\right\}$ decreases to zero.

7. Real Analysis 5 Solution

(a) When I get a copy of Jason Deblois's solution I will include it here.

(b) Let $\langle f, g\rangle = \int fg$. By Bessel's inequality,

$$\sum_{i=1}^{\infty} \langle g, e_m\rangle \;\leq\; \|g\|_2^2 \;<\; \infty,$$

since $g \in L^2$, so $\langle g, e_m\rangle \to 0$.

But then

$$\begin{aligned} |\langle g, f_m\rangle| &= |\langle g, f_m - e_m\rangle + \langle g, e_m\rangle| \\ &\leq |\langle g, f_m - e_m\rangle| + |\langle g, e_m\rangle| \to 0, \end{aligned}$$

since $\|f_m - g_m\| < 2^{-m/2}$ and so approaches 0, and since $\langle g, e_m\rangle \to 0$ from above.

(a) I claim that the answer that the prelim committee was looking for was that there exists a positive constant C such that for all $\lambda \geq 0$,

$$|\{g > \lambda\}| \left(-\log |\{g > \lambda\}|\right) < \frac{C}{\lambda}.$$

Here's my argument. We first look for an equivalent form for the condition on g:

$$\int e^f \, dx \leq 1 \Rightarrow \int gf \, dx < A,$$

so

$$\int e^{\log h} \, dx \leq 1 \Rightarrow \int g \log h \, dx < A \quad \text{if } h > 0, \quad h \in L^1;$$

that is,

$$\int h \, dx \leq 1 \Rightarrow \int g \log h \, dx < A \quad \text{if } h > 0, \quad h \in L^1. \tag{1}$$

We take Equation (1) as our starting point. [2]

Let

$$h = \alpha \chi_E + \beta \chi_{E^C}$$

with α, β, and a measurable set $E \subseteq [0,1]$ chosen so that

$$\alpha |E| + \beta(1 - |E|) = 1, \quad \alpha > 1, \beta > 0. \tag{2}$$

This means that

$$\beta = \frac{1 - \alpha |E|}{|E^C|} = \frac{1 - \alpha |E|}{1 - |E|}.$$

Then,

$$\int h \, dx = \int \alpha \chi_E + \beta \chi_{E^C} \, dx = \alpha |E| + \beta \left(1 - |E|\right) = 1.$$

The conditions on Equation (1) are satisfied since $h > 0, h \in L^1$, so

$$\begin{aligned}
\int g \log h \, dx &= \int_E g \log \alpha \, dx + \int_{E^C} g \log \beta \, dx \\
&= \log \alpha \int_E g \, dx + \log \beta \int_{E^C} g \, dx < A, \\
\Rightarrow \int_E g \, dx &< \frac{A - \log \beta \int_{E^C} g \, dx}{\log \alpha},
\end{aligned} \tag{3}$$

where we used the fact that $\log \alpha > 0$ since $\alpha > 1$.

Now let $E = \{g > \lambda\}$ and assume that λ is large enough that

$$\alpha = \frac{1}{2|E|} > 1.$$

[2]If I were to work this problem again, I would have taken the original statement as the starting point. For some reason, I found the transformed form less non-intuitive.

Then

$$\beta = \frac{1-\alpha\,|E|}{1-|E|} = \frac{1-\frac{1}{2|E|}\,|E|}{1-|E|} = \frac{1}{2(1-|E|)}$$

and so also $\beta > 0$ as required (and $\beta < 1$) so Equation (2) is satisfied and Equation (3) becomes

$$\begin{aligned}
\lambda\,|E| &\le \int_E g\,dx \\
&< \frac{A - \log\left(\frac{1}{2(1-|E|)}\right)\int_{E^C} g\,dx}{\log\left(1/(2\,|E|)\right)} \\
&= \frac{A + \log\left(2(1-|E|)\right)\int_{E^C} g\,dx}{-\log\left(2\,|E|\right)} \\
&\le \frac{A + \log 2\,\|g\|_1}{-\log\left(2\,|E|\right)} \\
\Rightarrow |E|\left(-\log\left(2\,|E|\right)\right) &< \frac{A + \log 2\,\|g\|_1}{\lambda}
\end{aligned}$$

and clearly, then, there exists a positive constant C such that

$$|E|\left(-\log|E|\right) < \frac{C}{\lambda}.$$

Comment: My condition on the function g is stronger than weak-L^1, but not as strong as L^1. This observation was what lead me to my approach to part (b).

(b) Claim: If $g \in L^p([0,1])$, $g \ge 0$, $1 < p \le \infty$, then g satisfies the properties of the function g of part (a).

Remark: This means that g needn't be in L^2–it could be in $L^{3/2}([0,1]) - L^2([0,1)$, for instance, which is nonempty.

Proof: Let q be such that $1/p + 1/q = 1$, and let h be as in part (a). Then for $1 < p < \infty$,

$$\begin{aligned}
\int g \log h\,dx &= \int_{\{h \le e^{q-1}\}} g \log h\,dx + \int_{\{h > e^{q-1}\}} g \log h\,dx \\
&\le \|g\|_1 (q-1) + \left(\int_{\{h > e^{q-1}\}} g^p\,dx\right)^{1/p} \left(\int_{\{h > e^{q-1}\}} (\log h)^q\,dx\right)^{1/q},
\end{aligned}$$

where we used the fact that $g \ge 0$ and $\log h \le q-1$ on the first integral, and where we used Hölder's inequality on the second integral. Also, $\log h >$

$\log(q-1) > 0$ on the second integral, so no absolute values are needed. Then,

$$\int g \log h \, dx \le \|g\|_1 (q-1) + \|g\|_p \left(\int_{\{h > e^{q-1}\}} (\log h)^q \, dx \right)^{1/q}. \tag{4}$$

Let $\varphi(x) = (\log x)^q$. Then brute calculation shows that

$$\varphi''(x) = \frac{q (\log x)^q (q - \log x - 1)}{x^2 (\log x)^2} < 0 \quad \text{for all } x > e^{q-1},$$

meaning $\varphi(x)$ is concave for $x > e^{q-1}$ and so by Jensen's inequality,

$$\begin{aligned} \int_{\{h > e^{q-1}\}} (\log h)^q \, dx &= \int_{\{h > e^{q-1}\}} \varphi \, dx \\ &\le \left(\int_{\{h > e^{q-1}\}} dx \right) \varphi \left(\frac{\int_{\{h > e^{q-1}\}} h \, dx}{\int_{\{h > e^{q-1}\}} dx} \right) \\ &= \left| \{h > e^{q-1}\} \right| \left(\log \left(\frac{\int_{\{h > e^{q-1}\}} h \, dx}{|\{h > e^{q-1}\}|} \right) \right)^q. \end{aligned}$$

Since, $\int_{\{h > e^{q-1}\}} h \, dx \le \int h \, dx \le 1$,

$$\int_{\{h > e^{q-1}\}} (\log h)^q \, dx \le \left| \{h > e^{q-1}\} \right| \left(-\log \left(\left| \{h > e^{q-1}\} \right| \right) \right)^q. \tag{5}$$

Let

$$\begin{aligned} &\psi : (0,1) \to [0, \infty), \\ &\psi(x) = x(-\log x)^q. \end{aligned}$$

Equation (5) can be written in terms of ψ as

$$\int_{\{h > e^{q-1}\}} (\log h)^q \, dx \le \psi \left(\left| \{h > e^{q-1}\} \right| \right). \tag{6}$$

We want to bound $\int_{\{h > e^{q-1}\}} (\log h)^q \, dx$ by the maximum value of ψ on its domain, $(0,1)$ (observing that $\left| \{h > e^{q-1}\} \right|$ lies in this domain). Since,

$$\psi'(x) = (-\log x)^q \left[1 + \frac{q}{\log x} \right]^q$$

is zero iff $x = e^{-q}$, ψ has exactly one extremum. More brute calculation gives

$$\psi''(x)\big|_{e^{-q}} = \frac{q(-\log x)^q (\log x + q - 1)}{x (\log x)^2}\Big|_{e^{-q}},$$

which is less than zero, since $\log e^{-q} + q - 1 = -1 < 0$. Therefore, $\psi(x)$ has a maximum value of $\psi(e^{-q}) = e^{-q}(-(-q))^q = e^{-q}q^q$. Then, from Equation (4) and Equation (6),

$$\begin{aligned}\int g \log h \, dx &\le \|g\|_1 (q-1) + \|g\|_p (e^{-q}q^q)^{1/q} \\ &= \|g\|_1 (q-1) + \|g\|_p \, q/e,\end{aligned}$$

which gives us $A = \|g\|_1 (q-1) + \|g\|_p \, q/e$ for part (a) when $1 < p < \infty$.

If $p = \infty$, then [3] Equation (4) becomes (since $q = 1$)

$$\begin{aligned}\int g \log h \, dx &\le \|g\|_1 (1-1) + \|g\|_\infty \int_{h>1} \log h \, dx \\ &\le \|g\|_\infty \|h\|_1 , \quad \text{since } |\log h| < |h| \text{ for } h > 1 \\ &\le \|g\|_\infty \quad \text{since } \|h\|_1 \le 1,\end{aligned}$$

and $A = \|g\|_\infty$ can be used in part (a).

[3]This case is not needed, but I am proving it for my own satisfaction.