FLOW MAP ESTIMATES FOR A NON-AUTONOMOUS VECTOR FIELD

Abstract. We give a proper argument for the spatial MOC of the flow map—and the inverse flow map—for a non-autonomous vector field having an Osgood MOC.

Let Ω be a domain in $\mathbb{R}^d$, $d \geq 2$, and let $v\colon [0,\infty) \times \Omega \to \mathbb{R}^d$ be a time-varying velocity field on Ω. Assume that v has an Osgood modulus of continuity (MOC), μ, in space, uniform over time. That is,

$$|v(t,x) - v(t,y)| \leq \mu(|x-y|) \tag{1}$$

for all $t \in [0,\infty)$ and all $x, y \in \Omega$, where $\mu\colon [0,\infty) \to [0,\infty)$ with $\mu(0) = 0$ is an increasing function satisfying

$$\int_0^1 \frac{dx}{\mu(x)} = \infty.$$

Proposition 1. *The vector field v has a unique continuous flow. More precisely, there exists a unique mapping X, continuous from $[0,\infty) \times \Omega$ to $\mathbb{R}^d$, such that*

$$X(t,x) = x + \int_0^t v(s, X(s,x))\, ds.$$

Let $\Gamma_t\colon [0,\infty) \to [0,\infty)$ be defined by $\Gamma_t(0) = 0$ and for $\delta > 0$ by

$$\int_\delta^{\Gamma_t(\delta)} \frac{dr}{\mu(r)} = t. \tag{2}$$

Then $\delta \mapsto \Gamma_t(\delta)$ is a MOC for $X(t,\cdot)$ for all $t \geq 0$; that is, for all x and y in $\mathbb{R}^2$

$$|X(t,x) - X(t,y)| \leq \Gamma_t(|x-y|). \tag{3}$$

Moreover, if we define $X^{-1}(t,x)$ so that $X(t, X^{-1}(t,x)) = x$ then $\delta \mapsto \Gamma_t(\delta)$ is also a MOC for $X^{-1}(t,\cdot)$.

Proof. The existence of the flow map is classical. To obtain a spatial MOC on the flow map, let $x, y \in \Omega$. Then

$$\begin{aligned}
|X(t,x) - X(t,y)| &= \left| x - y + \int_0^t \left(v(s, X(s,x)) - v(s, X(s,y))\right) ds \right| \\
&\leq |x-y| + \int_0^t |v(s, X(s,x)) - v(s, X(s,y))|\, ds \\
&\leq |x-y| + \int_0^t \mu(|X(s,x) - X(s,y)|)\, ds.
\end{aligned}$$

Applying Osgood's lemma, Lemma 3, gives (2, 3).

The bound on $X^{-1}(t,\cdot)$ follows by running the flow backward. For completeness, we give the argument explicitly, arguing as in part of the proof of Lemma 8.2 p. 318-319 of [2].

Suppose that a particle moving under the flow map is at position x at time t. Let $Y(\tau; t, x)$ be the position of that same particle at time $t - \tau$, where $0 \le \tau \le t$. Then

$$X^{-1}(t, x) = Y(t; t, x), \quad x = Y(0; t, x)$$

and

$$\frac{d}{d\tau} Y(\tau; t, x) = -v(t - \tau, Y(\tau; t, x)).$$

By the fundamental theorem of calculus,

$$Y(s; t, x) - x = \int_0^s \frac{d}{d\tau} Y(\tau; t, x)\, d\tau,$$

or,

$$Y(s; t, x) = x - \int_0^s v(t - \tau, Y(\tau; t, x))\, d\tau.$$

Thus,

$$\begin{aligned}
&|Y(s;t, x) - Y(s; t, y)| \\
&\quad = \left| x - y - \int_0^s \left(v(t - \tau, Y(\tau; t, x)) - v(t - \tau, Y(\tau; t, y)) \right) d\tau \right| \\
&\quad = |x - y| + \int_0^s |v(t - \tau, Y(\tau; t, x)) - v(t - \tau, Y(\tau; t, y))|\, d\tau \\
&\quad \le |x - y| + \int_0^s \mu(|Y(\tau; t, x) - Y(\tau; t, y)|)\, d\tau.
\end{aligned}$$

For any fixed $t > 0$, this bound applies for all $0 \le s \le t$. We can thus apply Osgood's lemma to conclude that

$$|Y(s; t, x) - Y(s; t, y)| \le \Gamma_s(|x - y|).$$

In particular, setting $s = t$ yields

$$|X^{-1}(t, x) - X^{-1}(t, y)| \le \Gamma_t(|x - y|).$$

□

Remark 2. Note the bound in (8.36) p. 315 of [2] on the MOC of the *forward* flow map in time can be improved to $\|v\|_{L^\infty} |t_1 - t_2|$ by a direct bound, because only one flow line is involved. The inverse flow map bound in (8.35) of [2] cannot be improved, because two flow lines are involved. There is no discrepancy in the forward and backward flow map bounds on the spatial MOC, because both involve two flow lines.

The following is Osgood's lemma, as stated in [1].

Lemma 3 (Osgood's lemma)**.** *Let L be a measurable nonnegative function and γ a nonnegative locally integrable function, each defined on the domain $[t_0, t_1]$. Let $\mu\colon [0, \infty) \to [0, \infty)$ be a continuous nondecreasing function, with $\mu(0) = 0$. Let $a \ge 0$, and assume that for all t in $[t_0, t_1]$,*

$$L(t) \le a + \int_{t_0}^t \gamma(s) \mu(L(s))\, ds. \tag{4}$$

If $a > 0$*, then*

$$\int_a^{L(t)} \frac{ds}{\mu(s)} \leq \int_{t_0}^t \gamma(s)\, ds.$$

If $a = 0$ *and* $\int_0^\infty ds/\mu(s) = \infty$*, then* $L \equiv 0$*.*

References

[1] J.-Y. Chemin, *Perfect incompressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 14, The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie. MR1688875 (2000a:76030) ↑2

[2] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002. MR1867882 (2003a:76002) ↑1, 2