ON KATO'S CONDITIONS FOR VANISHING VISCOSITY

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Abstract. Let u be a solution to the Navier-Stokes equations with viscosity ν in a bounded domain Ω in $\mathbb{R}^d$, $d \geq 2$, and let $\overline{u}$ be the solution to the Euler equations in Ω. In 1983 Tosio Kato showed that for sufficiently regular solutions, $u \to \overline{u}$ in $L^\infty([0, T]; L^2(\Omega))$ as $\nu \to 0$ if and only if $\nu \|\nabla u\|_X^2 \to 0$ as $\nu \to 0$, where $X = L^2([0, T] \times \Gamma_{c\nu})$, $\Gamma_{c\nu}$ being a layer of thickness $c\nu$ near the boundary. We show that Kato's condition is equivalent to $\nu \|\omega(u)\|_X^2 \to 0$ as $\nu \to 0$, where $\omega(u)$ is the vorticity (curl) of u, and is also equivalent to $\nu^{-1} \|u\|_X^2 \to 0$ as $\nu \to 0$.

1. Kato's Conditions for Vanishing Viscosity

The question of whether solutions of the Navier-Stokes equations converge to a solution of the Euler equations as the viscosity goes to zero—the so-called *vanishing viscosity* or *inviscid limit*—on a domain with boundary is a long-open problem in mathematical fluid mechanics. The necessary and sufficient conditions for the existence of this limit given by Tosio Kato in [1], along with an extension of the conditions by Temam and Wang in [4] and Wang in [5], probably represent the closest anyone has come to resolving this question.

Kato's key condition requires that the L^2-norm of the gradient of the velocity in a boundary layer of width proportional to the viscosity not blow up too rapidly as the viscosity vanishes (condition (iii′) in Theorem 1.1). Temam and Wang in [4] and [5] show that, at the expense of increasing the size of the boundary layer slightly, one need only consider the tangential derivatives of the tangential components of the velocity or the tangential derivatives of the normal components of the velocity. We leave the size of Kato's boundary layer unchanged, and show that the gradient of the velocity can be replaced by the vorticity in Kato's condition. We also establish

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another necessary and sufficient condition that the average energy in the boundary layer vanish with the viscosity. Both of these conditions have more immediate physical meaning than Kato's, though they may well be no easier to verify or refute. The necessity of our conditions follows easily from Kato's conditions; it is the sufficiency of the conditions that requires a modification of Kato's argument.

We now describe in detail Kato's result and our extension of it. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with C^2-boundary Γ, and let $\mathbf{n}$ be the outward normal vector to Γ. A classical solution $(\overline{u}, \overline{p})$ to the Euler equations satisfies,

$$(E) \quad \begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{p} = \overline{f} \text{ and } \operatorname{div} \overline{u} = 0 \text{ on } [0,T] \times \Omega, \\ \overline{u} \cdot \mathbf{n} = 0 \text{ on } [0,T] \times \Gamma, \text{ and } \overline{u} = \overline{u}^0 \text{ on } \{0\} \times \Omega, \end{cases}$$

where $\operatorname{div} \overline{u}^0 = 0$. These equations describe the motion of an incompressible fluid of constant density and zero viscosity.

We assume that $\overline{u}^0$ is in $C^{k+\epsilon}(\Omega)$, $\epsilon > 0$, where $k = 1$ for two dimensions and $k = 2$ for 3 and higher dimensions, and that $\overline{f}$ is in $C^1([0,t] \times \Omega)$ for all $t > 0$. Then as shown in [2] (Theorem 1 and the remarks on p. 508-509), there is some $T > 0$ for which there exists a unique solution

$$\overline{u} \text{ in } C^1([0,T]; C^{k+\epsilon}(\Omega)). \tag{1.1}$$

In two dimensions, T can be arbitrarily large, though it is only known that some nonzero T exists in three and higher dimensions.

The Navier-Stokes equations describe the motion of an incompressible fluid of constant density and positive viscosity ν. A classical solution to the Navier-Stokes equations can be defined in analogy to (E) by

$$(NS) \quad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f \text{ and } \operatorname{div} u = 0 \text{ on } [0,T] \times \Omega, \\ u = 0 \text{ on } [0,T] \times \Gamma, \text{ and } u = u^0_\nu \text{ on } \{0\} \times \Omega. \end{cases}$$

We will work, however, with weak solutions to the Navier-Stokes equations. (See, for instance, Chapter III of [3].) It follows, assuming that f is in $L^1([0,T]; L^2(\Omega))$, that for such solutions,

$$\begin{aligned} &(u(t), \phi(t)) - (u(0), \phi(0)) \\ &\qquad = \int_0^t \left[(u, u \cdot \nabla \phi) - \nu(\nabla u, \nabla \phi) + (f, \phi) + (u, \partial_t \phi) \right] \, dt \end{aligned} \tag{1.2}$$

for all ϕ in $C^1([0,T] \times \Omega) \cap C^1([0,T]; V)$. Here, $(\cdot, \cdot)$ is the L^2-inner product and V is the space of all divergence-free vector fields in $H^1_0(\Omega)$. We will also need the related function space H of divergence-free vector fields v in $L^2(\Omega)$ with $v \cdot \mathbf{n} = 0$ on Γ in the sense of a trace.

The advantage of using weak solutions for (NS) is that existence is known globally in time. Uniqueness, however, is only known to hold in two dimensions, so in dimension 3 and higher when we say that u is a solution to (NS) we mean that for each value of ν we choose one of possibly many solutions.

Theorem 1.1 is Theorem 1 of [1].

Theorem 1.1 (Kato)**.** *Assume that u_ν^0 is in H and that $\overline{u}^0$ is such that (1.1) holds. In addition, assume that*

(a) $u_\nu^0 \to \overline{u}^0$ *in* $L^2(\Omega)$ *as* $\nu \to 0$,
(b) f *is in* $L^1([0,T]; L^2(\Omega))$,
(c) $\|f - \overline{f}\|_{L^1([0,T];L^2(\Omega))} \to 0$ *as* $\nu \to 0$.

Then the following conditions are equivalent:

(i) $u(t) \to \overline{u}(t)$ *in* $L^2(\Omega)$ *as* $\nu \to 0$ *uniformly over* t *in* $[0,T]$,
(ii) $u(t) \to \overline{u}(t)$ *in* $L^2(\Omega)$ *as* $\nu \to 0$ *weakly for all* t *in* $[0,T]$,
(iii) $\nu \int_0^T \|\nabla u\|_{L^2(\Omega)}^2 \, dt \to 0$ *as* $\nu \to 0$,
(iii′) $\nu \int_0^T \|\nabla u\|_{L^2(\Gamma_{c\nu})}^2 \, dt \to 0$ *as* $\nu \to 0$,

where $\Gamma_{c\nu}$ *is the boundary strip of width* $c\nu$ *with* $c > 0$ *fixed but arbitrary.*

If $\overline{f} = 0$, *then the four conditions above are also equivalent to*

(iv) $u(T) \to \overline{u}(T)$ *in* $L^2(\Omega)$ *as* $\nu \to 0$ *weakly* .

It follows immediately from Lemma A.2 that ∇u can be replaced in condition (iii) by the vorticity $\omega(u)$ of (A.1). The same cannot be said immediately of condition (iii′), however, because we have no control over the value of u on the interior boundary of $\Gamma_{c\nu}$. Instead we reexamine Kato's proof of Theorem 1.1 to establish this, giving Theorem 1.2.

Theorem 1.2. *The following condition is equivalent to those of Theorem 1.1:*

(iii″) $\nu \int_0^T \|\omega(u)\|_{L^2(\Gamma_{c\nu})}^2 \, dt \to 0$ *as* $\nu \to 0$.

If, in addition, the solution $\overline{u}$ *is in* $C^1([0,T] \times C^2(\Omega))$ *then condition (iii′) and the conditions in Theorem 1.1 are equivalent to the following condition:*

(iii‴) $\nu^{-1} \int_0^T \|u\|_{L^2(\Gamma_{c\nu})}^2 \, dt \to 0$ *as* $\nu \to 0$.

Proof. That (iii′) ⇒ (iii″) is trivial; in Section 4 we prove (iii″) ⇒ (i). Applying Lemma A.1,

$$\nu^{-1} \int_0^T \|u\|_{L^2(\Gamma_{c\nu})}^2 \leq \nu^{-1} \int_0^T C\nu^2 \|\nabla u\|_{L^2(\Gamma_{c\nu})}^2 ,$$

so (iii′) ⇒ (iii‴). We prove that (iii‴) ⇒ (i) in Section 5, completing the circle of implications. □

Observe that Kato's conditions in Theorem 1.1 reduce the question of whether the vanishing viscosity limit holds to properties of the Navier-Stokes equations alone. Our improvement in (iii″) of Theorem 1.2 shows that only certain combinations of the derivatives of the velocity need be considered. Condition (iii‴) requires that the time integral of the energy in the boundary layer decrease faster than linearly with the viscosity or, viewed another way, that the average energy on $[0,T] \times \Gamma_{c\nu}$ vanish with the viscosity.

2. A priori estimates

The classical energy inequalities are

$$\|u(t)\|_{L^2(\Omega)}^2 + 2\nu \int_0^t \|\nabla u\|_{L^2(\Omega)}^2 \, dt \le \|u(0)\|_{L^2(\Omega)}^2 + 2\int_0^t (f, u)\, dt \tag{2.1}$$

for (NS) and

$$\|\overline{u}(t)\|_{L^2(\Omega)}^2 \le \|\overline{u}(0)\|_{L^2(\Omega)}^2 + 2\int_0^t (\overline{f}, \overline{u})\, dt \tag{2.2}$$

for (E). From these assumptions and our assumptions on f and $\overline{f}$ it follows easily that

$$\|u\|_{L^\infty([0,T];L^2(\Omega))} \le C. \tag{2.3}$$

It follows immediately from (2.1) that

$$\begin{aligned} &\nu \|\nabla u\|_{L^2([0,t];L^2(\Omega))}^2 \\ &\quad \le \frac{1}{2}\|u(0)\|_{L^2(\Omega)}^2 + \|u\|_{L^\infty([0,T];L^2(\Omega))} \|f\|_{L^1([0,T];L^2(\Omega))} \le C, \end{aligned} \tag{2.4}$$

and thus that

$$\begin{aligned} \nu^{1/2} \|\nabla u\|_{L^1([0,T];L^2(\Omega))} &\le \nu^{1/2} \|1\|_{L^2([0,T])} \|\nabla u\|_{L^2([0,T];L^2(\Omega))} \\ &= T^{1/2}\nu^{1/2} \|\nabla u\|_{L^2([0,T];L^2(\Omega))} \le C. \end{aligned} \tag{2.5}$$

3. Boundary layer

In [1], Kato constructs a boundary layer velocity: a time-varying velocity field v that is nonzero only within a distance $\delta > 0$ from Γ and that equals $\overline{u}$ on Γ. He first shows that there exists a matrix-valued function $\overline{a}$ near the boundary that is zero on Γ and such that

$$\overline{u} = \operatorname{div} \overline{a} = \partial_k \overline{a}_{jk} \text{ on } \Gamma,$$

where we use the implied summation convention. Kato's construction of $\overline{a}$ shows that $\overline{a}$ has no loss (though no gain) in regularity over $\overline{u}$. He then defines a function z in $C^\infty(\Omega)$ whose support lies in Γ_δ by

$$z(x) = \zeta(\rho(x)/\delta),$$

where $\zeta : [0,\infty) \to [0,1]$ is a smooth cutoff function with $\zeta(0) = 1$ and $\zeta(r) = 0$ for $r \ge 1$, and ρ is the distance from x to Γ. Finally, he lets

$$v = \operatorname{div}(z\overline{a}) = z \operatorname{div} \overline{a} + \overline{a} \cdot \nabla z.$$

Given the smoothness of $\overline{a}$ inherited from $\overline{u}$, it follows that v lies in the space $C^1([0,T]; C^{k-1}(\Omega))$, where k is as in (1.1). In dimensions 3 and higher,

v is in $C^1([0,T]\times\Omega)$, which is sufficient to derive the following bounds (see Equation (4.6) of [1]) with the help of Lemma A.1:

$$\begin{aligned} \|v\|_{L^\infty([0,T];L^2(\Omega))} &\le C\delta^{1/2}, \quad \|\partial_t v\|_{L^1([0,T];L^2(\Omega))} \le C\delta^{1/2}, \\ \|\nabla v\|_{L^\infty([0,T];L^2(\Omega))} &\le C\delta^{-1/2}, \qquad \|v\|_{L^\infty([0,T]\times\Omega)} \le C, \\ \|\nabla v\|_{L^\infty([0,T]\times\Omega)} &\le C\delta^{-1}. \end{aligned} \tag{3.1}$$

Similarly, when we assume that $\overline{u}$ is in $C^1([0,T];C^2(\Omega))$, we have

$$\|\Delta v\|_{L^\infty([0,T];L^2(\Omega))} \le C\delta^{-3/2}. \tag{3.2}$$

In dimension 2 we must construct v differently so as to not lose regularity over $\overline{u}$. We do this by employing the stream function for $\overline{u}$; that is, a function ψ such that $\overline{u} = \nabla^\perp\psi = (-\partial_2\psi, \partial_1\psi)$. Given ψ, we construct a new boundary layer stream function ψ_0 defined on Γ_δ by subtracting from ψ its constant value on the nearest component of Γ, so ψ_0 is zero on Γ and $\overline{u} = \nabla^\perp\psi_0$ on Γ_δ. Finally, we define the boundary layer velocity v by

$$v = \nabla^\perp(z\psi_0) = z\nabla^\perp\psi_0 + \psi_0\nabla^\perp z.$$

Because ψ_0 has a gain in regularity over $\overline{u}$ of one derivative, v is in $C^1([0,T]\times\Omega)$ and the estimates in (3.1) follow in a similar manner.

4. Proving (iii$''$) $\Rightarrow$ (i)

We follow Kato's approach in [1] to establish the energy inequality in (4.2), departing from his approach only in bounding the terms in $R(t)$. We give the complete argument, however, for the sake of being self-contained.

We now let $\delta = c\nu$. It follows from (2.1) and (2.2) that for all t in $[0,T]$,

$$\begin{aligned} \|u(t)-\overline{u}(t)\|^2_{L^2(\Omega)} &= \|u(t)\|^2_{L^2(\Omega)} + \|\overline{u}(t)\|^2_{L^2(\Omega)} - 2(u,\overline{u}) \\ &\le \|u(0)\|^2_{L^2(\Omega)} + 2\int_0^t (f,u)\,dt + \|\overline{u}(0)\|^2_{L^2(\Omega)} + 2\int_0^t (\overline{f},\overline{u})\,dt - 2(u,\overline{u}) \\ &= \alpha_1 + 2\int_0^t [(f,u)+(\overline{f},\overline{u})]\,dt + 2\,\|\overline{u}(0)\|^2_{L^2(\Omega)} - 2(u,\overline{u}-v), \end{aligned}$$

where

$$\alpha_1 = \|u(0)\|^2_{L^2(\Omega)} - \|\overline{u}(0)\|^2_{L^2(\Omega)} - 2(u,v).$$

But

$$|(u,v)| \le \|u\|_{L^\infty([0,T];L^2(\Omega))}\,\|v\|_{L^\infty([0,T];L^2(\Omega))} \le C\delta^{1/2}$$

by (2.3) and (3.1), so by assumption (a), $\alpha_1 \to 0$ uniformly over $[0,T]$ as $\nu \to 0$ since $\delta = c\nu$.

To handle the term $-2(u, \overline{u} - v)$, we use $\phi = \overline{u} - v$ as a test function in (1.2). This gives

$$\begin{aligned}(u, &\overline{u} - v) - (u(0), \overline{u}(0) - v(0)) \\ &= \int_0^t (u, u \cdot \nabla(\overline{u} - v)) - \nu(\nabla u, \nabla(\overline{u} - v)) + (f, \overline{u} - v) + (u, \partial_t(\overline{u} - v))\, dt,\end{aligned}$$

or after multiplying by -2,

$$\begin{aligned}-2(u, \overline{u} - v) + 2\|u(0)\|^2_{L^2(\Omega)} = \alpha_2 &+ \int_0^t [-2(u, u \cdot \nabla(\overline{u} - v)) \\ &+ 2\nu(\nabla u, \nabla(\overline{u} - v)) - 2(f, \overline{u}) - 2(u, \partial_t(\overline{u} - v))]\, dt.\end{aligned} \tag{4.1}$$

Here,

$$\alpha_2 = 2(u(0), u(0)) - 2(u(0), \overline{u}(0)) + 2(u(0), v(0)) + 2\int_0^t (f, v)\, dt,$$

for which

$$\begin{aligned}|\alpha_2| \le &2\|u(0)\|_{L^2(\Omega)} \|u(0) - \overline{u}(0)\|_{L^2(\Omega)} + 2\|u(0)\|_{L^2(\Omega)} \|v(0)\|_{L^2(\Omega)} \\ &+ \|f\|_{L^1([0,T];L^2(\Omega))} \|v\|_{L^\infty([0,T];L^2(\Omega))}.\end{aligned}$$

This vanishes as $\nu \to 0$ uniformly over $[0, T]$ by assumptions (a) and (b) and (3.1).

The last term in (4.1) is

$$\int_0^t [-2(u, \partial_t(\overline{u} - v))]\, dt = -2\int_0^t (u, \partial_t \overline{u})\, dt + 2\int_0^t (u, \partial_t v)\, dt.$$

Because $\overline{u}$ is a solution to (E),

$$\int_0^t (u, \partial_t \overline{u})\, dt = -\int_0^t [(u, \overline{u} \cdot \nabla \overline{u}) + (u, \overline{f})]\, dt.$$

Also, by (2.3) and (3.1),

$$\left| \int_0^t (u, \partial_t v) \right| \le \int_0^t \|u\|_{L^2(\Omega)} \|\partial_t v\|_{L^2(\Omega)} \le C\nu^{1/2}.$$

We conclude that for all t in $[0, T]$,

$$\begin{aligned}\|u(t) - \overline{u}(t)\|^2_{L^2(\Omega)} \le \alpha + 2\int_0^t &[(f - \overline{f}, u - \overline{u}) - (u, u \cdot \nabla(\overline{u} - v)) \\ &+ (u, \overline{u} \cdot \nabla \overline{u}) + \nu(\nabla u, \nabla(\overline{u} - v))]\, dt,\end{aligned}$$

where $\alpha \to 0$ as $\nu \to 0$. But,

$$\begin{aligned}((u - \overline{u}), &(u - \overline{u}) \cdot \nabla \overline{u}) \\ &= (u, u \cdot \nabla \overline{u}) - (u, \overline{u} \cdot \nabla \overline{u}) - (\overline{u}, u \cdot \nabla \overline{u}) + (\overline{u}, \overline{u} \cdot \nabla \overline{u}) \\ &= (u, u \cdot \nabla \overline{u}) - (u, \overline{u} \cdot \nabla \overline{u}).\end{aligned}$$

The vanishing of the two terms above follows from Green's theorem and the high regularity of $\overline{u}$. Thus,

$$\begin{aligned}
&\|u(t) - \overline{u}(t)\|^2_{L^2(\Omega)} \\
&\qquad \le \alpha + 2\int_0^t [(f - \overline{f}, u - \overline{u}) - ((u - \overline{u}), (u - \overline{u}) \cdot \nabla \overline{u}) \\
&\qquad\qquad + (u, u \cdot \nabla v) + \nu(\nabla u, \nabla(\overline{u} - v))]\, dt \\
&\qquad \le \alpha + \int_0^t R(t)\, dt + 2\int_0^t \int_\Omega |u - \overline{u}|^2 |\nabla \overline{u}|\, dt,
\end{aligned} \tag{4.2}$$

where

$$R(t) = 2(f - \overline{f}, u - \overline{u}) + 2\nu(\nabla u, \nabla(\overline{u} - v)) + 2(u, u \cdot \nabla v). \tag{4.3}$$

We can control all three terms in $R(t)$. For the first term,

$$\left|(f - \overline{f}, u - \overline{u})\right| \le \|f - \overline{f}\|_{L^2(\Omega)} \|u - \overline{u}\|_{L^2(\Omega)} \le C\|f - \overline{f}\|_{L^2(\Omega)},$$

so by assumption (c), $\int_0^t (f - \overline{f}, u - \overline{u}) \to 0$ as $\nu \to 0$.

For the second term in $R(t)$ we have

$$\begin{aligned}
\left|\nu \int_0^t (\nabla u, \nabla(\overline{u} - v))\right| &\le \nu \left|\int_0^t (\nabla u, \nabla \overline{u})\right| + \nu \left|\int_0^t (\nabla u, \nabla v)\right| \\
&= \nu \left|\int_0^t (\nabla u, \nabla \overline{u})\right| + 2\nu \left|\int_0^t (\omega(u), \omega(v))\right| \\
&\le \nu \int_0^t \|\nabla u\|_{L^2(\Omega)} \|\nabla \overline{u}\|_{L^2(\Omega)} + 2\nu \int_0^t \|\omega(u)\|_{L^2(\Gamma_\delta)} \|\nabla v\|_{L^2(\Gamma_\delta)} \\
&\le C\nu \|\nabla u\|_{L^1([0,t];L^2(\Omega))} + C\nu\delta^{-1/2} \|\omega(u)\|_{L^1([0,t];L^2(\Gamma_\delta))}.
\end{aligned}$$

Here we applied Lemma A.2 and (3.1). As $\nu \to 0$, the first term above vanishes by (2.5). For the second term, since $\delta = c\nu$,

$$\begin{aligned}
\nu\delta^{-1/2} \|\omega(u)\|_{L^1([0,t];L^2(\Gamma_\delta))} &= C\nu^{1/2} \|\omega(u)\|_{L^1([0,t];L^2(\Gamma_{c\nu}))} \\
&\le C\nu^{1/2} t^{1/2} \|\omega(u)\|_{L^2([0,t];L^2(\Gamma_{c\nu}))} \\
&= Ct^{1/2} \left(\nu \int_0^t \|\omega(u)\|^2_{L^2(\Gamma_{c\nu})}\right)^{1/2},
\end{aligned}$$

which vanishes by assumption (iii$''$).

For the third term in $R(t)$, we apply Lemma A.3, Lemma A.4, Lemma A.1, (3.1), and (2.4), to obtain

$$\begin{aligned}\left|\int_0^t (u, u\cdot\nabla v)\right| &= \left|\int_0^t (v, u\cdot\nabla u)\right| = 2\left|\int_0^t (v, u\cdot\omega(u))\right| \\ &\le 2\|v\|_{L^\infty([0,T]\times\Omega)}\int_0^t \|u\|_{L^2(\Gamma_\delta)}\|\omega(u)\|_{L^2(\Gamma_\delta)} \\ &\le C\delta\int_0^t \|\nabla u\|_{L^2(\Gamma_\delta)}\|\omega(u)\|_{L^2(\Gamma_\delta)} \\ &\le C\delta^{1/2}\|\nabla u\|_{L^2([0,T];L^2(\Gamma_\delta))}\delta^{1/2}\|\omega(u)\|_{L^2([0,T];L^2(\Gamma_\delta))} \\ &\le C\left(\delta\int_0^t \|\omega(u)\|^2_{L^2(\Gamma_\delta)}\right)^{1/2}.\end{aligned}$$

This also vanishes as $\nu \to 0$ by assumption (iii$''$) since $\delta = c\nu$.

Applying Gronwall's lemma to (4.2) as in [1] gives condition (i).

5. Proving (iii$'''$) $\Rightarrow$ (i)

The only change that is required in the proof of (iii$''$) $\Rightarrow$ (i) in Section 4 is the manner in which we bound the second and third term in $R(t)$ of (4.3).

To bound the second term in $R(t)$, we start as before with

$$\left|\nu\int_0^t (\nabla u, \nabla(\overline{u} - v))\right| \le \nu\left|\int_0^t (\nabla u, \nabla\overline{u})\right| + \nu\left|\int_0^t (\nabla u, \nabla v)\right|.$$

The first term vanishes as we showed in Section 4 without the use of any of the conditions. For the second term, we apply Green's theorem to give $(\nabla u, \nabla v) = -(u, \Delta v)$, which uses the vanishing of u on Γ. Then using (3.2),

$$\begin{aligned}\nu\left|\int_0^t (\nabla u, \nabla v)\right| &= \nu\left|\int_0^t (u, \Delta v)\right| \le C\nu\int_0^t \|u\|_{L^2(\Gamma_\delta)}\|\Delta v\|_{L^2(\Gamma_\delta)} \\ &\le C\nu^{-1/2}\int_0^t \|u\|_{L^2(\Gamma_\delta)} \le Ct^{1/2}\left(\nu^{-1}\int_0^t \|u\|^2_{L^2(\Gamma_\delta)}\right)^{1/2},\end{aligned}$$

so the second term in $R(t)$ vanishes with ν by condition (iii$'''$).

Using (3.1), we bound the third term in $R(t)$ by

$$\left|\int_0^t (u, u\cdot\nabla v)\right| \le \|\nabla v\|_{L^\infty([0,T]\times\Omega)}\int_0^t \|u\|^2_{L^2(\Gamma_\delta)} \le \frac{C}{\nu}\int_0^t \|u\|^2_{L^2(\Gamma_{c\nu})},$$

which also vanishes with ν by condition (iii$'''$).

Appendix A. Appendix

Lemma A.1 is a version of Poincaré's inequality.

Lemma A.1. *Let ψ be in $H^{1,p}(\Omega)$ for p in $[1,\infty]$ with $\psi = 0$ on Γ. There exists a constant C independent of p such that for all sufficiently small δ*

$$\|\psi\|_{L^p(\Gamma_\delta)} \le C\delta\|\nabla\psi\|_{L^p(\Gamma_\delta)}.$$

We define the vorticity $\omega(v)$ to be the $d \times d$ antisymmetric matrix

$$\omega(v) = \frac{1}{2}\left[\nabla v - (\nabla v)^T\right]. \tag{A.1}$$

This gives the useful identity,

$$\begin{aligned} 2\omega(u) \cdot \omega(v) &= \frac{1}{2}(\nabla u - (\nabla u)^T) \cdot (\nabla v - (\nabla v)^T) \\ &= \nabla u \cdot \nabla v - \nabla u \cdot (\nabla v)^T. \end{aligned} \tag{A.2}$$

Lemma A.2. *For all u and v in $H^1(\Omega)$ with $\operatorname{div} v = 0$ and such that $(u \cdot \nabla v) \cdot \mathbf{n} = 0$ on Γ,*

$$\int_\Omega \nabla u \cdot \nabla v = 2 \int_\Omega \omega(u) \cdot \omega(v).$$

Proof. Since $\operatorname{div} v = 0$, we have $\nabla u \cdot (\nabla v)^T = \partial_j u^i \partial_i v^j = \partial_j (u^i \partial_i v^j) = \operatorname{div}(u \cdot \nabla v)$, so if u and v are both in $C^\infty(\Omega)$ then

$$\begin{aligned} 2 \int_\Omega \omega(u) \cdot \omega(v) &= \int_\Omega \nabla u \cdot \nabla v - \nabla u \cdot (\nabla v)^T \\ &= \int_\Omega \nabla u \cdot \nabla v - \int_\Omega \operatorname{div}(u \cdot \nabla v) = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (u \cdot \nabla v) \cdot \mathbf{n} \\ &= \int_\Omega \nabla u \cdot \nabla v. \end{aligned}$$

The result then follows by the density of $C^\infty(\Omega)$ in $H^1(\Omega)$. □

Lemma A.3. *For all u in V and v in $C^1(\Omega)$,*

$$(u, u \cdot \nabla v) = -(v, u \cdot \nabla u).$$

Proof. First observe that both sides of this equality makes sense because of the regularity of u and v. Then

$$\begin{aligned} (u, u \cdot \nabla v) &= \int_\Omega u^i u^j \partial_j v^i = \int_\Omega u^j \partial_j (v^i u^i) - \int_\Omega u^j v^i \partial_j u^i \\ &= \int_\Omega u \cdot \nabla(u \cdot v) - \int_\Omega (u \cdot \nabla u) \cdot v \\ &= -\int_\Omega (\operatorname{div} u) u \cdot v + \int_\Gamma (u \cdot \mathbf{n}) u \cdot v - \int_\Omega (u \cdot \nabla u) \cdot v \\ &= -\int_\Omega (u \cdot \nabla u) \cdot v. \end{aligned}$$

In applying Green's theorem above, all that we required was that u and $\operatorname{div} u$ be in $L^2(\Omega)$ and that $u \cdot v$ be in $H^1(\Omega)$ (see, for instance, Theorem I.1.2 of [3]). □

Lemma A.4. *For all u in $H^1(\Omega)$ and v in $C^1(\Omega)$ with $\operatorname{div} v = 0$ on Ω and such that $(v \cdot \mathbf{n})u = 0$ on Γ,*

$$(v, u \cdot \nabla u) = 2(v, u \cdot \omega(u)).$$

Proof. Assume first that u is in $C^\infty(\Omega)$. Then

$$\int_\Omega (u \cdot \omega(u)) \cdot v = \frac{1}{2} \int_\Omega (u \cdot \nabla u) \cdot v - \frac{1}{2} \int_\Omega (u \cdot (\nabla u)^T) \cdot v.$$

But,

$$(u \cdot (\nabla u)^T) \cdot v = u^i \partial_j u^i v^j = \frac{1}{2} \partial_j (u^i u^i) v^j = \frac{1}{2} v \cdot \nabla |u|^2$$

so

$$\int_\Omega (u \cdot (\nabla u)^T) \cdot v = -\frac{1}{2} \int_\Omega \operatorname{div} v \, |u|^2 + \frac{1}{2} \int_\Gamma (v \cdot \mathbf{n}) \, |u|^2 = 0,$$

which, with the density of $C^\infty(\Omega)$ in $H^1(\Omega)$, gives the desired identity. □

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