THE LINEARIZED 3D EULER EQUATIONS WITH INFLOW, OUTFLOW

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Abstract. In 1983, Antontsev, Kazhikhov, and Monakhov published a proof of the existence and uniqueness of solutions to the 3D Euler equations in which on certain inflow boundary components fluid is forced into the domain while on other outflow components fluid is drawn out of the domain. A key tool they used was the linearized Euler equations in vorticity form. We extend their result on the linearized problem to multiply connected domains and establish compatibility conditions on the initial data that allow higher regularity solutions.

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1. Introduction

Motivating this work and that of [11] is the goal of obtaining higher regularity solutions to the 3D Euler equations when fluid enters the domain through certain boundary components and exits through others—so-called inflow, outflow or injection, suction boundary conditions. Letting Ω be a bounded domain in $\mathbb{R}^3$ and defining the time-space domain,

$$Q := (0, T) \times \Omega \text{ for a fixed but arbitrary } T > 0,$$

we can write these equations in the form,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega, \\ \mathbf{u} \cdot \boldsymbol{n} = U^{\boldsymbol{n}} & \text{on } [0, T] \times \Gamma, \\ \mathbf{u}^{\boldsymbol{\tau}} = \mathbf{h} & \text{on } [0, T] \times \Gamma_+. \end{cases} \tag{1.1}$$

Here Γ is the boundary of Ω; Γ_+ is the portion of the boundary on which inflow occurs; $\boldsymbol{n}$ is the outward unit normal vector; $U^{\boldsymbol{n}} < 0$ and $\mathbf{h}$ are prescribed boundary values; $\mathbf{u}^{\boldsymbol{\tau}}$ is the tangential component of $\mathbf{u}$; $\mathbf{u}_0$ is the initial velocity; $\mathbf{f}$ is the external forcing.

All proofs of existence of solutions to the Euler equations use some kind of approximation, encoded as a sequence or as the fixed point of an operator. As the basis for one such approximation, we study the linear problem,

$$\begin{cases} \partial_t \mathbf{Y} + \mathbf{u}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{u} = \mathbf{g} & \text{in } Q, \\ \mathbf{Y} = \mathbf{H} & \text{on } [0,T]\times\Gamma_+, \\ \mathbf{Y}(0) = \mathbf{Y}_0 & \text{on } \Omega. \end{cases} \tag{1.2}$$

In (1.2), $\mathbf{u}$ is given on Q, as are the initial value $\mathbf{Y}_0$ on Ω and the value $\mathbf{H}$ of $\mathbf{Y}$ on the inflow boundary. Should it happen that $\mathbf{Y} = \operatorname{curl}\mathbf{u}$ and $\mathbf{H}$ is the value of the vorticity generated by the Euler equations on the inflow boundary, then $\boldsymbol{\omega} := \mathbf{Y}$ would be the vorticity for a solution to the Euler equations, and $(1.2)_1$ would become the vorticity equation,

$$\partial_t\boldsymbol{\omega} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = \mathbf{g} := \operatorname{curl}\mathbf{f}. \tag{1.3}$$

We see, then, that (1.2) is a linearization of the vorticity formulation for (1.1).

Employing $C^\alpha(Q)$ solutions to (1.2), well-posedeness of (1.1) on simply connected domains for $\boldsymbol{\omega} \in C^\alpha(Q)$, $\alpha\in(0,1)$, was obtained in Chapter 4 of [2]. Here, we obtain, for any $N \geqslant 0$, a $C^{N,\alpha}(Q)^3$ solution to (1.2) on a multiply connected domain, which we use in [11] to obtain a solution to (1.1) for vorticity in $C^{N,\alpha}(Q)^3$ for any $N\geqslant 0$ on a multiply connected domain. Both the linearized (1.2) and the nonlinear (1.1) require suitable compatibility conditions on the initial data to allow regularity of the solution.

Therefore, although (1.1) motivates our work, we restrict our attention to (1.2).

The key difficulty. If $\mathbf{g}\equiv 0$, $(1.2)_1$ shows that $\mathbf{Y}$ is the pushforward (transport with stretching) of $\mathbf{Y}_0$ (we explain this in detail in Section 4). The trajectories of the flow map for $\mathbf{u}$ play a central role. If we assume that $\mathbf{u}\cdot\boldsymbol{n} = 0$ on the boundary then this is nearly the complete story other than accounting for forcing, and (1.2) is solved in an entirely classical manner.

With inflow of fluid from the boundary, however, we must insure that the values of $\mathbf{Y} = \mathbf{H}$ coming from the inflow meet seamlessly enough with those coming from the initial data $\mathbf{Y}_0$ so that the desired regularity of the solution is obtained. This is the primary complication we face in solving (1.2).

Inflow, outflow. We assume that $\Gamma := \partial\Omega$ has at least C^2 regularity and has a finite number of components, with Γ_+, Γ_-, Γ_0 a partition of the boundary components into those on which inflow, outflow, no-penetration boundary conditions hold, respectively. That is, defining

$$U^{\boldsymbol{n}} := \mathbf{u}\cdot\boldsymbol{n},$$

we require that

$$U^{\boldsymbol{n}} < 0 \text{ on } [0,T]\times\Gamma_+, \qquad U^{\boldsymbol{n}} > 0 \text{ on } [0,T]\times\Gamma_-, \qquad U^{\boldsymbol{n}} = 0 \text{ on } [0,T]\times\Gamma_0. \tag{1.4}$$

(We allow $\Gamma_0 = \emptyset$ or $\Gamma_0 = \Gamma$—see Remark 7.2.) Moreover, $\operatorname{div}\mathbf{u} = 0$ imposes the constraint,

$$\int_{\Gamma_+} U^{\boldsymbol{n}} = -\int_{\Gamma_-} U^{\boldsymbol{n}}. \tag{1.5}$$

Throughout, we fix $\alpha\in(0,1)$.

Linear problem as a tool. With sufficient regularity we interpret (1.2) classically, but for our lowest regularity solutions $(1.2)_1$ must be treated weakly, as equality in the sense of distributions on Q. In all cases, we will construct and treat $\mathbf{Y}$ as a Lagrangian solution to (1.2), though to even define what we mean by such solutions will require the development of some technology because of the inflow of $\mathbf{H}$ from Γ_+ (this leads ultimately to Definition 5.4).

We can view (1.3) as a special case of (1.2), in which $\mathbf{Y} = \boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ and $\mathbf{H}$ is derived from the pressure, as done in [2, 11]. In (1.3), $\boldsymbol{\omega}$ is a curl and so, in particular, is divergence-free, whereas this is not assumed for $\mathbf{Y}$ in (1.2). We will show, however, that if $\operatorname{div} \mathbf{Y}_0 = 0$ and $\mathbf{H}$ satisfies the condition in (1.14), obtained formally be restricting $(1.2)_1$ to Γ_+, then $\operatorname{div} \mathbf{Y}(t)$ will be zero for $t \in [0, T]$. Moreover, we show that if $\mathbf{Y}_0$ is in the range of the curl then $\mathbf{Y}(t)$ remains in the range of the curl for $t \in [0, T]$, which requires additional work only because Γ has multiple components (unless, perhaps, $\Gamma = \Gamma_0$).

The analysis of (1.2) in [2] focused on $C^\alpha(Q)$ regularity. We are concerned with obtaining $C^{N,\alpha}(Q)$ regularity of solutions to both the linear (in this paper) and nonlinear (in [11]) problems for any integer $N \geqslant 0$. To accomplish this, we must discover the right compatibility conditions on the initial data and on $\mathbf{H}$. For $N = 0$ the conditions, obtained in [2], are simply that $\mathbf{H}(0) = \mathbf{Y}_0$ on Γ_+. We will show that these conditions have a natural generalization to all $N \geqslant 0$, most cleanly stated below in the necessary and sufficient conditions in (1.11).

Some function spaces. Let V be an open subset of $\mathbb{R}^d$, $d \geqslant 1$. We define the classical Hölder space $C^\alpha(V)$ to be all measurable real-valued functions on V for which

$$\|f\|_{C^\alpha(\Omega)} := \|f\|_{L^\infty(\Omega)} + \|f\|_{\dot{C}^\alpha(\Omega)} < \infty, \quad \|f\|_{\dot{C}^\alpha(V)} := \sup_{x \neq y \in V} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For any integer $N \geqslant 0$ we define the Banach space $C^{N,\alpha}(V)$ with the norm

$$\|f\|_{C^{N,\alpha}(V)} := \sum_{|\gamma| \leqslant N} \|D^\gamma f\|_{L^\infty} + \sum_{|\gamma| = N} \|D^\gamma f\|_{\dot{C}^\alpha(V)}.$$

We also allow f to be vector- or matrix-valued, but will not make a notational distinction.

For the time-space domain Q, we define

$$\mathring{C}^{N+1,\alpha}(Q) := \{\mathbf{v} \colon Q \to \mathbb{R} \colon \partial_t^j D_{\mathbf{x}}^\gamma \mathbf{v} \in C^\alpha(Q)^3, j + |\gamma| \leqslant N + 1, j \leqslant N\},$$

endowed with the natural norm based upon its regularity. That is, $\mathring{C}^{N+1,\alpha}(Q)$ is the same as $C^{N+1,\alpha}(Q)$, but with one less time than spatial derivative of regularity.

We call $\beta \in C^{N+1,\alpha}([0, T] \times \Gamma)$ a *proper inflow, outflow boundary value* if it satisfies the same conditions as $U^{\boldsymbol{n}}$ does in (1.4) and (1.5). We then define the affine spaces

$$\begin{aligned} C_{\sigma,\beta}^{N+1,\alpha}(Q) &:= \{\mathbf{v} \in C^{N+1,\alpha}(Q) \colon \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \boldsymbol{n} = \beta \text{ on } [0, T] \times \Gamma\}, \\ \mathring{C}_{\sigma,\beta}^{N+1,\alpha}(Q) &:= \{\mathbf{v} \in \mathring{C}^{N+1,\alpha}(Q) \colon \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \boldsymbol{n} = \beta \text{ on } [0, T] \times \Gamma\}. \end{aligned} \tag{1.6}$$

Primarily, we will utilize the following:

$$\begin{aligned} &C_\sigma^{N+1,\alpha}(\Omega) := \{\mathbf{v} \in C^{N+1,\alpha}(\Omega) \colon \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \boldsymbol{n} = U^{\boldsymbol{n}} \text{ on } \Gamma\}, \\ &C_\sigma^{N+1,\alpha}(Q) := C_{\sigma,U^{\boldsymbol{n}}}^{N+1,\alpha}(Q), \qquad \mathring{C}_\sigma^{N+1,\alpha}(Q) := \mathring{C}_{\sigma,U^{\boldsymbol{n}}}^{N+1,\alpha}(Q), \end{aligned} \tag{1.7}$$

where we suppose that $U^{\boldsymbol{n}}$ is at least as regular as $C^{N+1,\alpha}([0, T] \times \Gamma)$. Observe that only the normal component of $\mathbf{v}$ is specified on the boundary.

We will also use the classical space,

$$H := \{\mathbf{v} \in L^2(\Omega)^3 \colon \operatorname{div} \mathbf{v} = 0, \ \mathbf{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\} = H_0 \oplus H_c, \tag{1.8}$$

where the L^2-orthogonal subspaces H_c, H_0 of H are defined by

$$H_c := \{\mathbf{v} \in H : \operatorname{curl} \mathbf{v} = 0\}, \quad H_0 := H_c^\perp. \tag{1.9}$$

Regularity assumptions on the data. We specify the regularity of the initial data, boundary value $U^{\boldsymbol{n}}$, $\mathbf{H}$ on inflow, forcing $\mathbf{g}$, and velocity field $\mathbf{u}$, as follows:

Definition 1.1. *We say the data has regularity N for integer $N \geqslant 0$ if the following hold:*

- Γ *is* $C^{\max\{N+2,3\},\alpha}$, $U^{\boldsymbol{n}} \in C^{N+1,\alpha}([0,T] \times \Gamma)$;
- $\mathbf{g} \in C^\alpha(Q)$ *if* $N = 0$, $\mathbf{g} \in \mathring{C}^{N,\alpha}(Q)$ *if* $N \geqslant 1$;
- $\mathbf{Y}_0 \in C^{N,\alpha}(\Omega)$, $\mathbf{H} \in C^{N,\alpha}([0,T] \times \Gamma_+)$;
- $\mathbf{u} \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$.

In [2], $\mathbf{H}$ and $U^{\boldsymbol{n}}$ are assumed to have one more derivative of regularity than we have assumed here for data regularity $N = 0$. Higher regularity of $\mathbf{H}$ is required, as we will see in Theorem 1.3, to insure that $\mathbf{Y}(t)$ remains in the range of the curl if $\mathbf{Y}_0$ is in the range of the curl. Since [2] analyzes solutions to the Euler equations, such higher regularity is needed. Moreover, the need to properly control the pressure for the nonlinear problem, which is directly related to the production of vorticity on the boundary, requires higher regularity of both $\mathbf{H}$ and $U^{\boldsymbol{n}}$ in [2] (and in [11]). But that is not an issue for the linear problem we treat here.

For most of the analysis we make, Γ being $C^{N+2,\alpha}$ is sufficient. In the proof of Lemma 2.6, however, which we apply only for data regularity $N = 0$, we need one more derivative of regularity. See Remark 2.7.

Compatibility conditions. To obtain the regularity of solutions to (1.2), we need to impose compatibility conditions. We define the conditions for $N = 0$ and $N = 1$ as

$$\begin{aligned} &\mathrm{cond}_0: \quad \mathbf{H}(0) = \mathbf{Y}_0 \text{ on } \Gamma_+, \\ &\mathrm{cond}_1: \quad \mathrm{cond}_0 \text{ and } \partial_t \mathbf{H}|_{t=0} + \mathbf{u}_0 \cdot \nabla \mathbf{Y}_0 - \mathbf{Y}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{g}(0) = 0 \text{ on } \Gamma_+, \end{aligned} \tag{1.10}$$

where $\mathbf{u}_0 := \mathbf{u}(0)$. We can view cond_1 formally as saying that $(1.2)_1$ holds at time zero on Γ_+, where $\mathbf{Y}_0 = \mathbf{H}(0)$ by cond_0. Indeed, we could write cond_N for all $N \geqslant 1$ suggestively as

$$\mathrm{cond}_N: \quad \partial_t^j \mathbf{H}|_{t=0} = \partial_t^j \mathbf{Y}|_{t=0} \text{ on } \Gamma_+ \text{ for all } 0 \leqslant j \leqslant N, \tag{1.11}$$

where we replace $\partial_t \mathbf{Y}$ by the form it would have were it a solution to $(1.2)_1$. Or, spelled out just a little more,

$$\mathrm{cond}_N: \quad \mathrm{cond}_{N-1} \text{ and } \partial_t^N \mathbf{H}|_{t=0} + \partial_t^{N-1}[\mathbf{u} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{u} - \mathbf{g}]_{t=0} = 0 \text{ on } \Gamma_+.$$

For $N = 2$, for instance, we would first write,

$$\begin{aligned} 0 &= \partial_t^2 \mathbf{H}|_{t=0} + \partial_t[\mathbf{u} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{u} - \mathbf{g}]_{t=0} \\ &= \partial_t^2 \mathbf{H}(0) + \partial_t \mathbf{u}(0) \cdot \nabla \mathbf{Y}_0 + \mathbf{u}_0 \cdot \nabla \partial_t \mathbf{Y}(0) - \partial_t \mathbf{Y}(0) \cdot \nabla \mathbf{u}_0 - \mathbf{Y}_0 \cdot \nabla \partial_t \mathbf{u}(0) - \partial_t \mathbf{g}(0), \end{aligned}$$

then replace each $\partial_t \mathbf{Y}(0)$ with $\mathbf{g}(0) - \mathbf{u}_0 \cdot \nabla \mathbf{Y}_0 + \mathbf{Y}_0 \cdot \nabla \mathbf{u}_0$, the value it would have were $(1.2)_1$ to hold.

Types of solutions. We will be concerned with both Lagrangian solutions (Definition 5.4) and with Eulerian solutions, classical as well as weak, as in Definition 1.2. In this definition, $\mathbf{Y}$ has more than sufficient time and boundary regularity to avoid the need to enforce the initial and boundary conditions weakly. With sufficient regularity, $\operatorname{div}(\mathbf{Y} \otimes \mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{Y}$ (using $\operatorname{div} \mathbf{u} = 0$). Since we only assume $\mathbf{Y} \in C^\alpha(Q)$, $\operatorname{div}(\mathbf{Y} \otimes \mathbf{u})$ is defined in the sense of distributions, given that $\mathbf{Y} \otimes \mathbf{u}$ is an integrable function.

Definition 1.2. *We say that* $\mathbf{Y} \in C^\alpha(Q)$ *is a weak (Eulerian) solution to (1.2) if* $\mathbf{Y} = \mathbf{H}$ *on* $[0,T] \times \Gamma_+$, $\mathbf{Y}(0) = \mathbf{Y}_0$, *and* $\partial_t \mathbf{Y} + \operatorname{div}(\mathbf{Y} \otimes \mathbf{u}) - \mathbf{Y} \cdot \nabla \mathbf{u} = \mathbf{g}$ *in* $\mathcal{D}'(Q)$.

In short, we will find that the Lagrangian solution is a weak Eulerian solution if $\mathbf{Y}$ is in the range of the curl, and weak Eulerian solutions lying in the range of the curl are unique. We have a particular concern over the Lagrangian versus Eulerian solution because we use both formulations when we apply our results in [11]—specifically, when $\mathbf{Y}$ is the vorticity of some vector field. The vast majority of the estimates come from the Lagrangian formulation, but the velocity formulation, which is also needed, is recovered from the weak Eulerian formulation, not from the Lagrangian, so they need to be the same solution.

Main results. Our main results are Theorems 1.3 and 1.4.

Theorem 1.3. *Assume that the data has regularity* N *for some* $N \geqslant 0$ *and that* cond_N *holds. There exists a solution* $\mathbf{Y}$ *to (1.2) in* $C^{N,\alpha}(Q)$ *with*

$$\|\mathbf{Y}\|_{C^{k,\alpha}(Q)} \leqslant C(T)\left[\|\mathbf{u}\|_{\mathring{C}_\sigma^{k+1,\alpha}(Q)} + \|\mathbf{H}\|_{C^{k,\alpha}([0,T]\times\Gamma_+)} + \|\mathbf{g}\|_{C^{\max\{k-1,0\},\alpha}(Q)} T\right] \tag{1.12}$$

for all $0 \leqslant k \leqslant N$, *where* $C(T)$ *also depends upon* $\|\mathbf{Y}_0\|_{C^{k,\alpha}(\Omega)}$. *The solution* $\mathbf{Y}$ *is Lagrangian and weak Eulerian for all* $N \geqslant 0$. *For* $N \geqslant 1$, *the solution is also the unique classical Eulerian solution.*

Moreover, suppose that $\mathbf{Y}_0$ *and* $\mathbf{g}(t)$ *for all* t *are in the range of the curl,*

$$\mathbf{H} \in C^{\max\{N,1\},\alpha}([0,T] \times \Gamma_+), \tag{1.13}$$

and (our notation is defined in (1.16))

$$\partial_t H^{\boldsymbol{n}} + \operatorname{div}_\Gamma[H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} = 0 \text{ on } (0,T] \times \Gamma_+. \tag{1.14}$$

Then $\mathbf{Y}(t)$ *remains in the range of the curl for all* $t \in [0,T]$; *also, for* $N = 0$, $\mathbf{Y}$ *is the unique weak Eulerian solution.*

We make a few comments on Theorem 1.3:

- In (1.14), $\operatorname{div}_\Gamma$ is the divergence-operator along Γ_+ (see Section 7).
- We give the definition of a Lagrangian solution in Definition 5.4. It is not entirely classical because values of $\mathbf{H}$ are brought into the domain from the inflow boundary.
- Because of (1.13), both $\mathbf{g} \cdot \boldsymbol{n}$ and $\partial_t H^{\boldsymbol{n}}$ are in $C^\alpha([0,T] \times \Gamma_+)$ in (1.14). Hence, it is implicit in (1.14) that $\operatorname{div}_\Gamma[H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}]$ is in $C^\alpha([0,T] \times \Gamma_+)$.
- Restricted to $t = 0$, (1.14) is the normal component of cond_1, as we can see by examining the proof of Proposition 7.1.

We also have a velocity formulation of (1.2), as given in Theorem 1.4.

Theorem 1.4. *Assume that the data has regularity* N *for some* $N \geqslant 0$ *and that* cond_N, *(1.13), and (1.14) hold. Let* $\mathbf{f} \in \mathring{C}^{N+1,\alpha}(Q) \cap C([0,T]; H_0)$ *and set* $\mathbf{g} := \operatorname{curl} \mathbf{f}$. *Let* $\boldsymbol{\omega} = \mathbf{Y}$ *and* $\boldsymbol{\omega}_0 = \mathbf{Y}_0$. *There exists a unique* $\mathbf{v} \in C_\sigma^{N+1,\alpha}(Q)$ *with* $\operatorname{curl} \mathbf{v} = \boldsymbol{\omega}$ *and mean-zero pressure field* π *with* $\nabla \pi \in C^{N,\alpha}(Q)$ *for which*

$$\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{u} \cdot (\nabla \mathbf{v})^T + \nabla \pi = \mathbf{f}. \tag{1.15}$$

The harmonic component of $\mathbf{v}$ *is given explicitly in Lemma 8.1.*

Suppose, further, that $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$. *Then setting* $p = \pi - (1/2)|\mathbf{u}|^2$, $(\mathbf{u}, p)$ *satisfies* $(1.1)_{1-4}$ *(but not* $(1.1)_5$*) with an additional, harmonic forcing term. If, further,* $\mathbf{v} = \mathbf{u}$ *then this additional term does not appear and the solution is unique, even for* $N = 0$.

Remark 1.5. *Theorem 1.4 shows that in velocity formulation, the solution to the linearized Euler equations is unique, even for* $N = 0$*. The last paragraph of Theorem 1.4 considers what would happen were the linear solution to be a solution to the Euler equations; it does not establish the existence of such a solution—that is done in* [11]*. Also, because of how* $\mathbf{v}$ *is recovered from the vorticity,* $\mathbf{v}$ *and* $\mathbf{u}$ *have matching normal components on the boundary, but will not, in general, have matching tangential components.*

Prior work. The primary reference for our work is Chapter 4 of [2]. Although we express things differently, the material in Sections 3 to 5, 7, and 8 clearly bears the imprint of [2]. Our motivation in this work and [11] is to ultimately extend the results of Chapter 4 of [2], which are for simply connected domains and $N = 0$ regularity, to obtain solutions to the Euler equations with $N \geqslant 0$ regularity with suitable compatibility conditions.

Section 1.4 of [15] contains an extensive survey of results, both 2D and 3D, related to the problem we are studying here. Petcu [17] presents a version of the argument in Chapter 4 of [2] specialized to a 3D channel with inflow and outflow constant in space and time.

Boyer and Fabrie [4, 5] (in particular, see the final two chapters of [5]) treat an analogous setup to ours for transport without stretching, studying weak solutions. They do not restrict inflow and outflow to lie on full components of the boundary, which would seem to make classical solutions impossible to obtain. In a related vein, see also the recent works [6, 16]. We also mention the works [8, 9, 12] on different linear problems.

Finally, we note that the need for the higher regularity results of Theorems 1.3 and 1.4 is explicitly stated in [20, 10], where such results are used to obtain high-order expansions of solutions to the Navier-Stokes equation asymptotically in terms of the viscosity.

Organization of this paper. We start by developing some necessary tools. In Section 2 we describe how to recover a velocity in $C^{N+1,\alpha}_\sigma(\Omega)$ from a vorticity in $C^{N,\alpha}(\Omega)$ and introduce the concepts we need to treat multiply connected domains. In Section 3, we develop the properties of the flow map, which we will need throughout the rest of the paper.

We develop the idea of a pushforward of a velocity field with boundary conditions on inflow, a non-classical construct, in Section 4, using it in Section 5 to define what we mean by a Lagrangian solution. We give the core of the proof of Theorem 1.3 in Propositions 5.7 and 5.9, though we defer to Section 6 a key part of it, giving an equivalent form of cond_1: it is of a much different flavor than the rest of the proof and would only distract from it.

The proof that if $\mathbf{Y}_0$ lies in the range of the curl then $\mathbf{Y}(t)$ remains in that range for $t > 0$ is given in Section 7. The results in Sections 5 and 7, specifically Propositions 5.9 and 7.1, together yield Theorem 1.3. The proof of Theorem 1.4 is presented in Section 8. In the appendices, we list some standard Hölder space estimates and give details on the continuity of the Biot-Savart law, which we referenced in Section 2.

On notation. Our notation, while fairly standard, has a few subtleties. If M is a matrix, M^i_k refers to the entry in row i, column k of M; v^i refers to the i^{th} entry in the vector $\mathbf{v}$, which we always treat as a column vector for purposes of multiplication. If M and N are the same size matrices then $M \cdot N := M^i_k N^i_k$, where here, as always, we use implicit summation notation. If $\mathbf{u}$ and $\mathbf{v}$ are vectors then the matrix $\mathbf{u} \otimes \mathbf{v}$ has components $[\mathbf{u} \otimes \mathbf{v}]^i_k := u^i v^k$. We define the divergence of a matrix row-by-row, so $\operatorname{div} M$ is the column vector with components $[\operatorname{div} M]^i = \partial_k M^i_k$. Hence, $[\operatorname{div}[\mathbf{u} \otimes \mathbf{v}]]^i = \operatorname{div}[\mathbf{u} \otimes \mathbf{v}]^i = \partial_k(u^i v^k)$, where ∂_k is the derivative with respect to the k^{th} spatial variable. For any vector field $\mathbf{v}$ defined on Γ, we define its normal and tangential components,

$$v^{\boldsymbol{n}} := \mathbf{v} \cdot \boldsymbol{n}, \quad \mathbf{v}^{\boldsymbol{n}} := v^{\boldsymbol{n}} \boldsymbol{n}, \quad \mathbf{v}^{\boldsymbol{\tau}} := \mathbf{v} - \mathbf{v}^{\boldsymbol{n}}. \tag{1.16}$$

The operators ∇, div are the gradient, divergence with respect to the spatial variables only. For vector fields $\mathbf{u}$ and $\mathbf{v}$, we will interchangeably write $\mathbf{u} \cdot \nabla \mathbf{v}$ and $\nabla \mathbf{v}\, \mathbf{u}$, each of which is a vector whose i^{th} component is $u^k \partial_k v^i$. When applied to a function η that includes two time variables (as our flow maps will), we write $\partial_{t_1} \eta$, $\partial_{t_2} \eta$ to mean the derivative with respect to the first, second time variable. We will sometimes treat time-space as a four-dimensional variable, defining the operator

$$D := (\partial_t, \nabla), \tag{1.17}$$

noting that if $\mathbf{v}$ is a vector field then $D\mathbf{v}$ is a 3×4 matrix field.

Finally, $H^{k,p}(\Omega)$ and $H^s(\Omega)$ are the standard L^p- and L^2-based Sobolev spaces.

2. Recovering velocity from vorticity

In this section, we develop the operator in (2.2) that we will use to recover an appropriate divergence-free vector field from its vorticity (curl), with its regularity given in Lemma 2.4. Other than in the proof of one technical, but important, result that we will defer to Appendix B, we will need only a few facts regarding the Hodge decomposition for multiply connected domains, which we now summarize.

Assume that Ω is connected and that Γ is at least C^2-regular. Let $\Gamma_1, \dots, \Gamma_{b+1}$, be the $b+1$ components of Γ with Γ_{b+1} the boundary of the unbounded component of Ω^C. We define the *external flux* of $\boldsymbol{\omega}$ through Γ_i as

$$\Phi_i^\Gamma(\boldsymbol{\omega}) := \int_{\Gamma_i} \boldsymbol{\omega} \cdot \boldsymbol{n}. \tag{2.1}$$

As we discuss in Appendix B, the space H_0 consists of those elements of H having vanishing *internal* fluxes, a characterization first given by Helmholtz (see the historical comments in [7]). For our purposes, we do not need an explicit characterization of these spaces, only the definitions of the spaces H, H_c, and H_0 in (1.8) and (1.9) and the fact that H_c is finite-dimensional. Employing elliptic regularity theory, Lemma 2.1 easily follows:

Lemma 2.1. *Assume that Γ is $C^{n,\alpha}$-regular, $n \geqslant 2$, and let X be any function space that contains $C^{n,\alpha}(\Omega)^3$. For any $\mathbf{v} \in H$,*

$$\|P_{H_c}\mathbf{v}\|_X \leqslant C(X)\|\mathbf{v}\|_H$$

and if also $\mathbf{v} \in X$ then

$$\|\mathbf{v}\|_X \leqslant \|P_{H_0}\mathbf{v}\|_X + C(X)\|\mathbf{v}\|_H, \quad \|P_{H_0}\mathbf{v}\|_X \leqslant \|\mathbf{v}\|_X + C(X)\|\mathbf{v}\|_H.$$

For any $\mathbf{v} \in H = H_0 \oplus H_c$, we call $P_{H_c}\mathbf{v}$ the *harmonic component or part of* $\mathbf{v}$. (Note that $\Delta \mathbf{v}_c = 0$ for any $\mathbf{v}_c \in H_c$, though unless Ω is simply connected, there are also $\mathbf{v} \in H_0$ for which $\Delta \mathbf{v} = 0$.)

The following is a classical trace theorem (see Theorem 1.2 p. 7 of [21]):

Lemma 2.2. *There is a continuous trace from the space of L^2 vector fields on Ω having divergence in $L^2(\Omega)$ to $H^{-\frac{1}{2}}(\Gamma)$.*

We write $\operatorname{curl} H^1(\Omega)^3$ for the image of $H^1(\Omega)^3$ under the curl operator and say that a vector field is *in the range of the curl* if it lies in $\operatorname{curl} H^1(\Omega)^3$. We can characterize it as in Theorem 2.3.

Theorem 2.3. *We have,*

$$\operatorname{curl} H^1(\Omega)^3 = \operatorname{curl}(H_0 \cap H^1(\Omega)^3) = \{\boldsymbol{\omega} \in L^2(\Omega) \colon \operatorname{div} \boldsymbol{\omega} = 0, \Phi_i^\Gamma(\boldsymbol{\omega}) = 0 \text{ for all } i\}.$$

Moreover, there exists a continuous operator K from $\operatorname{curl} H^1(\Omega)^3 \subseteq L^2(\Omega)^3$ to $H_0 \cap H^1(\Omega)^3$ with the property that $\mathbf{u} = K[\boldsymbol{\omega}]$ is the unique element of $H_0 \cap H^1(\Omega)^3$ for which $\operatorname{curl} \mathbf{u} = \boldsymbol{\omega}$.

Proof. This follows from Theorems 3.5 and 3.12 of [1]. Observe that Lemma 2.2 allows the external flux $\Phi_i^\Gamma(\boldsymbol{\omega})$ to be defined. □

To recover a velocity field $\mathbf{u}$ satisfying $\mathbf{u} \cdot \boldsymbol{n} = U^{\boldsymbol{n}}$, we let $\boldsymbol{\mathcal{V}} = \nabla\varphi$, where φ is the unique mean-zero solution to

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega, \\ \nabla\varphi \cdot \boldsymbol{n} = U^{\boldsymbol{n}} & \text{on } \Gamma. \end{cases}$$

Observe that $\operatorname{div} \boldsymbol{\mathcal{V}} = 0$, $\operatorname{curl} \boldsymbol{\mathcal{V}} = 0$, and $\boldsymbol{\mathcal{V}} \cdot \boldsymbol{n} = U^{\boldsymbol{n}}$ on Γ. (Note that if $U^{\boldsymbol{n}} = 0$ then $\boldsymbol{\mathcal{V}} \equiv 0$.) Then we define

$$K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}] := K[\boldsymbol{\omega}] + \boldsymbol{\mathcal{V}}. \tag{2.2}$$

Define the solution space for vorticity,

$$V_\sigma^{N,\alpha}(Q) := \{\boldsymbol{\omega} \in C^{N,\alpha}(Q)^3 \colon \boldsymbol{\omega}(t) \in \operatorname{curl} H^1(\Omega)^3 \text{ for all } t \in [0, T]\}.$$

From Theorem 2.3, $\boldsymbol{\omega} \in V_\sigma^{N,\alpha}(Q)$ is equivalent to $\boldsymbol{\omega} \in C^{N,\alpha}(Q)$ lying in the range of the curl.

Lemma 2.4. *Assume the data has regularity $N \geqslant 0$. Then $K_{U^{\boldsymbol{n}}}$ maps $C^{N,\alpha}(\Omega) \cap \operatorname{curl} H^1(\Omega)^3$ continuously onto $\mathring{C}_\sigma^{N+1,\alpha}(Q) \cap (H_0 + \boldsymbol{\mathcal{V}}(t))$ and maps $V_\sigma^{N,\alpha}(Q)$ continuously onto $\mathring{C}_\sigma^{N+1,\alpha}(Q) \cap C([0, T]; H_0 + \boldsymbol{\mathcal{V}}(t))$.*

Proof. Follows from (2.2), the regularity of $\boldsymbol{\mathcal{V}}$, Lemma 2.1, and Corollary B.2. □

We also have the Helmholtz decomposition, which we use to establish the continuity of the Leray projector. We give its proof, since the continuity of the decomposition in Hölder spaces, especially for $N = 0$, is not particularly accessible in the literature.

Lemma 2.5. *Assume Γ is $C^{N+1,\alpha}$ for $N \geqslant 0$. Given any $\boldsymbol{\omega} \in C^{N,\alpha}(\Omega)^3$, we have $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} + \nabla q$ for some unique $\mathbf{v} \in H_0 \cap C^{N+1,\alpha}(\Omega)^3$ and mean-zero $q \in C^{N+1,\alpha}(\Omega)$, the maps from $\boldsymbol{\omega}$ to $\mathbf{v}$ and $\boldsymbol{\omega}$ to q being continuous in their respective spaces. Moreover, the Leray projector, P_H, is continuous as a map from $C^{N,\alpha}(\Omega)^3$ to $C^{N,\alpha}(\Omega)^3 \cap H$.*

Proof. Let $\boldsymbol{\omega} = P_H\boldsymbol{\omega} + \nabla q$ be the Leray-Helmholtz decomposition of $\boldsymbol{\omega}$. By Lemma 2.4 (applied using $\boldsymbol{\mathcal{V}} \equiv 0$), $P_H\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ for some unique $\mathbf{v} \in H_0 \cap C^{N+1,\alpha}(\Omega)^3$, with $\boldsymbol{\omega} \mapsto \mathbf{v}$ being a continuous map. Hence, $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} + \nabla q$, and the maps, $\boldsymbol{\omega} \mapsto \mathbf{v}$, $\boldsymbol{\omega} \mapsto q$, and $\boldsymbol{\omega} \mapsto P_H\boldsymbol{\omega}$ have the stated continuity. □

Lemma 2.6. *Assume that the data has $N = 0$ regularity. There exists a sequence $(U_i^{\boldsymbol{n}})$ of proper inflow, outflow boundary values and a sequence $(\mathbf{u}_i)$ in $C^{2,\alpha}_{\sigma,U_i^{\boldsymbol{n}}}(Q)$ converging to $\mathbf{u}$ in $\mathring{C}^{1,\alpha}(Q)$. (The space $C^{2,\alpha}_{\sigma,U_i^{\boldsymbol{n}}}(Q)$ is defined in (1.6).)*

Proof. Let $(U_i^{\boldsymbol{n}})$ be a sequence of vector fields mollified along Γ so that $U_i^{\boldsymbol{n}} \in C^{2,\alpha}([0, T] \times \Gamma)$ with $U_i^{\boldsymbol{n}} \to U^{\boldsymbol{n}}$ in $C^{1,\alpha}([0, T] \times \Gamma)$. This is possible, since we assumed that Γ is $C^{2,\alpha}$. Then for all sufficiently large i, $U_i^{\boldsymbol{n}}$ will satisfy the same conditions as $U^{\boldsymbol{n}}$ does in (1.4) and, after possibly adjusting the value of $U_i^{\boldsymbol{n}}$ on one boundary component by adding a constant value c_i to it, with $c_i \to 0$, each $U_i^{\boldsymbol{n}}$ will also satisfy (1.5). Let $\boldsymbol{\mathcal{V}}_i = \nabla \varphi_i$, where φ_i solves

$$\begin{cases} \Delta \varphi_i = 0 & \text{in } \Omega, \\ \nabla \varphi_i \cdot \boldsymbol{n} = U_i^{\boldsymbol{n}} & \text{on } \Gamma. \end{cases}$$

Then $\boldsymbol{\mathcal{V}}_i \in C^{2,\alpha}(Q)$ with $\boldsymbol{\mathcal{V}}_i \to \boldsymbol{\mathcal{V}}$ in $C^{1,\alpha}(Q)$ by elliptic regularity theory.

We have $\mathbf{w} := \mathbf{u} - \boldsymbol{\mathcal{V}} \in \mathring{C}^{1,\alpha}_{\sigma,0}(Q)$, which is the space $\mathring{C}^{1,\alpha}_{\sigma}(Q)$, but with $\mathbf{w} \cdot \boldsymbol{n} = 0$ on Γ. Extend $\mathbf{w}$ to $\mathring{C}^{1,\alpha}(\mathbb{R} \times \mathbb{R}^3)$ using an extension operator like that in Theorem 5', chapter VI of [18]. Mollify $\mathbf{w}$ in time and space and apply the Leray projector P_H, giving, via Lemma 2.5 a sequence $\mathbf{w}_i \in C^{2,\alpha}_{\sigma,0}(Q)$ with $\mathbf{w}_i \to \mathbf{w}$ in $\mathring{C}^{1,\alpha}_{\sigma,0}(Q)$.

Finally, let $\mathbf{u}_i = \mathbf{w}_i + \boldsymbol{\mathcal{V}}_i$. □

Remark 2.7. *In the proof of Lemma 2.6, we applied Lemma 2.5 with $N = 2$, which required that Γ (through Corollary B.2) be $C^{3,\alpha}$. This is the only place in this paper in which we required higher than a $C^{2,\alpha}$ boundary for data regularity 0.*

3. The flow map

In this section, we assume that for some $N \geqslant 0$ and fixed $T > 0$, $\mathbf{u} \in \mathring{C}^{N+1,\alpha}_{\sigma}(Q)$. We also assume that Γ is $C^{N+2,\alpha}$-regular. We will obtain estimates related to the flow map for $\mathbf{u}$.

For convenience, we first extend $\mathbf{u}$ to be defined on all of $\mathbb{R} \times \mathbb{R}^3$ using an extension operator like that in Theorem 5', chapter VI of [18]. This extension need not be divergence-free. This extension will allow us to use classical results on flow maps without undue concern as to their domain and codomain, though it is only the value of $\mathbf{u}$ on $\overline{Q}$ that ultimately concerns us.

Define $\eta \colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ to be the unique flow map for $\mathbf{u}$, so that

$$\begin{aligned} &\partial_{t_2} \eta(t_1, t_2; \mathbf{x}) = \mathbf{u}(t_2, \eta(t_1, t_2; \mathbf{x})), \quad \eta(t_1, t_1; \mathbf{x}) = \mathbf{x}; \\ &\eta(t_1, t_2; \mathbf{x}) = \mathbf{x} + \int_{t_1}^{t_2} \mathbf{u}(s, \eta(t_1, s; \mathbf{x}))\, ds. \end{aligned} \tag{3.1}$$

That is, $\eta(t_1, t_2; \mathbf{x})$ is the position that a particle starting at time t_1 at position $\mathbf{x} \in \mathbb{R}^3$ will be at time t_2 as it moves in the velocity field $\mathbf{u}$. We allow t_2 to be greater than, equal to, or less than t_1, accounting for movement forward and backward in time. (The properties of flow lines within Ω, which are all we ultimately care about, do not depend upon the specific extension of $\mathbf{u}$ we employ.) We also have

$$\begin{aligned} \partial_{t_1} \eta(t_1, t_2; \mathbf{x}) &= -\mathbf{u}(t_1, \eta(t_1, t_1; \mathbf{x})) + \int_{t_1}^{t_2} \partial_{t_1} \mathbf{u}(s, \eta(t_1, s; \mathbf{x}))\, ds \\ &= -\mathbf{u}(t_1, \mathbf{x}) + \int_{t_1}^{t_2} \nabla \mathbf{u}(s, \eta(t_1, s; \mathbf{x})) \partial_{t_1} \eta(t_1, s; \mathbf{x})\, ds. \end{aligned} \tag{3.2}$$

Moreover, by the very definition of the flow map (and its uniqueness),

$$\eta(t_2, t_3; \eta(t_1, t_2; \mathbf{x})) = \eta(t_1, t_3; \mathbf{x}). \tag{3.3}$$

Lemma 3.1 shows that for fixed t_2 and $\mathbf{x}$, η is, in a sense, transported by itself.

Lemma 3.1. *We have,*

$$\partial_{t_1} \eta(t_1, t_2; \mathbf{x}) + \mathbf{u}(t_1, \mathbf{x}) \cdot \nabla \eta(t_1, t_2; \mathbf{x}) = 0.$$

Proof. From (3.3), $\mathbf{y} = \eta(t_1, t_2; \eta(t_2, t_1; \mathbf{y}))$; taking d/dt_1 of both sides of this identity,

$$\begin{aligned}0 &= \frac{d}{dt_1}\eta(t_1, t_2; \eta(t_2, t_1; \mathbf{y}))\\ &= \partial_1\eta(t_1, t_2; \eta(t_2, t_1; \mathbf{y})) + \nabla\eta(t_1, t_2; \eta(t_2, t_1; \mathbf{y}))\partial_{t_1}\eta(t_2, t_1; \mathbf{y})\\ &= \partial_1\eta(t_1, t_2; \eta(t_2, t_1; \mathbf{y})) + \mathbf{u}(t_1, \eta(t_2, t_1; \mathbf{y})) \cdot \nabla\eta(t_1, t_2; \eta(t_2, t_1; \mathbf{y})).\end{aligned}$$

Setting $\mathbf{x} = \eta(t_2, t_1; \mathbf{y})$ gives the result. □

For any $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3$ let

- $\boldsymbol{\gamma}(t, \mathbf{x})$ be the point on Γ_+ at which the flow line through $(t, \mathbf{x})$ intersects with Γ_+;
- let $\tau(t, \mathbf{x})$ be the time at which that intersection occurs.

For all $\mathbf{x} \in \Omega$, $\tau(t, \mathbf{x}) \leqslant t$. Because we extended $\mathbf{u}$, τ will always be defined, as long as we allow $\tau(t, \mathbf{x}) = -\infty$, but $\boldsymbol{\gamma}(t, \mathbf{x})$ is defined only when $\tau(t, \mathbf{x})$ is finite, and is meaningful only when $\tau(t, \mathbf{x}) \geqslant 0$.

We define the hypersurface,

$$S := \{(t, \mathbf{x}) \in \overline{Q} \colon \tau(t, \mathbf{x}) = 0\}$$

and the open sets $U_\pm \subset Q$,

$$\begin{aligned}U_- &:= \{(t, \mathbf{x}) \in (0, T) \times \Omega \colon \tau(t, \mathbf{x}) < 0\},\\ U_+ &:= \{(t, \mathbf{x}) \in (0, T) \times \Omega \colon \tau(t, \mathbf{x}) > 0\}.\end{aligned}$$

For $t \in [0, T]$, we also define the sections, $S(t)$ and $U_\pm(t)$:

$$S(t) := \{\mathbf{x} \colon (t, \mathbf{x}) \in S\}, \quad U_\pm(t) := \{\mathbf{x} \colon (t, \mathbf{x}) \in U_\pm\}.$$

This is illustrated in 2D in Figure 3.1.

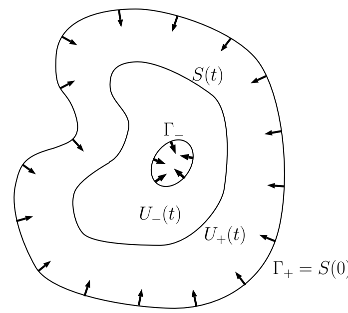

Figure 3.1

The hypersurface S consists of all points $(t, \mathbf{x})$ whose flow lines originated on Γ_+ at time zero; on U_- these flow lines originated in Ω at time zero; on U_+ the flow lines originated on Γ_+ at positive time. Observe that $\boldsymbol{\gamma}(t, \mathbf{x})$ is only meaningful on $\overline{U}_+$.

By virtue of (3.3), we have,

$$\begin{aligned}\eta(\tau(t, \mathbf{x}), t; \boldsymbol{\gamma}(t, \mathbf{x})) &= \mathbf{x},\\ \eta(t, \tau(t, \mathbf{x}); \mathbf{x}) &= \boldsymbol{\gamma}(t, \mathbf{x}).\end{aligned} \tag{3.4}$$

Also note that $\tau(t, \mathbf{x}) = t$ and $\boldsymbol{\gamma}(t, \mathbf{x}) = \mathbf{x}$ when $\mathbf{x} \in \Gamma_+$.

Remark 3.2. *We will often drop the $(t, \mathbf{x})$ arguments on τ and $\boldsymbol{\gamma}$ for brevity.*

The regularity of η given in Lemma 3.3 follows from entirely classical arguments, so we omit its proof. Note that η has one more derivative of time regularity (in both time variables) than $\mathbf{u}$, making up for the loss of time regularity of $\mathring{C}^{N+1,\alpha}_\sigma(Q)$ from that of $C^{N+1,\alpha}_\sigma(Q)$.

Lemma 3.3. *The flow map $\eta \in C^{N+1,\alpha}([0, T]^2 \times \mathbb{R}^3)$.*

In Lemma 3.4, we show that the hypersurface S has the regularity of the velocity field, and give conditions for when a function, regular on $U_\pm$ separately, can be glued together to obtain a regular function on all of $[0, T] \times \Omega$.

Lemma 3.4. *The set S is $C^{N+1,\alpha}$ as a hypersurface in $[0, T] \times \mathbb{R}^3$. There is a $T^* > 0$ depending only upon $\|\mathbf{u}\|_{C^1(Q)}$ for which $\partial_t\tau > 0$ on $\overline{U}_+$, while $\mathbf{u}(t)$ remains transversal to $S(t)$ and $S(t) \subseteq \Omega$ for all $t \in [0, T^*]$. If $g \in C^{k,\alpha}(\overline{U}_-)$ and $g \in C^{k,\alpha}(\overline{U}_+ \setminus \{0\} \times \Gamma_+)$ for some $k \leqslant N+1$ with $D^\beta g$ continuous on S for all $|\beta| = k$ then $g \in C^{k,\alpha}(Q)$ with a norm no larger than the larger of its $C^{k,\alpha}$ norms on $U_\pm$.*

Proof. To simplify notation, we will make the argument as though Γ_+ has one component, the result for multiple components following simply by summing the estimates over each component.

Because Γ_+ is a $C^{N+1,\alpha}$ surface, we can write $\Gamma_+ := \varphi_0^{-1}(0)$ for some $\varphi_0 \in C^{N+1,\alpha}(\mathbb{R}^3)$, which we can choose so that $\nabla \varphi_0|_{\varphi_0^{-1}(0)} = \boldsymbol{n}$ on Γ_+. Letting $\varphi_t(\mathbf{x}) = \varphi_0(\eta(t, 0; \mathbf{x}))$, we have $S(t) = \varphi_t^{-1}(0)$. Then by Lemma A.2,

$$\|\varphi_t\|_{C^{N,\alpha}(\Omega)} \leqslant \|\varphi_0\|_{C^{N,\alpha}(\mathbb{R}^3)} \left[1 + \|\eta(t, 0; \mathbf{x})\|_{C^{N+1}(\Omega)}\right]^{N+1}.$$

Since $S(t)$ is the surface Γ_+ transported to time t by $\eta(0, \cdot; \cdot) \in C^{N+1,\alpha}([0, T] \times \mathbb{R}^3)$, this shows that $S(t)$ is $C^{N+1,\alpha}$ as a surface in Ω.

Now let $\varphi(t, \mathbf{x}) := \varphi_t(\mathbf{x})$, so $\varphi \colon [0, T] \times \Omega \to \mathbb{R}$. Then $S = \varphi^{-1}(0)$ and we see, using also the regularity of η from Lemma 3.3, that S is $C^{N+1,\alpha}$.

Since φ is transported by the flow, we have $\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = 0$. Since $\nabla \varphi$ is normal to the surface $S(t)$, $\mathbf{u}(t)$ remains transversal to $S(t)$ as long as $\partial_t \varphi \neq 0$. But,

$$|\partial_t \varphi|_{t=0}| = |\mathbf{u}_0 \cdot \nabla \varphi_0|_{\varphi_0^{-1}(0)}| = |\mathbf{u}_0 \cdot \boldsymbol{n}| \geqslant U_{min} := \min_{\Gamma_+} |U^{\boldsymbol{n}}| > 0$$

on Γ_+, and the regularity of $\mathbf{u}$ then assures that $|\partial_t \varphi(t)| > 0$, at least up to some finite time $T^* > 0$; hence, $\mathbf{u}(t)$ remains transversal to $S(t)$ and $\partial_t \tau > 0$ on $\overline{U}_+$ for all $t \in [0, T^*]$. If necessary, we can always decrease T^* so that $S(t) \subseteq \Omega$ for all $t \in [0, T^*]$.

For the regularity of g, let $h := D^\beta g$ for any $|\beta| = k$. Then $h \in C^\alpha(U_\pm)$ and is continuous on S. Hence, a simple application of the triangle inequality shows that we can glue $g|_{U_\pm}$ together along S to obtain $h \in C^\alpha(Q)$, giving $g \in C^{k,\alpha}(Q)$.

More explicitly, let $\mathbf{y}_\pm := (t_\pm, \mathbf{x}_\pm) \in \overline{U}_\pm$, and let $\mathbf{y} := (t, \mathbf{x})$ be any point in S along the line segment from $\mathbf{y}_-$ to $\mathbf{y}_+$. Let $h = D^\beta g$, which we note is defined everywhere on S. Then

$$\begin{aligned}
\frac{|h(\mathbf{y}_-) - h(\mathbf{y}_+)|}{|\mathbf{y}_- - \mathbf{y}_+|^\alpha} &= \frac{|h(\mathbf{y}_-) - h(\mathbf{y}) + h(\mathbf{y}) - h(\mathbf{y}_+)|}{|\mathbf{y}_- - \mathbf{y}_+|^\alpha} \\
&\leqslant \frac{|h(\mathbf{y}_-) - h(\mathbf{y})|}{|\mathbf{y}_- - \mathbf{y}_+|^\alpha} + \frac{|h(\mathbf{y}) - h(\mathbf{y}_+)|}{|\mathbf{y}_- - \mathbf{y}_+|^\alpha} \\
&= \lambda^\alpha \frac{|h(\mathbf{y}_-) - h(\mathbf{y})|}{|\mathbf{y}_- - \mathbf{y}|^\alpha} + (1 - \lambda)^\alpha \frac{|h(\mathbf{y}) - h(\mathbf{y}_+)|}{|\mathbf{y} - \mathbf{y}_+|^\alpha} \\
&\leqslant \lambda^\alpha \|h\|_{\dot{C}^\alpha(U_-)} + (1 - \lambda)^\alpha \|h\|_{\dot{C}^\alpha(U_+)} \leqslant \max\{\|h\|_{\dot{C}^\alpha(U_-)}, \|h\|_{\dot{C}^\alpha(U_+)}\},
\end{aligned}$$

where

$$\lambda = \frac{|\mathbf{y}_- - \mathbf{y}|}{|\mathbf{y}_- - \mathbf{y}_+|} \implies 1 - \lambda = \frac{|\mathbf{y}_+ - \mathbf{y}|}{|\mathbf{y}_- - \mathbf{y}_+|},$$

where the equality for $1-\lambda$ holds because we chose $\mathbf{y}$ to be colinear with $\mathbf{y}_\pm$. Similar estimates hold for all the norms. □

Lemma 3.5. *Both τ and $\boldsymbol{\gamma}$ are transported by the flow map for $\mathbf{u}$; that is,*

$$\partial_t \tau + \mathbf{u} \cdot \nabla \tau = 0, \quad \partial_t \boldsymbol{\gamma} + \mathbf{u} \cdot \nabla \boldsymbol{\gamma} = 0, \tag{3.5}$$

and we have the identities (recall the definition of the operator D in (1.17)),

$$\begin{aligned} \partial_t \tau &= -\frac{\partial_{t_1}\eta(t,\tau;\mathbf{x})\cdot \boldsymbol{n}(\boldsymbol{\gamma})}{U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma})}, \qquad \nabla\tau = -\frac{(\nabla\eta(t,\tau;\mathbf{x}))^T\boldsymbol{n}(\boldsymbol{\gamma})}{U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma})}, \\ D\tau &= -\frac{(D\eta(t,t_2;\mathbf{x})|_{t_2=\tau})^T\boldsymbol{n}(\boldsymbol{\gamma})}{U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma})} \end{aligned} \tag{3.6}$$

and

$$D\boldsymbol{\gamma} = D\eta(t,t_2;\mathbf{x})|_{t_2=\tau} + \mathbf{u}(\tau,\boldsymbol{\gamma})\otimes D\tau. \tag{3.7}$$

Moreover, τ, $\boldsymbol{\gamma}$ lie in $C^{N+1,\alpha}(\overline{U}_+\setminus\{0\}\times\Gamma_+)$.

Proof. The quanitites τ and $\boldsymbol{\gamma}$ are transported by the flow map for $\mathbf{u}$ as in (3.5) because they are constant along flow lines by their definition.

Taking the spatial derivatives, $\partial_{\mathbf{x}_\ell}$, of both sides of $(3.4)_2$ gives (recall Remark 3.2)

$$\partial_{x_\ell}\eta^k(t,\tau;\mathbf{x}) = \partial_{t_2}\eta^k(t,\tau;\mathbf{x})\partial_{x_\ell}\tau + \partial_{x_\ell}\eta^k(t,\tau;\mathbf{x}) = \partial_{x_\ell}\gamma^k,$$

or, using $(3.1)_1$,

$$\partial_{t_2}\eta(t,\tau;\mathbf{x})\otimes\nabla\tau + \nabla\eta(t,\tau;\mathbf{x}) = \mathbf{u}(\tau,\boldsymbol{\gamma})\otimes\nabla\tau + \nabla\eta(t,\tau;\mathbf{x}) = \nabla\boldsymbol{\gamma}. \tag{3.8}$$

Taking the time derivative of both sides of $(3.4)_2$ gives

$$\partial_{t_1}\eta(t,\tau;\mathbf{x}) + \partial_t\tau\partial_{t_2}\eta(t,\tau;\mathbf{x}) = \partial_{t_1}\eta(t,\tau;\mathbf{x}) + \partial_t\tau\,\mathbf{u}(\tau,\boldsymbol{\gamma}) = \partial_t\boldsymbol{\gamma}. \tag{3.9}$$

We can write (3.8) and (3.9) together in the form (3.7). Now, $(D\boldsymbol{\gamma})^T\boldsymbol{n} = 0$ because $\boldsymbol{\gamma}$ always lies on the boundary component Γ_+. Thus (recall the comment following (1.17)),

$$0 = (D\eta(t,t_2;\mathbf{x})|_{t_2=\tau})^T\boldsymbol{n} + (D\tau\otimes\mathbf{u}(\tau,\boldsymbol{\gamma}))\boldsymbol{n} = (D\eta(t,t_2;\mathbf{x})|_{t_2=\tau})^T\boldsymbol{n} + U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma}))D\tau.$$

(Note that $D\eta$ is a 4×3 matrix field.) From this, each of the expressions in (3.6) follow.

Then (3.6) gives the regularity of τ, and (3.7) yields the regularity of $\boldsymbol{\gamma}$. □

Lemma 3.6. *We have,*

$$\partial_{t_2}\nabla\eta(t_1,t_2;\mathbf{z}) = \nabla\mathbf{u}(t_2,\eta(t_1,t_2;\mathbf{z}))\nabla\eta(t_1,t_2;\mathbf{z}) \text{ on } Q.$$

Proof. Using (3.1),

$$\begin{aligned} \partial_{t_2}\nabla\eta(t_1,t_2;\mathbf{z}) &= \partial_{t_2}\nabla_{\mathbf{z}}\eta(t_1,t_2;\mathbf{z}) = \nabla_{\mathbf{z}}\partial_{t_2}\eta(t_1,t_2;\mathbf{z}) = \nabla_{\mathbf{z}}\mathbf{u}(t_2,\eta(t_1,t_2;\mathbf{z})) \\ &= \nabla\mathbf{u}(t_2,\eta(t_1,t_2;\mathbf{z}))\nabla\eta(t_1,t_2;\mathbf{z}). \end{aligned}$$

□

4. The pushforward

In this section we describe how to extend the classical idea of the pushforward of the vorticity as a vector field to incorporate the generation of vorticity on the boundary. We start with a very brief overview of transport and the pushforward, specifically in flat space, but paying attention at the very beginning to issues of regularity. We then explain how we extend the pushforward to incorporate vorticity generation on the boundary, which we will use in Section 5 to produce a solution to (1.2).

Our focus is on analysis, the regularity of our operations, rather than their geometric meaning. For a very readable treatment of what the pushforward means geometrically in the context of fluid mechanics we refer the reader to Sections 2.2 and 3.1 of [3].

Transport. Before turning to the pushforward of a vector field, we review some basic facts regarding scalar transport—the pushforward of a scalar field. We define the transport operator applied to a scalar field $f(t, \mathbf{x})$ by

$$L_t f := \partial_t f + \nabla_{\mathbf{u}} f, \tag{4.1}$$

where $\nabla_{\mathbf{u}}$ is the directional derivative with respect to $\mathbf{u}$. When f has sufficient regularity, $\nabla_{\mathbf{u}} f = \mathbf{u} \cdot \nabla f$. As long as f is, say, continuous, we can always write, for any fixed $s \in \mathbb{R}$,

$$L_t(f(t, \eta(s,t;\mathbf{x}))) = \frac{d}{dt} f(t, \eta(s,t;\mathbf{x})). \tag{4.2}$$

We will also apply the transport operator to a vector field, component-by-component.

Pushforward. The *pushforward* of a vector field $\mathbf{X}_s$ by η from time s to time $t \in [0,T]$ is

$$(\eta(s,t)_* \mathbf{X}_s)(t, \mathbf{x}) := \mathbf{X}_s(\eta(t,s;\mathbf{x})) \cdot \nabla \eta(s,t;\eta(t,s;\mathbf{x})). \tag{4.3}$$

This presupposes that we stay within the domain of η and $\mathbf{X}$. We define the associated *pushforward operator* L by

$$L\mathbf{X} := L_t \mathbf{X} - \mathbf{X} \cdot \nabla \mathbf{u}.$$

Lemma 4.1. *For a vector field* $\mathbf{X} \in C^{N,\alpha}(Q)$ *for any* $N \geqslant 0$, $L\mathbf{X} = 0$.

Proof. Holding s and $\mathbf{x}$ fixed while applying (4.2) to each component of $\mathbf{X}(t,x) := (\eta(s,t)_* \mathbf{X}_s)(t,\mathbf{x})$ and appealing to Lemma 3.6, we have

$$\begin{aligned}
L_t(\mathbf{X}(t, \eta(s,t;\mathbf{x})) &= \frac{d}{dt} \mathbf{X}(t, \eta(s,t;\mathbf{x})) \\
&= \frac{d}{dt} \left[\mathbf{X}_s(\eta(t,s;\eta(s,t;\mathbf{x}))) \cdot \nabla \eta(s,t;\eta(t,s;\eta(s,t;\mathbf{x}))) \right] \\
&= \frac{d}{dt} \left[\mathbf{X}_s(\mathbf{x}) \cdot \nabla \eta(s,t;\mathbf{x}) \right] = \mathbf{X}_s(\mathbf{x}) \cdot \frac{d}{dt} \nabla \eta(s,t;\mathbf{x}) \\
&= \mathbf{X}_s(\mathbf{x}) \cdot (\nabla \mathbf{u}(t, \eta(s,t;\mathbf{x})) \nabla \eta(s,t;\mathbf{x})) \\
&= (\mathbf{X}_s(\mathbf{x}) \cdot \nabla \eta(s,t;\mathbf{x})) \cdot \nabla \mathbf{u}(t, \eta(s,t;\mathbf{x})) = (\mathbf{X} \cdot \nabla \mathbf{u})(t, \eta(s,t;\mathbf{x})).
\end{aligned}$$

□

In geometric language, Lemma 4.1 tells us that the pushforward of a velocity field is Lie-transported (Lie-advected) by the flow (as in (3.9) of [3]).

If $N \geqslant 1$, we can write the conclusion of Lemma 4.1 applied to a potential solution $\mathbf{Y}$ as

$$\partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} = \mathbf{Y} \cdot \nabla \mathbf{u},$$

and we see the connection with (1.2). The limitation, of course, is that this form of the pushforward does not account for inflow from the boundary (nor forcing, which we will consider later).

But if we let $\mathbf{Y}(t) = \eta(0,t)_* \mathbf{Y}_0$, we can see that the value of $\mathbf{Y}_0$ completely determines the value of $\mathbf{Y}$ on U_-. So other than the domain of $\mathbf{Y}$ shrinking in time as the flow sweeps Γ_+ through Ω (see Figure 3.1), the pushforward applies in an essentially classical way on U_-. Indeed, Lemma 4.1, which is easily localized to U_-, gives that $L\mathbf{Y} = 0$ on U_-.

Inflow from the boundary. By contrast, on U_+, we are assigned the value of $\mathbf{Y} = \mathbf{H}$ only on Γ_+, so that the values of $\mathbf{Y}$ on U_+ are entirely populated by the values of $\mathbf{H}$ on $[0, T] \times \Gamma_+$. Yet we wish the expression for the pushforward in (4.3) to continue to hold on U_+, as it yields $(1.2)_1$, but in the end we must connect the value of $\mathbf{Y}(t, \mathbf{x})$ on U_+ to the time and place its value originated from on Γ_+. This is the main purpose of the functions τ and $\boldsymbol{\gamma}$ that we defined in Section 3, and leads to the following definition:

Definition 4.2. *Let $\mathbf{X}_0 \in C^\alpha(\Omega)$ and $\mathbf{H} \in C^\alpha([0, T] \times \Gamma_+)$. Define the pushforward of $\mathbf{X}_0$ by η on Q with boundary value $\mathbf{H}$ by*

$$\mathbf{X}(t, \mathbf{x}) := \begin{cases} (\eta(0, t)_* \mathbf{X}_0)(t, \mathbf{x}) & \text{on } U_-, \\ (\eta(\tau(t, \mathbf{x}), t)_* \mathbf{H}(\tau(t, \mathbf{x}))(t, \mathbf{x}) & \text{on } U_+. \end{cases}$$

Written out more fully, for $(t, \mathbf{x}) \in U_+$,

$$\begin{aligned} \mathbf{X}(t, \mathbf{x}) &= \mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x})) \cdot \nabla\eta(\tau(t, \mathbf{x}), t; \eta(t, \tau(t, \mathbf{x}); \mathbf{x})) \\ &= \mathbf{H}(\tau(t, \mathbf{x}), \boldsymbol{\gamma}(t, \mathbf{x})) \cdot \nabla\eta(\tau(t, \mathbf{x}), t; \boldsymbol{\gamma}(t, \mathbf{x})). \end{aligned}$$

Lemma 4.3. *With $\mathbf{X}$ as in Definition 4.2, $L\mathbf{X} = 0$ on U_+.*

Proof. Let $(t, \mathbf{x}) \in U_+$ and fix $s \geqslant \tau(t, \mathbf{x})$. Because τ and $\boldsymbol{\gamma}$ are constant along flow lines, so that $\tau(t, \eta(s, t; \mathbf{x})) = \tau(s, \mathbf{x})$ and $\boldsymbol{\gamma}(t, \eta(s, t; \mathbf{x})) = \boldsymbol{\gamma}(s, \mathbf{x})$, the verification of $L\mathbf{X} = 0$ on U_+ parallels the calculation that led to Lemma 4.1, though its expression is a little more complex. Holding s and $\mathbf{x}$ fixed, we apply (4.2) on U_+, to give

$$\begin{aligned} L_t(\mathbf{X}(t, \eta(s, t; \mathbf{x})) &= \frac{d}{dt} \mathbf{X}(t, \eta(s, t; \mathbf{x})) \\ &= \frac{d}{dt} \left[\mathbf{H}(\tau(t, \eta(s, t; \mathbf{x})), \boldsymbol{\gamma}(t, \eta(s, t; \mathbf{x}))) \cdot \nabla\eta(\tau(t, \eta(s, t; \mathbf{x})), t; \boldsymbol{\gamma}(t, \eta(s, t; \mathbf{x}))) \right] \\ &= \frac{d}{dt} \left[\mathbf{H}(\tau(s, \mathbf{x}), \boldsymbol{\gamma}(s, \mathbf{x})) \cdot \nabla\eta(\tau(s, \mathbf{x}), t; \boldsymbol{\gamma}(s, \mathbf{x})) \right] \\ &= \mathbf{H}(\tau(s, \mathbf{x}), \boldsymbol{\gamma}(s, \mathbf{x})) \cdot \partial_{t_2} \nabla\eta(\tau(s, \mathbf{x}), t; \boldsymbol{\gamma}(s, \mathbf{x})) \\ &= \mathbf{H}(\tau(s, \mathbf{x}), \boldsymbol{\gamma}(s, \mathbf{x})) \cdot \left[\nabla\mathbf{u}(t, \eta(\tau(s, \mathbf{x}), t; \boldsymbol{\gamma}(s, \mathbf{x}))) \nabla\eta(\tau(s, \mathbf{x}), t; \boldsymbol{\gamma}(s, \mathbf{x})) \right] \\ &= \left[\mathbf{H}(\tau(s, \mathbf{x}), \boldsymbol{\gamma}(s, \mathbf{x})) \cdot \nabla\eta(\tau(s, \mathbf{x}), t; \boldsymbol{\gamma}(s, \mathbf{x})) \right] \nabla\mathbf{u}(t, \eta(\tau(s, \mathbf{x}), t; \boldsymbol{\gamma}(s, \mathbf{x}))) \\ &= \left[\mathbf{H}(\tau(t, \eta(s, t; \mathbf{x})), \boldsymbol{\gamma}(t, \eta(s, t; \mathbf{x}))) \cdot \nabla\eta(\tau(t, \eta(s, t; \mathbf{x})), t; \boldsymbol{\gamma}(t, \eta(s, t; \mathbf{x}))) \right] \\ &\qquad \nabla\mathbf{u}(t, \eta(\tau(t, \eta(s, t; \mathbf{x})), t; \boldsymbol{\gamma}(t, \eta(s, t; \mathbf{x})))) \\ &= \left[\mathbf{H}(\tau(t, \mathbf{z}), \boldsymbol{\gamma}(t, \mathbf{z})) \cdot \nabla\eta(\tau(t, \mathbf{z}), t; \boldsymbol{\gamma}(t, \mathbf{z})) \right] \nabla\mathbf{u}(t, \eta(\tau(t, \mathbf{z}), t; \boldsymbol{\gamma}(t, \mathbf{z})))|_{\mathbf{z} = \eta(s, t; \mathbf{x})} \\ &= (\mathbf{X} \cdot \nabla\mathbf{u})(t, \eta(s, t; \mathbf{x})). \end{aligned}$$

This shows that $L\mathbf{X} = 0$ on U_+. (Note that assuming higher regularity for $\mathbf{H}$ was not required for this proof.) □

5. Lagrangian and Eulerian solutions

To define a Lagrangian solution, we first explain how to handle forcing. We then show, in Proposition 5.7, how to obtain the regularity of Lagrangian solutions from the compatibility condition cond_N. Finally, we relate our Lagrangian solutions to Eulerian solutions in Proposition 5.9.

Assumption 5.1. *$T^* = T$, where T^* is as in Lemma 3.4.*

Remark 5.2. *In Remark 5.10, we show how to drop Assumption 5.1.*

Forcing. To treat forcing, which we have so far not considered, we use a version of Duhamel's principle. We define $\mathbf{G}$ is as in (3.21) Chapter 4 of [2],

$$\mathbf{G}(t, \mathbf{x}) := \int_{\overline{\tau}(t,\mathbf{x})}^{t} (\eta(s,t)_* \mathbf{g}(s))(t, \mathbf{x})\, ds, \tag{5.1}$$

where $\overline{\tau}(t, \mathbf{x}) = \max\{0, \tau(t, \mathbf{x})\}$.

Proposition 5.3. *Assume that* $\mathbf{g} \in C^\alpha(Q)$ *and* $\mathbf{u} \in \mathring{C}^{1,\alpha}_\sigma(Q)$. *Then* $\mathbf{G} \in C^\alpha(Q)$ *and* $L\mathbf{G} = \mathbf{g}$ *on* Q *weakly. If* $\mathbf{g} \in \mathring{C}^{N,\alpha}(U_\pm) \cap C^\alpha(Q)$ *and* $\mathbf{u} \in \mathring{C}^{N+1,\alpha}_\sigma(Q)$ *for* $N \geqslant 1$ *then* $\mathbf{G} \in C^{N,\alpha}(U_\pm)$ *with*

$$\partial_t \mathbf{G}(t, \mathbf{x}) = \mathbf{g}(t, \mathbf{x}) - (\eta(\overline{\tau}(t, \mathbf{x}), t)_* \mathbf{g}(\overline{\tau}(t, \mathbf{x})))(t, \mathbf{x}) \partial_t \overline{\tau}(t, \mathbf{x}) + \int_{\overline{\tau}(t,\mathbf{x})}^{t} \partial_t((\eta(s,t)_* \mathbf{g}(s))(t, \mathbf{x}))\, ds,$$

$$\nabla \mathbf{G}(t, \mathbf{x}) = -(\eta(\overline{\tau}(t, \mathbf{x}), t)_* \mathbf{g}(\overline{\tau}(t, \mathbf{x})))(t, \mathbf{x}) \otimes \nabla \overline{\tau}(t, \mathbf{x}) + \int_{\overline{\tau}(t,\mathbf{x})}^{t} \nabla_x((\eta(s,t)_* \mathbf{g}(s))(t, \mathbf{x}))\, ds$$

for all $(t, \mathbf{x}) \in Q$, *noting that the terms involving* $\partial_t \overline{\tau}(t, \mathbf{x})$ *and* $\nabla \overline{\tau}(t, \mathbf{x})$ *have a potential singularity along* S.

Proof. The expressions for $\partial_t \mathbf{G}$ and $\nabla \mathbf{G}$ follow from applying the chain rule for integrals to (5.1), as does

$$\begin{aligned} L\mathbf{G} &= (\eta(t,t)_* \mathbf{g}(t))(t, \mathbf{x}) - (\eta(\overline{\tau}(t, \mathbf{x}), t)_* \mathbf{g}(\overline{\tau}(t, \mathbf{x})))(t, \mathbf{x}) \left(\partial_t + \mathbf{u} \cdot \nabla\right) \overline{\tau}(t, \mathbf{x}) \\ &\quad + \int_{\overline{\tau}(t,\mathbf{x})}^{t} L(\eta(s,t)_* \mathbf{g}(s))(t, \mathbf{x})\, ds \\ &= \mathbf{g}(t, \mathbf{x}) - (\eta(\overline{\tau}(t, \mathbf{x}), t)_* \mathbf{g}(\overline{\tau}(t, \mathbf{x})))(t, \mathbf{x}) \left(\partial_t + \mathbf{u} \cdot \nabla\right) \overline{\tau}(t, \mathbf{x}), \end{aligned}$$

since $L(\eta(s,t)_* \mathbf{g}(s))(t, \mathbf{x}) \equiv 0$ for all s. Now, $\overline{\tau}$ either equals 0 or $\tau(t, \mathbf{x})$, but either way, by virtue of Lemma 3.5, we see that its value is transported along flow lines. Hence, also $(\partial_t + \mathbf{u} \cdot \nabla) \overline{\tau}(t, \mathbf{x}) = 0$, and we conclude that $L\mathbf{G} = \mathbf{g}$—though only weakly because $\partial_t \mathbf{G}$ and $\nabla \mathbf{G}$ are discontinuous along S. The regularity of $\mathbf{G}$ follows in a very classical way, employing the lemmas in Appendix A. □

Define,

$$\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0(t, \mathbf{x}) := \eta(t, 0; \mathbf{x}) \text{ on } \overline{U}_-, \tag{5.2}$$

$$B_- = B_-(t, \mathbf{x}) := \nabla \eta(0, t; \eta(t, 0; \mathbf{x})) = \nabla \eta(0, t; \boldsymbol{\gamma}_0) \text{ on } \overline{U}_-, \tag{5.3}$$

$$B_+ = B_+(t, \mathbf{x}) := \nabla \eta(\tau, t; \eta(t, \tau; \mathbf{x})) = \nabla \eta(\tau, t; \boldsymbol{\gamma}) \text{ on } \overline{U}_+ \setminus \{0\} \times \Gamma_+. \tag{5.4}$$

Definition 5.4 (Lagrangian solution to (1.2))**.** *Define* $\mathbf{Y}$ *by* $\mathbf{Y}|_{U_\pm} = \mathbf{Y}_\pm$, *where*

$$\begin{aligned} \mathbf{Y}_-(t, \mathbf{x}) &:= B_- \mathbf{Y}_0(\boldsymbol{\gamma}_0) + \mathbf{G}_-(t, \mathbf{x}), \\ \mathbf{Y}_+(t, \mathbf{x}) &:= B_+ \mathbf{H}(\tau, \boldsymbol{\gamma}) + \mathbf{G}_+(t, \mathbf{x}), \\ \mathbf{G}_-(t, \mathbf{x}) &:= \int_0^t (\eta(s,t)_* \mathbf{g}(s))(t, \mathbf{x})\, ds, \\ \mathbf{G}_+(t, \mathbf{x}) &:= \int_{\tau(t,\mathbf{x})}^{t} (\eta(s,t)_* \mathbf{g}(s))(t, \mathbf{x})\, ds. \end{aligned} \tag{5.5}$$

We say that $\mathbf{Y}$ *is the Lagrangian solution to (1.2).*

Lemma 5.5. *Assuming data regularity* $N \geqslant 0$, $\mathbf{Y}_\pm \in C^{N,\alpha}(U_\pm)$.

Proof. This follows directly from the definition of the pushforward, the regularity assumed on $\mathbf{Y}_0$ and $\mathbf{H}$, the regularity of η, τ, $\boldsymbol{\gamma}$, and $\mathbf{G}$ given by Lemmas 3.3 and 3.5 and Proposition 5.3, and the lemmas in Appendix A. □

We will obtain uniqueness of solutions later in Proposition 5.9, but we will find a need in the proof of Proposition 5.7 for a weaker kind of uniqueness (that is, with stronger assumptions) that allows us to know that certain Eulerian solutions are, in fact, Lagrangian solutions. That is the purpose of Lemma 5.6.

Lemma 5.6. *Assume that the data has regularity $N \geqslant 1$, $\mathbf{Y} \in C^{1,\alpha}(U_\pm)$ and $\mathbf{Y} \in C^\alpha(Q)$, $\mathbf{Y} = \mathbf{H}$ on $[0,T] \times \Gamma_+$, $\mathbf{Y}(0) = \mathbf{Y}_0$ on Ω for some $\mathbf{H} \in C^{N,\alpha}([0,T] \times \Gamma_+)$, and $\mathbf{Y}_0 \in C^{N,\alpha}(\Omega)$. If*

$$\partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{u} = \mathbf{g} \text{ on } U_\pm \tag{5.6}$$

then $\mathbf{Y}$ is the Lagrangian solution given by (5.5).

Proof. From Lemmas 4.1, 4.3, and 5.5 the Lagrangian solution given by (5.5) satisfies (5.6), so we need only show uniqueness of solutions to (5.6) with the same values of $\mathbf{H}$ and $\mathbf{Y}(0)$. Since (5.6) is linear, this is equivalent to showing that given vanishing values of $\mathbf{H}$, $\mathbf{Y}(0)$, and forcing, the only solution is $\mathbf{Y} \equiv 0$.

We suppose, then, that

$$\begin{cases} \partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{u} = 0 & \text{in } U_+ \cup U_-, \\ \mathbf{Y} = 0 & \text{on } [0,T] \times \Gamma_+, \\ \mathbf{Y}(0) = 0 & \text{on } \Omega. \end{cases} \tag{5.7}$$

Multiplying by $\mathbf{Y}$ and integrating over $U_+ \cup U_-$, we have

$$\frac{d}{dt} \|\mathbf{Y}\|^2_{L^2(\Omega)} = -2(\mathbf{u} \cdot \nabla \mathbf{Y}, \mathbf{Y})_{L^2(U_+(t))} - 2(\mathbf{u} \cdot \nabla \mathbf{Y}, \mathbf{Y})_{L^2(U_-(t))} + 2(\mathbf{Y} \cdot \nabla \mathbf{u}, \mathbf{Y})_{L^2(\Omega)},$$

using that $S(t)$ has measure zero. But,

$$\begin{aligned} -2(\mathbf{u} \cdot \nabla \mathbf{Y}, &\mathbf{Y})_{L^2(U_+(t))} - 2(\mathbf{u} \cdot \nabla \mathbf{Y}, \mathbf{Y})_{L^2(U_-(t))} \\ &= -(\mathbf{u}, \nabla |\mathbf{Y}|^2)_{L^2(U_+(t))} - (\mathbf{u}, \nabla |\mathbf{Y}|^2)_{L^2(U_-(t))} \\ &= -\int_{S(t)} (\mathbf{u} \cdot \boldsymbol{n}) |\mathbf{Y}|^2 + \int_{S(t)} (\mathbf{u} \cdot \boldsymbol{n}) |\mathbf{Y}|^2 - \int_{\Gamma_-} U^{\boldsymbol{n}} |\mathbf{Y}|^2 \leqslant 0. \end{aligned}$$

We used here that $\mathbf{Y} \in C^\alpha(Q)$, so the two boundary integrals over $S(t)$, the shared portion of $\partial U_+(t)$ and $\partial U_-(t)$, properly oriented cancel. Moreover, we used that $\mathbf{Y}$ has sufficient regularity to integrate the term $(\mathbf{u}, \nabla |\mathbf{Y}|^2)$ by parts over U_+ and U_- separately. Using that $(\mathbf{Y} \cdot \nabla \mathbf{u}, \mathbf{Y})_{L^2(\Omega)} \leqslant \|\nabla \mathbf{u}\|_{L^\infty(\Omega)} \|\mathbf{Y}\|^2_{L^2(\Omega)}$ and applying Grönwall's lemma yields $\mathbf{Y} \equiv 0$ in Q, establishing uniqueness. □

Regularity across S. The solution can be decomposed as in (5.5) because (1.2) is a linear problem, the velocity field $\mathbf{u}$ and so the flow map η being given. We see that formally (1.2) holds; however, this applies only distributionally on $U_\pm$ separately until we can establish sufficient regularity of $\mathbf{Y}$ across S, as we will see in the proof of Proposition 5.7. Before turning to its proof, however, let us consider some of the difficulties in treating it.

For transport or transport-stretching equations, much of the analysis is done in more-or-less Lagrangian form, following the solution along flow lines. There, the natural operator that emerges is L_t or L. The regularity of $L_t \mathbf{Y}_0$ is obtained quite easily in this way, but it does not allow the regularity of $\partial_t \mathbf{Y}$ and $\mathbf{u} \cdot \nabla \mathbf{Y}$ to be treated separately. Indeed, even without inflow, $\mathbf{Y}_0$ discontinuous along, say, a curve is no obstacle to obtaining solutions in which $L\mathbf{Y}_0 = 0$, so we should not expect to be able to separate them.

Nonetheless, classically, the regularity of the pushforward of $\partial_t \mathbf{Y}$ and so, as we shall see, $\nabla \mathbf{Y}$ is fairly easily tied directly to the regularity of $\mathbf{Y}_0$. The difficulty we face, is that while $\mathbf{Y}_-$ is a classical pushforward off a domain from time zero, $\mathbf{Y}_+$ is a pushforward off of a surface into a domain, and the behavior of higher time derivatives of $\mathbf{Y}_+$ on the interface S is not as easily connected to its behavior at time zero or to the behavior of $\mathbf{Y}_\pm$ on S.

Closely related to this is that for $N \geqslant 1$ and nonzero forcing, solutions $\mathbf{Y} \in C^{N,\alpha}(Q)$ do not require that both terms in the decomposition of $\mathbf{Y}_\pm$ in (5.5) be in $C^{N,\alpha}(Q)$. Indeed, we can see from Proposition 5.3 that $\mathbf{G}$ will never have higher than $C^\alpha(Q)$ regularity unless we impose the strong condition that $\partial_t^j \mathbf{g}(0)$ vanish on Γ_+ for all $1 \leqslant j \leqslant N$.

In the proof of Proposition 5.7 we reference Proposition 6.1, which we prove in the next section, because it is of a different character than the rest of the proof.

Proposition 5.7. *Assume that the data has regularity N as in Definition 1.1 for some $N \geqslant 0$ and let $\mathbf{Y}$ be the Lagrangian solution to (1.2) as in Definition 5.4. Then $\mathbf{Y} \in C^{N,\alpha}(Q)$ if and only if* cond_N *holds.*

Proof. First observe that $\mathbf{Y}_\pm \in C^{N,\alpha}(U_\pm)$ follows from Lemma 5.5 without the need for any additional conditions. Also, $\mathbf{Y} \in C^{N,\alpha}(Q)$ gives cond_N, as we can most readily see from the form of cond_N in (1.11).

We assume, then, that cond_N holds and will prove that $\mathbf{Y} \in C^{N,\alpha}(Q)$.

Assume first that $N = 0$. By Proposition 5.3, $G \in C^\alpha(Q)$, and we can see from (5.5) that along the hypersurface S, where $\tau(t, \mathbf{x}) = 0$ and $B_+ = B_-$, the two expressions for $\mathbf{Y}(t, \mathbf{x})$ agree if and only if $\mathbf{H}(0) = \mathbf{Y}_0$ on Γ_+. It follows from Lemma 3.4 that $\mathbf{Y} \in C^\alpha(Q)$, completing the proof for $N = 0$.

Now assume that $N = 1$. As already observed, $\mathbf{Y} \in C^{1,\alpha}(U_\pm)$. By Lemma 3.4, then, we see that $\partial_t \mathbf{Y} \in C^\alpha(Q)$ if and only if $\partial_t \mathbf{Y}_- = \partial_t \mathbf{Y}_+$ along S and $\nabla \mathbf{Y} \in C^\alpha(Q)$ if and only if $\nabla \mathbf{Y}_- = \nabla \mathbf{Y}_+$ along S: both conditions hold by Proposition 6.1 since we assumed that cond_1 holds. Hence, $\mathbf{Y} \in C^{1,\alpha}(Q)$, and we can write $\partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} = \mathbf{Y} \cdot \nabla \mathbf{u} + \mathbf{g}$ on all of Q. This completes the proof for $N = 1$.

Now assume $N = 2$ data regularity. Then $\mathbf{Y} \in C^{2,\alpha}(U_\pm)$, as already observed for general N, and by the $N = 1$ result we know that $\mathbf{Y} \in C^{1,\alpha}(Q)$.

Next, let $\mathbf{Z} := \partial_t \mathbf{Y}$, so $\mathbf{Z} \in C^{1,\alpha}(U_\pm)$ and $\mathbf{Z} \in C^\alpha(Q)$. Then

$$\partial_t \mathbf{Z} + \mathbf{u} \cdot \nabla \mathbf{Z} - \mathbf{Z} \cdot \nabla \mathbf{u} = \mathbf{h} := \partial_t \mathbf{g} - \partial_t \mathbf{u} \cdot \nabla \mathbf{Y} + \mathbf{Y} \cdot \nabla \partial_t \mathbf{u} \in \mathring{C}^{1,\alpha}(U_\pm) \cap C^\alpha(Q). \tag{5.8}$$

Equality holds classically on $U_\pm$, so, in fact, $\mathbf{Z}$ is the Lagrangian solution as in Definition 5.4 with forcing $\mathbf{h}$, initial value $\partial_t \mathbf{Y}(0)$, and $\mathbf{Z} = \partial_t \mathbf{H}$ on $[0, T] \times \Gamma_+$. This follows from Lemma 5.6 or, with much greater trouble, directly from applying ∂_t to the expression for $\mathbf{Y}$ in (5.5).

We can see that cond_2 for $\mathbf{Y}$ with forcing $\mathbf{g}$ and boundary value $\mathbf{H}$ is cond_1 for $\mathbf{Z}$ with forcing $\mathbf{h}$ and boundary value $\partial_t \mathbf{H}$, so Proposition 6.1 gives that $\partial_t \mathbf{Z}_- = \partial_t \mathbf{Z}_+$ on S and $\nabla \mathbf{Z}_- = \nabla \mathbf{Z}_+$ on S. Applying Lemma 3.4, this, in turn, gives $\partial_t \mathbf{Y} = \mathbf{Z} \in C^{1,\alpha}(Q)$.

Now, we cannot immediately apply this same argument to $\mathbf{W} := \partial_j \mathbf{Y}$, since cond_2 for $\mathbf{Y}$ does not become cond_1 for $\mathbf{W}$. But we can take ∂_j of $(1.2)_1$, giving

$$\partial_t \mathbf{W} + \mathbf{u} \cdot \nabla \mathbf{W} - \mathbf{W} \cdot \nabla \mathbf{u} = \mathbf{j} := \partial_j \mathbf{g} - \partial_j \mathbf{u} \cdot \nabla \mathbf{Y} + \mathbf{Y} \cdot \nabla \partial_j \mathbf{u} \in \mathring{C}^{1,\alpha}(U_\pm) \cap C^\alpha(Q). \tag{5.9}$$

Since we already know that $\partial_t \mathbf{W} = \partial_t \partial_j \mathbf{Y} = \partial_j \mathbf{Z} \in C^\alpha(Q)$, we have $\mathbf{u} \cdot \nabla \mathbf{W} \in C^\alpha(Q)$. But cond_0 and the $N = 2$ data regularity give that $\nabla \mathbf{W}$ is C^α-continuous along $S(t)$. Letting $Q^* = (0, T^*) \times \Omega$, where $T^* > 0$ is given by Lemma 3.4, the transversality of $\mathbf{u} \in C^{3,\alpha}(Q^*)$

across $S(t)$ gives that $\nabla \mathbf{W}$ is C^α-continuous in the direction perpendicular to $S(t)$. We conclude that $\nabla \mathbf{W} = \nabla \partial_j \mathbf{Y} \in C^\alpha(Q^*)$ and so $\nabla^2 \mathbf{Y} \in C^\alpha(Q^*)$. Combined with the regularity of $\mathbf{Z}$, we have $\mathbf{Y} \in C^{2,\alpha}(Q^*)$

We now have the regularity to return to (5.9) and view it as the solution to

$$\begin{cases} \partial_t \mathbf{W} + \mathbf{u} \cdot \nabla \mathbf{W} - \mathbf{W} \cdot \nabla \mathbf{u} = \mathbf{j} & \text{in } Q, \\ \mathbf{W} = \mathbf{J} & \text{on } [0,T] \times \Gamma_+, \\ \mathbf{W}(0) = \partial_j \mathbf{Y}_0 & \text{on } \Omega, \end{cases} \tag{5.10}$$

where $\mathbf{J} := \partial_j \mathbf{Y}$. Since we know that $\mathbf{W} \in C^{1,\alpha}(Q^*)$, it follows from Proposition 6.1 that cond_1 holds for $\mathbf{W}$. But it then follows again from Proposition 6.1 that, in fact, $\mathbf{W} \in C^{1,\alpha}(Q)$. Combined with the regularity of $\mathbf{Z}$, we can conclude that $\mathbf{Y} \in C^{2,\alpha}(Q)$.

This argument inducts to any value of $N \geqslant 3$. □

In the proof of Proposition 5.7, we reduced the $N \geqslant 2$ case inductively to the $N = 1$ case, using the convenient form of cond_N in (1.11). Formally, we could reduce all the way to the $N = 0$ case (removing the need for Section 6 altogether). In (5.8), however, $\mathbf{Z}$ would only lie in $C^\alpha(U_\pm)$ and $\mathbf{h}$ at best would lie in the negative Hölder space $C^{\alpha-1}(Q)$, which we do not have the tools to handle.

It is not hard to see that cond_0 is the Rankine-Hugoniot condition that allows $\mathbf{Y}$ as given by Definition 5.4 to be a weak Eulerian solution to (1.2). There is a somewhat delicate issue regarding regularity, however, as in the classical case one typically considers C^1 solutions, while for $N = 0$, $\mathbf{Y}$ is only C^α. Therefore, we present a proof in Lemma 5.8, which we will need to obtain Eulerian solutions in Proposition 5.9.

Lemma 5.8. *Assume data regularity $N = 0$ and let $\mathbf{Y}$ be the Lagrangian solution to (1.2) as in Definition 5.4. If* cond_0 *holds then $\mathbf{Y}$ is a weak Eulerian solution to (1.2).*

Proof. Let $(\mathbf{u}_i)$ be the sequence in $C^{2,\alpha}_{\sigma, U_i^n}(Q)$ approximating $\mathbf{u} \in \mathring{C}^{1,\alpha}_\sigma(Q)$ given by Lemma 2.6. Also let $(\mathbf{Y}_{i,0})$ be a sequence in $C^{1,\alpha}(\Omega)$ with $\mathbf{Y}_{i,0} \to \mathbf{Y}_0$ in $C^\alpha(\Omega)$. Let $\mathbf{Y}_i$ be the Lagrangian solution to (1.2) as in Definition 5.4 with $\mathbf{u}_i$ in place of $\mathbf{u}$ and $\mathbf{Y}_{i,0}$ in place of $\mathbf{Y}_0$ (leaving $\mathbf{H}$ and $\mathbf{g}$ unchanged). In what follows, i subscripts on quantities, such as $U_{i,\pm}$, will refer to the corresponding quantity for $\mathbf{u}$, $\mathbf{Y}$ applied with $\mathbf{u}_i$, $\mathbf{Y}_i$.

Neither cond_0 nor cond_1 need hold for $\mathbf{Y}_i$, so we only have $\mathbf{Y}_{i,\pm} \in C^{1,\alpha}(U\pm)$, but Lemmas 4.1, 4.3, and 5.5 do give, as in the proof of Lemma 5.6, that

$$\partial_t \mathbf{Y}_i + \mathbf{u}_i \cdot \nabla \mathbf{Y}_i - \mathbf{Y}_i \cdot \nabla \mathbf{u}_i = \mathbf{g} \text{ on } U_{i,\pm}.$$

Let $\varphi \in \mathcal{D}(Q)$. Multiplying the above equation by φ and integrating by parts gives

$$((\mathbf{Y}_i, \partial_t \varphi)) + ((\mathbf{Y}_i \cdot \nabla \mathbf{u}_i, \varphi)) + ((\mathbf{g}, \varphi)) + \int_0^T \int_{S_i(t)} (\mathbf{u}_i \cdot \boldsymbol{n})(\mathbf{Y}_{i,-} - \mathbf{Y}_{i,+}) \cdot \varphi = 0,$$

where $((\cdot,\cdot))$ is the pairing in $\mathcal{D}'(Q), \mathcal{D}(Q)$ and where we have arbitrarily chosen $\boldsymbol{n}$ on $S_i(t)$ to be outward from $U_{i,+}(t)$.

Because $\mathbf{G}_i \in C^\alpha(Q)$ by Proposition 5.3, we have $|\mathbf{G}_{i,+} - \mathbf{G}_{i,-}| \to 0$ on $S_i(t)$. Also,

$$\begin{aligned} B_{i,-} \mathbf{Y}_{i,0}(\boldsymbol{\gamma}_{i,0}) - B_{i,+} \mathbf{H}(\tau_i, \boldsymbol{\gamma}_i) &= B_{i,-}(\mathbf{Y}_{i,0}(\boldsymbol{\gamma}_{i,0}) - \mathbf{H}(\tau_i, \boldsymbol{\gamma}_i)) + (B_{i,-} - B_{i,+}) \mathbf{H}(\tau_i, \boldsymbol{\gamma}_i) \\ &= B_{i,-}(\mathbf{Y}_{i,0}(\boldsymbol{\gamma}_i) - \mathbf{H}(0, \boldsymbol{\gamma}_i)) \text{ on } S_i(t), \end{aligned}$$

since on $S_i(t)$, $\tau_i = 0$, $\boldsymbol{\gamma}_{i,0} = \boldsymbol{\gamma}_i$, and $B_{i,-} = B_{i,+}$, without the need for cond_0 to hold. But $\mathbf{Y}_{i,0} \to \mathbf{Y}_0$ in $C^\alpha(\Omega)$ and $\mathbf{Y}_0 = \mathbf{H}$ on Γ_+, and $\boldsymbol{\gamma}_i(t, \mathbf{x}) \in \Gamma_+$ for $(t, \mathbf{x}) \in S$, so we see that

$$B_{i,-} \mathbf{Y}_{i,0}(\boldsymbol{\gamma}_{i,0}) - B_{i,+} \mathbf{H}(\tau_i, \boldsymbol{\gamma}_i) \to 0 \text{ on } S_i(t).$$

Hence, $\mathbf{Y}_{i,-} - \mathbf{Y}_{i,+} \to 0$ on $S_i(t)$. We conclude from $\mathbf{u}_i \to \mathbf{u}$ in $\mathring{C}^{1,\alpha}_\sigma(Q)$ that in the limit,

$$((\mathbf{Y}, \partial_t \varphi)) + ((\mathbf{Y} \cdot \nabla \mathbf{u}, \varphi)) + ((\mathbf{g}, \varphi)) = 0,$$

which shows that $\mathbf{Y}$ is a weak Eulerian solution to (1.2). □

Proposition 5.9. *Assume that the data has regularity N as in Definition 1.1 for some $N \geqslant 0$ and let $\mathbf{Y}$ be the Lagrangian solution to (1.2) as in Definition 5.4. Assume that* cond_N *holds. The estimate in (1.12) holds and $\mathbf{Y}$ is a weak Eulerian solution as in Definition 1.2. Moreover, $\mathbf{Y}$ is the unique classical solution in $C^{N,\alpha}(Q)$ to (1.2) when $N \geqslant 1$.*

Proof. Since cond_N holds, we know from Proposition 5.7 that $\mathbf{Y}$ is a Lagrangian solution to (1.2). The bound in (1.12) follows from the Lagrangian form of $\mathbf{Y}$ in (5.5), applying Lemma 3.4 and Proposition 5.3. That $\mathbf{Y}$ is a weak Eulerian solution for $N = 0$ follows from Lemma 5.8. For $N \geqslant 1$, all the terms in (1.2) are defined classically, so $\mathbf{Y}$ is a classical and so weak solution to (1.2).

To prove uniqueness of Eulerian solutions for $N \geqslant 1$, suppose that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are two solutions to (1.2). Letting $\mathbf{Y} = \mathbf{Y}_1 - \mathbf{Y}_2$, we see that

$$\begin{cases} \partial_t \mathbf{Y} + \mathbf{u} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{Y} = 0 & \text{on } [0, T] \times \Gamma_+, \\ \mathbf{Y}(0) = 0 & \text{on } \Omega. \end{cases}$$

Since $N \geqslant 1$, we have enough regularity to multiply by $\mathbf{Y}$, and integrate over Ω using that $\mathbf{Y} = 0$ on the inflow boundary, to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{Y}\|_{L^2}^2 &\leqslant \|\nabla \mathbf{u}\|_{L^\infty(Q)} \|\mathbf{Y}\|_{L^2}^2 - (\mathbf{u} \cdot \nabla \mathbf{Y}, \mathbf{Y}) \leqslant C\|\mathbf{Y}\|_{L^2}^2 - \frac{1}{2}(\mathbf{u}, \nabla |\mathbf{Y}|^2) \\ &= C\|\mathbf{Y}\|_{L^2}^2 - \frac{1}{2}\int_{\Gamma_+} U^{\boldsymbol{n}} |\mathbf{Y}|^2 - \frac{1}{2}\int_{\Gamma_-} U^{\boldsymbol{n}} |\mathbf{Y}|^2 \leqslant C\|\mathbf{Y}\|_{L^2}^2, \end{aligned} \tag{5.11}$$

since $|\mathbf{Y}|^2 = 0$ on Γ_+ and $U^{\boldsymbol{n}} > 0$ on Γ_-. Applying Grönwall's lemma gives $\mathbf{Y} = 0$ so $\mathbf{Y}_1 = \mathbf{Y}_2$. □

Remark 5.10. *We have obtained the results of this section under Assumption 5.1 or, equivalently, we have obtained these results up to time T^*. To extend beyond T^*, we reset time so that T^* becomes time zero. Because compatibility conditions hold for solutions at all positive times, we know that* cond_N *holds at the new time zero, and extend the result for another T^* time interval, repeating this process as needed. This allows us to drop Assumption 5.1.*

6. Regularity condition for $N = 1$

In this section we prove Proposition 6.1, which is the key to obtaining $\mathbf{Y} \in C^{N,\alpha}(Q)$.

Proposition 6.1. *Assume the data has $N = 1$ regularity, though we only assume that $\mathbf{g} \in \mathring{C}^{1,\alpha}(U_\pm) \cap C^\alpha(Q)$, and assume that* cond_0 *holds. Let $\mathbf{Y}$ be the Lagrangian solution as in Definition 5.4. Then $D\mathbf{Y}_- = D\mathbf{Y}_+$ on S if and only if* cond_1 *holds. (Recall that D is defined in (1.17).)*

The direct calculation showing that $D\mathbf{Y}_+ = D\mathbf{Y}_-$ on S if cond_1 holds is quite lengthy. Instead, we use the indirect method below. But first, we develop some necessary tools.

We know from the proof of Proposition 5.7 that regularity of $\mathbf{Y}$ on Q derives from its regularity across the hypersurface S. To assess such regularity, we would like to be able to treat $\mathbf{Y}_\pm$ uniformly on the same domain, but their domains overlap only on S. Therefore, we extend $\mathbf{Y}_-$ to all of Q by extending $\mathbf{Y}_0$ to $C^{1,\alpha}(\mathbb{R}^3)$ and $\mathbf{g}$ to $C^\alpha([0,T]\times\mathbb{R}^3)$. Then because, as in Section 3, we have extended $\mathbf{u}$ and so η, the expression for $\mathbf{Y}_-$ in (5.5) is defined on all of Q and lies in $C^{1,\alpha}(Q)$ as long as $\mathbf{u}\in\mathring{C}^{2,\alpha}_\sigma(Q)$. Moreover, it follows that $\partial_t\mathbf{Y}_- + \mathbf{u}\cdot\nabla\mathbf{Y}_- - \mathbf{Y}_-\cdot\nabla\mathbf{u} = \mathbf{g}$ on Q.

Because S is at least a $C^{2,\alpha}$ hypersurface in Q by Lemma 3.4, one-sided derivatives of $\mathbf{Y}$ in time and space up to order $N=1$ exist on $U_\pm$. Hence, $\mathbf{Y}_-\in C^{1,\alpha}(\overline{U}_-)$ with $\mathbf{Y}_+\in C^{1,\alpha}(\overline{U}_+\setminus\{0\}\times\Gamma_+)$. We remove $\{0\}\times\Gamma_+$ from $\overline{U}_+$, since there are no spatial derivatives of $\mathbf{Y}_+$ normal to the boundary at time zero. The time derivatives, however, are defined and continuous on all of $\overline{U}_+$.

In any case, one-sided derivatives of $\mathbf{Y}_\pm$ exist on S, so we can freely take derivatives of both $\mathbf{Y}_-$ and $\mathbf{Y}_+$ on $\overline{U}_+\setminus\{0\}\times\Gamma_+$. Restricted to S, any calculation for $\mathbf{Y}_-$ that depends only upon its $C^{1,\alpha}(\overline{U}_-)$ regularity is independent of the extension when restricted to S. Hence, the calculations on S that follow do not depend upon the manner in which we extend $\mathbf{Y}_0$ and $\mathbf{g}$. These extensions also allow us to make sense of $\boldsymbol{\gamma}_0$, B_-, and B_+ of (5.2) through (5.4) on $\overline{U}_+\setminus\{0\}\times\Gamma_+$, along with the matrix-valued function,

$$M = M(t,\mathbf{x}) := B_-^{-1} - B_+^{-1} \text{ on } \overline{U}_+\setminus\{0\}\times\Gamma_+. \tag{6.1}$$

Since

$$I = \nabla\mathbf{x} = \nabla_{\mathbf{x}}(\eta(t_1,t_2;\eta(t_2,t_1;\mathbf{x}))) = \nabla\eta(t_1,t_2;\eta(t_2,t_1;\mathbf{x}))\nabla\eta(t_2,t_1;\mathbf{x}),$$

we have

$$\nabla\eta(t_1,t_2;\eta(t_2,t_1;\mathbf{x})) = (\nabla\eta(t_2,t_1;\mathbf{x}))^{-1}.$$

Hence,

$$B_+^{-1} = \nabla\eta(t,\tau;\mathbf{x}), \quad B_-^{-1} = \nabla\eta(t,0;\mathbf{x}). \tag{6.2}$$

In what follows, we will apply the chain rule quite liberally, and will avoid much redundancy by largely employing the time-space derivative operator D of (1.17).

Since $\tau=0$ and $\boldsymbol{\gamma}=\boldsymbol{\gamma}_0$ on S, we can write (3.7) as

$$D\boldsymbol{\gamma} = D\boldsymbol{\gamma}_0 + \mathbf{u}_0(\boldsymbol{\gamma}_0)\otimes D\tau \text{ on } S. \tag{6.3}$$

On $\overline{U}_+\setminus\{0\}\times\Gamma_+$, we can see from the chain rule that

$$\begin{aligned}\partial_t M &= \partial_t(\nabla\eta(t,0;\mathbf{x})) - \partial_t(\nabla\eta(t,\tau;\mathbf{x}))\\ &= \partial_{t_1}\nabla\eta(t,0;\mathbf{x}) - \partial_{t_1}\nabla\eta(t,\tau;\mathbf{x}) - \partial_{t_2}\nabla\eta(t,\tau;\mathbf{x})\partial_t\tau,\\ \nabla M &= \nabla_{\mathbf{x}}(\nabla\eta(t,0;\mathbf{x})) - \nabla_{\mathbf{x}}(\nabla\eta(t,\tau;\mathbf{x}))\\ &= \nabla\nabla\eta(t,0;\mathbf{x}) - \partial_{t_2}\nabla\eta(t,\tau;\mathbf{x})\otimes\nabla\tau - \nabla\nabla\eta(t,\tau;\mathbf{x}).\end{aligned}$$

But, from Lemma 3.6, on S, where $\tau=0$, $\partial_{t_2}\nabla\eta(t,\tau;\mathbf{x}) = \nabla\mathbf{u}(0,\eta(t,0;\mathbf{x}))\nabla\eta(t,0;\mathbf{x}) = \nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)B_-^{-1}$, so we have,

$$\begin{aligned}&\partial_t M = -\partial_t\tau\nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)B_-^{-1}, \quad \nabla M = -\nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)B_-^{-1}\otimes\nabla\tau,\\ &DM = -\nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)B_-^{-1}\otimes D\tau \text{ on } S.\end{aligned} \tag{6.4}$$

Letting $\mathbf{Z}_\pm := \mathbf{Y}_\pm - \mathbf{G}_\pm$, we have, from (5.5),

$$B_-^{-1}\mathbf{Z}_-(t,\mathbf{x}) = \mathbf{Y}_0(\boldsymbol{\gamma}_0), \quad B_+^{-1}\mathbf{Z}_+(t,\mathbf{x}) = \mathbf{H}(\tau,\boldsymbol{\gamma}). \tag{6.5}$$

Using the expressions for $\partial_t G$ and ∇G in Proposition 5.3,

$$\begin{aligned}\partial_t(\mathbf{G}_- - \mathbf{G}_+)(t,\mathbf{x}) &= (\eta(0,t)_*\mathbf{g}(0))(t,\mathbf{x})\partial_t\tau(t,\mathbf{x}) = B_-\mathbf{g}(0,\boldsymbol{\gamma}_0)\partial_t\tau(t,\mathbf{x}) \text{ on } S,\\ \nabla(\mathbf{G}_- - \mathbf{G}_+)(t,\mathbf{x}) &= (\eta(0,t)_*\mathbf{g}(0))(t,\mathbf{x}) \otimes \nabla\tau = B_-\mathbf{g}(0,\boldsymbol{\gamma}_0) \otimes \nabla\tau \text{ on } S,\end{aligned}$$

or,

$$D(\mathbf{G}_- - \mathbf{G}_+)(t,\mathbf{x}) = B_-\mathbf{g}(0,\boldsymbol{\gamma}_0) \otimes D\tau \text{ on } S. \tag{6.6}$$

Observe that by virtue of Proposition 5.3, our assumptions on $\mathbf{g}$ were enough to obtain (6.6).

Proof of Proposition 6.1. Because B_-^{-1} is invertible, $D\mathbf{Y}_- = D\mathbf{Y}_+$ on S iff $B_-^{-1}D(\mathbf{Y}_+ - \mathbf{Y}_-) = 0$. Then, using (6.5) and (6.6),

$$\begin{aligned}B_-^{-1}D(\mathbf{Y}_+ - \mathbf{Y}_-) &= B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) + B_-^{-1}D(\mathbf{G}_+ - \mathbf{G}_-)\\ &= B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) - (B_-^{-1}B_-)\mathbf{g}(0,\boldsymbol{\gamma}_0) \otimes D\tau = B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) - \mathbf{g}(0,\boldsymbol{\gamma}_0) \otimes D\tau\end{aligned}$$

on S. Also on S,

$$B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) = D(B_-^{-1}(\mathbf{Z}_+ - \mathbf{Z}_-)) - (DB_-^{-1})(\mathbf{Z}_+ - \mathbf{Z}_-) = D(B_-^{-1}(\mathbf{Z}_+ - \mathbf{Z}_-)),$$

since $\mathbf{Z}_+ - \mathbf{Z}_- = \mathbf{H}(\tau,\boldsymbol{\gamma}) - \mathbf{Y}_0(\boldsymbol{\gamma}_0) = 0$ on S by cond$_0$. So,

$$\begin{aligned}B_-^{-1}D(\mathbf{Z}_+ - \mathbf{Z}_-) &= D(B_+^{-1}\mathbf{Z}_+ - B_-^{-1}\mathbf{Z}_-) + D((B_-^{-1} - B_+^{-1})\mathbf{Z}_+)\\ &= D[\mathbf{H}(\tau,\boldsymbol{\gamma}) - \mathbf{Y}_0(\boldsymbol{\gamma}_0)] + D(M\mathbf{Z}_+) \text{ on } S.\end{aligned} \tag{6.7}$$

Then,

$$D[\mathbf{H}(\tau,\boldsymbol{\gamma}) - \mathbf{Y}_0(\boldsymbol{\gamma}_0)] = \partial_{t_1}\mathbf{H}(\tau,\boldsymbol{\gamma}) \otimes D\tau + \nabla_\Gamma\mathbf{H}(\tau,\boldsymbol{\gamma})D\boldsymbol{\gamma} - \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)D\boldsymbol{\gamma}_0$$

on S. But, using (6.3),

$$\begin{aligned}\nabla_\Gamma\mathbf{H}(\tau,\boldsymbol{\gamma})D\boldsymbol{\gamma} - \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)D\boldsymbol{\gamma}_0 &= [\nabla_\Gamma\mathbf{H}(\tau,\boldsymbol{\gamma}) - \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)]D\boldsymbol{\gamma} + \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)[D\boldsymbol{\gamma} - D\boldsymbol{\gamma}_0]\\ &= [\nabla_\Gamma\mathbf{H}(\tau,\boldsymbol{\gamma}) - \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)]D\boldsymbol{\gamma} + \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)(\mathbf{u}_0(\boldsymbol{\gamma}_0) \otimes D\tau) \text{ on } S.\end{aligned}$$

Now, $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t,\mathbf{x})$ always lies on Γ_+, so

$$[\nabla_\Gamma\mathbf{H}(\tau,\boldsymbol{\gamma}) - \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)]D\boldsymbol{\gamma} = 0,$$

where we used cond$_0$ along with $N = 1$ regularity. Hence, on S,

$$D[\mathbf{H}(\tau,\boldsymbol{\gamma}) - \mathbf{Y}_0(\boldsymbol{\gamma}_0)] = [\partial_{t_1}\mathbf{H}(\tau,\boldsymbol{\gamma}) + \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)\mathbf{u}_0(\boldsymbol{\gamma}_0)] \otimes D\tau. \tag{6.8}$$

And, using (6.4), the vanishing of M on S, and cond$_0$ once more,

$$\begin{aligned}D(M\mathbf{Z}_+) &= MD\mathbf{Z}_+ + DM\mathbf{Z}_+ = -[\nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)B_-^{-1} \otimes D\tau]\mathbf{Z}_+\\ &= -\nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)B_-^{-1}\mathbf{Z}_- \otimes D\tau = -\nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)\mathbf{Y}_0(\boldsymbol{\gamma}_0) \otimes D\tau \text{ on } S.\end{aligned}$$

Combined, these calculations give

$$B_-^{-1}D(\mathbf{Y}_+ - \mathbf{Y}_-) = [\partial_{t_1}\mathbf{H}(\tau,\boldsymbol{\gamma}) + \nabla\mathbf{Y}_0(\boldsymbol{\gamma}_0)\mathbf{u}_0(\boldsymbol{\gamma}_0) - \nabla\mathbf{u}_0(\boldsymbol{\gamma}_0)\mathbf{Y}_0(\boldsymbol{\gamma}_0) - \mathbf{g}(0,\boldsymbol{\gamma}_0)] \otimes D\tau$$

on S. Recalling that B_- is always invertible, if cond$_1$ holds then the right-hand side vanishes so $D(\mathbf{Y}_+ - \mathbf{Y}_-) = 0$ on S. Conversely, if the left-hand side vanishes then since $\partial_t\tau > 0$ up to at least time T^*, we know that $D\tau$ does not vanish up to T^*, so cond$_1$ must hold. □

7. Vorticity

Proposition 7.1. *Assume that for some $N \geqslant 0$ the data has regularity N as in Definition 1.1,* cond_N *holds,* $\mathbf{Y}_0$ *and* $\mathbf{g}(t)$ *for all* $t \in [0, T]$ *are in the range of the curl, and (1.13) and (1.14) hold. Then the solution* $\mathbf{Y}$ *to (1.2) given by Proposition 5.9 (unique classical for* $N \geqslant 1$*, Lagrangian for all* $N \geqslant 0$*) is in the range of the curl for the lifetime of the solution.*

Remark 7.2. *Proposition 7.1 along with Proposition 5.9 (and Remark 5.10) give a complete proof of Theorem 1.3. We note that if* $\Gamma_0 = \Gamma$*, the classical setting of impermeable boundary conditions, no vorticity is transported off of the boundary, so* U_- *is all of* Q *and many of the flow map constructs, such as* S*,* τ*, and* $\boldsymbol{\gamma}$*, are unnecessary.*

In this section, we make use of the div_Γ operator, so let us introduce the few facts we will need about it, referring the interested reader to Appendix B of [11] for more details (including its explicit expression in coordinates on the boundary). First, we define ∇_Γ, the gradient in the tangent space to Γ, by requiring that for any $f \in C^1(\Gamma)$ and any continuously differentiable curve $\mathbf{x}(s)$ on Γ parameterized by arc length,

$$\nabla_\Gamma f \cdot \mathbf{x}'(0) = \lim_{s \to 0} \frac{f(\mathbf{x}(s)) - f(\mathbf{x}(0))}{s}.$$

We then define div_Γ by duality, requiring that for any $f \in C^1(\Gamma)$, $\mathbf{v} \in C^1(\Gamma)^d$,

$$\int_\Gamma \mathbf{v} \cdot \nabla_\Gamma f = - \int_\Gamma \text{div}_\Gamma \mathbf{v} \, f. \tag{7.1}$$

See, for instance, Proposition 2.2 of [19].

We also need the following two identities (again, see Appendix B of [11]): For all $f \in C^1(\Gamma)$,

$$\text{div}_\Gamma(f\mathbf{v}) = f \, \text{div}_\Gamma \mathbf{v} + \mathbf{v} \cdot \nabla_\Gamma f \tag{7.2}$$

and for any C^1 vector field $\mathbf{v}$ defined in an epsilon neighborhood $\mathcal{N}$ of Γ,

$$\text{div}\, \mathbf{v} = \text{div}_\Gamma \mathbf{v}|_\Gamma + \nabla(\mathbf{v} \cdot \boldsymbol{n}) \cdot \boldsymbol{n} + \kappa \mathbf{v} \cdot \boldsymbol{n}, \tag{7.3}$$

where $\kappa = \kappa_1 + \kappa_2$ is the mean curvature and $\boldsymbol{n}$ is extended orthogonally into $\mathcal{N}$.

Proof of Proposition 7.1. From Theorem 2.3, we know that $\mathbf{Y}(t)$ will be in the range of the curl if and only if it is divergence-free and has vanishing external flux through each boundary component, as defined in (2.1).

That $\text{div}\, \mathbf{Y}_-$ is transported on U_- is essentially classical. It comes from taking the divergence of $(1.2)_1$, which yields, after a few calculations, that (since $\text{div}\, \mathbf{g} = \text{div}\, \text{curl}\, \mathbf{f} = 0$),

$$\partial_t \, \text{div}\, \mathbf{Y} + \mathbf{u} \cdot \nabla \, \text{div}\, \mathbf{Y} = 0. \tag{7.4}$$

This holds in weak form when $N = 0$, but still gives $\text{div}\, \mathbf{Y} = 0$ on $\overline{U}_-$.

To show that $\text{div}\, \mathbf{Y} = 0$ on $\overline{U}_+$, we need only show that $\text{div}\, \mathbf{Y} = 0$ on Γ_+, because (7.4) holds on all of $\overline{Q}$ and all flow lines on $\overline{U}_+$ originate on $[0, T] \times \Gamma_+$. Complicating things, however, is that values of $\mathbf{Y}$ are generated on Γ_+ by $\mathbf{H}$.

Let us start by assuming that $N \geqslant 1$. Then because

$$\text{div}[\mathbf{Y} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{Y}]^i = \partial_j[Y^i u^j - u^i Y^j] = Y^i \, \text{div}\, \mathbf{u} + [\mathbf{u} \cdot \nabla \mathbf{Y}]^i - u^i \, \text{div}\, \mathbf{Y} - [\mathbf{Y} \cdot \nabla \mathbf{u}]^i,$$

$(1.2)_1$ gives

$$\partial_t \mathbf{Y} + \text{div}[\mathbf{Y} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{Y}] + (\text{div}\, \mathbf{Y})\, \mathbf{u} = \mathbf{g}. \tag{7.5}$$

Restricting (7.5) to $[0,T] \times \Gamma_+$ and taking the inner product with $\boldsymbol{n}$, we have,

$$U^{\boldsymbol{n}} \operatorname{div} \mathbf{Y} = \mathbf{g} \cdot \boldsymbol{n} - \partial_t \mathbf{Y} \cdot \boldsymbol{n} - \operatorname{div}[\mathbf{Y} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{Y}] \cdot \boldsymbol{n} \text{ on } [0,T] \times \Gamma_+. \tag{7.6}$$

Using Lemma 7.4, below, the condition for $\operatorname{div} \mathbf{Y} = 0$ on $[0,T] \times \Gamma_+$ and hence on all of Q can be written (noting that $U^{\boldsymbol{n}}$ never vanishes on $[0,T] \times \Gamma_+$),

$$\partial_t H^{\boldsymbol{n}} + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} = 0 \text{ on } [0,T] \times \Gamma_+, \tag{7.7}$$

which is (1.14). We conclude that $\operatorname{div} \mathbf{Y} = 0$ on $\overline{U}_+$ and hence on Q. (Observe that $\operatorname{div} \mathbf{Y}_0 = 0$ on $\overline{\Omega}$ was important to conclude that $\operatorname{div} \mathbf{Y}(t) = 0$ on $S(t)$.)

It follows from (7.7) that the external flux through any component Γ_ℓ of Γ_+ vanishes, since $\operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}]$ integrates to zero on Γ_ℓ, as does $\mathbf{g} \cdot \boldsymbol{n}$ by Theorem 2.3. Then, because $\partial_t Y^n = \partial_t H^{\boldsymbol{n}} \in C^\alpha([0,T] \times \Gamma_+)$, even for $N = 0$, we have

$$\frac{d}{dt} \Phi_\ell^\Gamma(\mathbf{Y}(t)) = \frac{d}{dt} \int_{\Gamma_\ell} \left[\mathbf{g} \cdot \boldsymbol{n} - \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] \right] = 0.$$

Hence, $\Phi_\ell^\Gamma(\mathbf{Y}) = 0$, since $\mathbf{Y}_0$ is in the range of the curl.

For $N \geqslant 1$, we have sufficient regularity to obtain the vanishing of the external fluxes through any boundary component Γ_ℓ. Returning to (7.5), using that $\operatorname{div} \mathbf{Y} = 0$ and Lemma 7.4, we have, for any boundary component Γ_ℓ,

$$\partial_t Y^n + \operatorname{div}_\Gamma [Y^n \otimes \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \otimes \mathbf{Y}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} = 0 \text{ on } [0,T] \times \Gamma_\ell.$$

Now we have sufficient regularity to make the same calculation we made on components of Γ_+ to obtain $\Phi_\ell^\Gamma(\mathbf{Y}) = 0$.

Now suppose that $N = 0$. We will make an approximation argument like that in the proof of Lemma 5.8, but without approximating $\mathbf{Y}_0$.

Let $(\mathbf{u}_i)$ be the sequence in $C^{2,\alpha}_{\sigma, U_i^n}(Q)$ with $\mathbf{u}_i \to \mathbf{u}$ in $\mathring{C}^{1,\alpha}_\sigma(Q)$ given by Lemma 2.6. Let $\mathbf{Y}_i$ be the Lagrangian solution to (1.2) as in Definition 5.4 with $\mathbf{u}_i$ in place of $\mathbf{u}$, and let $U_{i,\pm}$ be the $U_\pm$ sets corresponding to $\mathbf{u}_i$. Because we did not change $\mathbf{Y}_0$, cond_0 is satisfied, though cond_1 need not be, so we only have that $\mathbf{Y}_i \in C^{1,\alpha}(U_{i,+})$ and, from Proposition 5.7 applied for $N = 0$, that $\mathbf{Y}_i \in C^\alpha(Q)$. We have sufficient regularity of $\mathbf{Y}_i$ on $U_{i,+}$, however, to conclude, after applying Lemma 7.4 and using that $\mathbf{Y}_i = \mathbf{H}$ on $[0,T] \times \Gamma_+$, that

$$\partial_t H^{\boldsymbol{n}} + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}_i^{\boldsymbol{\tau}} - U_i^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] + (\operatorname{div} \mathbf{Y}_i) U_i^{\boldsymbol{n}} = \mathbf{g} \cdot \boldsymbol{n},$$

holds classically for all $t > 0$. It follows that

$$\begin{aligned}
(\operatorname{div} \mathbf{Y}_i) U_i^{\boldsymbol{n}} &= \mathbf{g} \cdot \boldsymbol{n} - \partial_t H^{\boldsymbol{n}} - \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}_i^{\boldsymbol{\tau}} - U_i^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] \\
&= \mathbf{g} \cdot \boldsymbol{n} - \partial_t H^{\boldsymbol{n}} - \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] + \operatorname{div}_\Gamma [(U_i^{\boldsymbol{n}} - U^{\boldsymbol{n}}) \mathbf{H}^{\boldsymbol{\tau}}] \\
&\qquad + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} (\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{u}_i^{\boldsymbol{\tau}})] \\
&= \operatorname{div}_\Gamma [(U_i^{\boldsymbol{n}} - U^{\boldsymbol{n}}) \mathbf{H}^{\boldsymbol{\tau}}] + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} (\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{u}_i^{\boldsymbol{\tau}})] \\
&= (U_i^{\boldsymbol{n}} - U^{\boldsymbol{n}}) \operatorname{div}_\Gamma \mathbf{H}^{\boldsymbol{\tau}} + \nabla_\Gamma (U_i^{\boldsymbol{n}} - U^{\boldsymbol{n}}) \cdot \mathbf{H}^{\boldsymbol{\tau}} \\
&\qquad + H^{\boldsymbol{n}} \operatorname{div}_\Gamma (\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{u}_i^{\boldsymbol{\tau}}) + \nabla_\Gamma H^{\boldsymbol{n}} \cdot (\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{u}_i^{\boldsymbol{\tau}}),
\end{aligned} \tag{7.8}$$

where we used the assumption that (1.14) holds. But $\mathbf{u}_i \to \mathbf{u}$ in $\mathring{C}^{1,\alpha}_\sigma(Q)$ so, in particular, $\mathbf{u}_i \to \mathbf{u}$ in $C^\alpha([0,T]; C^{1,\alpha}(\Gamma_+))$. Hence, taking advantage of (1.13), $\operatorname{div} \mathbf{Y}_i \to 0$ in $C^\alpha(\Gamma_+)$, and arguing as above using transport, $\|\operatorname{div} \mathbf{Y}_i\|_{C^\alpha(U_{i,+})} \to 0$.

Let $\varphi \in \mathcal{D}(U_+)$. The convergence $\mathbf{u}_i \to \mathbf{u}$ in $\mathring{C}^{1,\alpha}_\sigma(Q)$ gives that $\operatorname{supp} \varphi \subseteq U_{i,+}$ for all sufficiently large i. We can see, then, from the form of $\mathbf{Y}_\pm$ and $\mathbf{Y}_{i,\pm}$ coming from (5.5), that $\mathbf{Y}_i \to \mathbf{Y}$ in $C^\alpha(\operatorname{supp} \varphi)$. Then,

$$(\operatorname{div}(\mathbf{Y}_i - \mathbf{Y}), \varphi) = -(\mathbf{Y}_i - \mathbf{Y}, \nabla \varphi) \to 0.$$

But also $(\operatorname{div} \mathbf{Y}_i, \varphi) \to 0$ since $\|\operatorname{div} \mathbf{Y}_i\|_{C^\alpha(U_{i,+})} \to 0$, so it follows that as a distribution, $\operatorname{div} \mathbf{Y} = 0$ on U_+.

The vanishing of the external fluxes on components of Γ_+ holds directly for $N = 0$ as noted above. Finally, the form of $\mathbf{Y}_\pm$ and $\mathbf{Y}_{i,\pm}$ coming from (5.5) also shows that $\mathbf{Y}_i \to \mathbf{Y}$ in $C^\alpha(V)$ for any V compactly supported in $U_- \cup ([0,T] \times (\Gamma_- \cup \Gamma_0))$, from which the vanishing of all external fluxes of $\mathbf{Y}$ follows, so, in fact, $\mathbf{Y}(t)$ remains in the range of the curl. ◻

Remark 7.3. *In the proof above we used that $\mathbf{H} \in C^{1,\alpha}([0,T] \times \Gamma_+)$ and $\mathbf{g} \in C^\alpha(Q)$ for both $N = 0$ and $N = 1$, so that we could make the approximation argument by only approximating $\mathbf{u}$ with a higher-regularity sequence. If for $N = 0$ we assumed, as would seem natural, only that $\mathbf{H} \in C^\alpha([0,T] \times \Gamma_+)$, we would also need to approximate $\mathbf{H}$ by some $\mathbf{H}_i$. But because of the factor $\nabla_\Gamma H^{\boldsymbol{n}}$ in (7.8), it would be difficult, if not impossible, to successfully complete the approximation argument to conclude that $\operatorname{div} \mathbf{Y} = 0$ when $N = 0$.*

We used Lemma 7.4 in the proof of Proposition 7.1.

Lemma 7.4. *Assume that $N \geqslant 1$. We have*

$$\operatorname{div}[\mathbf{Y} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{Y}] \cdot \boldsymbol{n} = \operatorname{div}_\Gamma[Y^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{Y}^{\boldsymbol{\tau}}] \tag{7.9}$$

with

$$Y^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{Y}^{\boldsymbol{\tau}} = H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}} \text{ on } \Gamma_+.$$

On any boundary component Γ_ℓ,

$$\int_{\Gamma_\ell} \operatorname{div}[\mathbf{Y} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{Y}] \cdot \boldsymbol{n} = \int_{\Gamma_\ell} \operatorname{div}_\Gamma[Y^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{Y}^{\boldsymbol{\tau}}] = 0. \tag{7.10}$$

Proof. Let $M := \mathbf{Y} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{Y}$, an anti-symmetric matrix-valued function. Then, working in Cartesian coordinates,

$$\operatorname{div} M \cdot \boldsymbol{n} = \partial_k M^i_k n^i = \partial_k (M^i_k n^i) - M^i_k \partial_k n^i = \operatorname{div}(M \cdot \boldsymbol{n}) - M^i_k \partial_k n^i = \operatorname{div}(M \cdot \boldsymbol{n}).$$

Here, $M^i_k \partial_k n^i = 0$ because M is antisymmetric while $\nabla \boldsymbol{n}$ is symmetric (since, as in the proof of Lemma 3.4, we can write locally, $\Gamma = \varphi^{-1}(0)$ for some $\varphi \in C^{1,\alpha}(\mathbb{R}^3)$, which we can choose so that $\nabla \varphi|_{\varphi_0^{-1}(0)} = \boldsymbol{n}$ on Γ). By (7.3),

$$\operatorname{div}(M \cdot \boldsymbol{n}) = \operatorname{div}_\Gamma(M \cdot \boldsymbol{n})|_\Gamma + \nabla((M \cdot \boldsymbol{n}) \cdot \boldsymbol{n}) \cdot \boldsymbol{n} + \kappa (M \cdot \boldsymbol{n}) \cdot \boldsymbol{n} = \operatorname{div}_\Gamma(M \cdot \boldsymbol{n})|_\Gamma,$$

since $(M \cdot \boldsymbol{n}) \cdot \boldsymbol{n} = M^j_i n^i n^j = -M^i_j n^j n^i = -M^j_i n^i n^j$, so $\nabla((M \cdot \boldsymbol{n}) \cdot \boldsymbol{n}) \cdot \boldsymbol{n} = \kappa (M \cdot \boldsymbol{n}) \cdot \boldsymbol{n} = 0$.

Finally, $M \cdot \boldsymbol{n} = Y^{\boldsymbol{n}} \mathbf{u} - U^{\boldsymbol{n}} \mathbf{Y}$, and we obtain (7.9).

Integrating by parts along the boundary using (7.1) gives (7.10), completing the proof. ◻

8. Velocity

In this section, we prove Theorem 1.4. We will show that the operator $\boldsymbol{\mathcal{V}}_c$ of Lemma 8.1 recovers the harmonic component of the velocity field corresponding to a solution to (1.2).

For a vorticity field $\boldsymbol{\mu} \in C^\alpha(Q)$, define the matrix-valued function,

$$\boldsymbol{\Omega}(\boldsymbol{\mu}) := \nabla K[\boldsymbol{\mu}] - (\nabla K[\boldsymbol{\mu}])^T,$$

noting that the nonzero components of $\boldsymbol{\Omega}(\boldsymbol{\mu})$ are those of $\pm \boldsymbol{\mu}$.

Lemma 8.1. *Assume the data has regularity $N \geqslant 0$ and let K be the Biot-Savart operator of Theorem 2.3. For a velocity field $\mathbf{u}$ and vorticity field $\boldsymbol{\mu}$, define the vector field,*

$$(\boldsymbol{\mathcal{V}}_c(\mathbf{u}, \boldsymbol{\mu}))(t) := P_{H_c}\mathbf{u}_0 + \int_0^t P_{H_c}\mathbf{f}(s)\,ds - \int_0^t P_{H_c}P_H\left(\mathbf{u}(s)\cdot\boldsymbol{\Omega}(\boldsymbol{\mu})(s)\right)\,ds.$$

Then $\boldsymbol{\mathcal{V}}_c \colon \mathring{C}_\sigma^{N+1,\alpha}(Q) \times C^{N,\alpha}(Q) \to C^{N+1,\alpha}(Q) \cap C([0,T]; H_c)$.

Proof. From the regularity of $\mathbf{u}$ and $\boldsymbol{\mu}$, $\mathbf{u}(s)\cdot\boldsymbol{\Omega}(\boldsymbol{\mu})(s) \in C^{N,\alpha}(Q)$. By Lemmas 2.1 and 2.5, it follows that $P_{H_c}P_H\left(\mathbf{u}(s)\cdot\boldsymbol{\Omega}(\boldsymbol{\mu})(s)\right) \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$. The integrals in time along with the assumed regularity of $\mathbf{u}_0$ and $\mathbf{f}$ then give that $\boldsymbol{\mathcal{V}}_c(\mathbf{u}, \boldsymbol{\mu}) \in C_\sigma^{N+1,\alpha}(Q)$. ☐

Proof of Theorem 1.4. Letting $\boldsymbol{\omega} = \mathbf{Y}$, we write (1.2) as

$$\begin{cases} \partial_t\boldsymbol{\omega} + \operatorname{div}(\boldsymbol{\omega}\otimes\mathbf{u}) - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = \mathbf{g} & \text{in } Q, \\ \boldsymbol{\omega} = \mathbf{H} & \text{on } [0,T]\times\Gamma_+, \\ \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 & \text{on } \Omega. \end{cases} \tag{8.1}$$

Now, $(8.1)_1$ holds at least in the sense of elements of $\mathcal{D}'(Q)$. We know from Theorem 1.3 that $\boldsymbol{\omega}$ is in the range of the curl, so from (2.2), we know that $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$, where

$$\mathbf{v} = K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}] + \mathbf{v}_c = \overline{\mathbf{v}} + \mathbf{v}_c + \boldsymbol{\mathcal{V}}, \quad \overline{\mathbf{v}}(t) := K[\boldsymbol{\omega}(t)] \in H_0,$$

where we defer the free choice of $\mathbf{v}_c(t) \in H_c$ until later.

A direct calculation gives

$$\operatorname{div}(\boldsymbol{\omega}\otimes\mathbf{u}) - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = \operatorname{curl}(\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot(\nabla\mathbf{v})^T).$$

Thus,

$$\operatorname{curl}\left(\partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot(\nabla\mathbf{v})^T - \mathbf{f}\right) = 0,$$

equality holding in $\mathcal{D}'(Q)$. We conclude, since $\boldsymbol{\omega} = \operatorname{curl}\overline{\mathbf{v}} = \operatorname{curl}\mathbf{v}$, that

$$\partial_t\mathbf{v} + \mathbf{u}\cdot\boldsymbol{\Omega}(\boldsymbol{\omega}) - \mathbf{f} = \partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot(\nabla\mathbf{v})^T - \mathbf{f} = -\nabla\pi + \mathbf{z}, \tag{8.2}$$

where $\mathbf{z}(t) \in H_c$ and $\nabla\pi(t) \in L^2(\Omega)$. Since $\boldsymbol{\mathcal{V}}$ is a gradient, $P_H\boldsymbol{\mathcal{V}} = 0$, so we see that $P_{H_0}P_H\mathbf{v} = \overline{\mathbf{v}}$. Letting $\overline{\mathbf{f}} = P_{H_0}\mathbf{f}$, $\mathbf{f}_c = P_{H_c}\mathbf{f}$, we first apply P_H to both sides of (8.2), giving

$$\partial_t(\overline{\mathbf{v}} + \mathbf{v}_c) + P_H\left(\mathbf{u}\cdot\boldsymbol{\Omega}(\boldsymbol{\omega})\right) - \mathbf{f} = \mathbf{z} \tag{8.3}$$

then apply P_{H_0}, P_{H_c} to give

$$\begin{gathered} \partial_t\overline{\mathbf{v}} + P_{H_0}P_H\left(\mathbf{u}\cdot\boldsymbol{\Omega}(\boldsymbol{\omega})\right) - \overline{\mathbf{f}} = 0, \\ \partial_t\mathbf{v}_c + P_{H_c}P_H\left(\mathbf{u}\cdot\boldsymbol{\Omega}(\boldsymbol{\omega})\right) - \mathbf{f}_c = \mathbf{z}. \end{gathered} \tag{8.4}$$

We now choose $\mathbf{v}_c(t) = \boldsymbol{\mathcal{V}}_c(\mathbf{u}, \boldsymbol{\omega})(t)$, from which $\mathbf{z} = 0$ follows, and (8.2) becomes (1.15).

Now suppose that $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$. Then also $\operatorname{curl}\mathbf{u} = \operatorname{curl}\mathbf{v}$, so $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{w}(t) \in H_c \cap \mathring{C}_{\sigma,0}^{N+1,\alpha}(Q)$, noting that $\mathring{C}_{\sigma,0}^{N+1,\alpha}(Q)$ of (1.6) is defined as $\mathring{C}_\sigma^{N+1,\alpha}(Q)$, but with $\mathbf{u}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, and where we appealed to Lemma 2.4. Also, $\mathbf{u}\cdot\boldsymbol{\Omega}(\boldsymbol{\omega}) = \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot(\nabla\mathbf{u})^T$, since the off-diagonal components of the antisymmetric $\nabla\mathbf{w} - (\nabla\mathbf{w})^T$ come from the components of $\operatorname{curl}\mathbf{w} = 0$. Hence, (1.15) becomes

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot(\nabla\mathbf{u})^T + \nabla\pi = \partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} + \partial_t\mathbf{w},$$

where $p = \pi - (1/2)|\mathbf{u}|^2$. If, further, $\mathbf{v} = \mathbf{u}$ then $\partial_t\mathbf{w} = 0$, and we recover (1.1) with $\mathbf{u}\cdot\boldsymbol{n} = U^{\boldsymbol{n}}$ on $[0,T]\times\Gamma$.

From Lemma 8.1, $\mathbf{v}_c \in C^{N+1,\alpha}(Q) \cap C([0,T]; H_c)$. From Lemma 2.4, $\overline{\mathbf{v}} = K_{U^n}[\boldsymbol{\omega}] \in \mathring{C}_\sigma^{N+1,\alpha}(Q)$. And from $(8.4)_1$ and Lemma 2.5, $\partial_t \overline{\mathbf{v}} \in C^{N,\alpha}(Q)$, which gives the additional time continuity to conclude that $\mathbf{v} \in C_\sigma^{N+1,\alpha}(Q)$. Returning to (8.2) (where now $\mathbf{z} = 0$), we conclude that $\nabla \pi \in C^{N,\alpha}(Q)$.

To prove uniqueness, suppose that $\partial_t \mathbf{v}_j + \mathbf{u} \cdot \nabla \mathbf{v}_j - \mathbf{u} \cdot (\nabla \mathbf{v}_j)^T + \nabla \pi_j = \mathbf{f}$ for $j = 1, 2$. Letting $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, we have $\partial_t \mathbf{v} + \mathbf{u} \cdot (\nabla \mathbf{v} - (\nabla \mathbf{v})^T) + \nabla \pi = 0$, where $\pi = \pi_1 - \pi_2$. But also, $\operatorname{curl} \mathbf{v} = \boldsymbol{\omega} - \boldsymbol{\omega} = 0$ so $\nabla \mathbf{v} - (\nabla \mathbf{v})^T = 0$, and we see that $\partial_t \mathbf{v} = -\nabla \pi$. Then $\mathbf{v} \cdot \boldsymbol{n} = 0$ so $\mathbf{v} \in H$, meaning that it must be that $\nabla \pi = 0$ and hence $\partial_t \mathbf{v} = 0$. Finally, since $\mathbf{v}(0) = \mathbf{u}_0 - \mathbf{u}_0 = 0$, we conclude that $\mathbf{v} = 0$ on Q, giving uniqueness. □

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Appendix A. Hölder space lemmas

We collect here a number of estimates in Hölder spaces.

For any $V \subseteq \mathbb{R}^d$, $d \geqslant 1$, define the classical Hölder space, $C^\alpha(V)$, with the norm

$$\|f\|_{C^\alpha(V)} := \|f\|_{L^\infty(V)} + \sup_{\mathbf{y}_1 \neq \mathbf{y}_2 \in V} \frac{|f(\mathbf{y}_1) - f(\mathbf{y}_2)|}{|\mathbf{y}_1 - \mathbf{y}_2|^\alpha}.$$

Lemma A.1. *Let $f, g \in C^\alpha(U)$. Then*

$$\begin{aligned}
\|fg\|_{C^\alpha} &\leqslant \|f\|_{C^\alpha} \|g\|_{C^\alpha}, \\
\|fg\|_{\dot{C}^\alpha} &\leqslant \|f\|_{L^\infty} \|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty} \|f\|_{\dot{C}^\alpha}, \\
\|fg\|_{C^\alpha} &\leqslant \|f\|_{L^\infty} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty} \|f\|_{\dot{C}^\alpha} \\
&\leqslant \|f\|_{L^\infty} \|g\|_{C^\alpha} + \|g\|_{L^\infty} \|f\|_{C^\alpha}.
\end{aligned}$$

Proof. These are all classical. □

Lemma A.2. *Let U, V be open subsets of Euclidean spaces, $\alpha \in (0, 1]$, and $k \geqslant 1$ an integer. If $f \in C^{k,\alpha}(U)$ and $g \in C^{k+1,\alpha}(V)$ with $g(V) \subseteq U$ then*

$$\begin{aligned}
&\|f \circ g\|_{\dot{C}^\alpha(V)} \leqslant \|f\|_{\dot{C}^\alpha(U)} \|g\|_{Lip(V)}^\alpha, \\
&\|f \circ g\|_{C^\alpha(V)} \leqslant \|f\|_{L^\infty(U)} + \|f\|_{\dot{C}^\alpha(U)} \|g\|_{Lip(V)}^\alpha \leqslant \|f\|_{C^\alpha(U)} \left[1 + \|g\|_{Lip(V)}^\alpha\right], \qquad \text{(A.1)} \\
&\|f \circ g\|_{C^{k,\alpha}(V)} \leqslant C(k) \|f\|_{C^{k,\alpha}(U)} \left[1 + \|g\|_{C^{k+1}(V)}\right]^{k+1},
\end{aligned}$$

where Lip is the homogeneous Lipschitz semi-norm and $\dot{C}^\alpha$ is the homogeneous Hölder norm.

Proof. These bounds are all classical. □

Appendix B. The continuity of the Biot-Savart law

The operator K of Theorem 2.3 allows us to recover a divergence-free vector field in $H_0 \cap H^1(\Omega)^3$ from its curl, $\boldsymbol{\omega}$. We need, however, to obtain estimates on $K[\boldsymbol{\omega}]$ in terms of $\boldsymbol{\omega}$ in various norms. To do that, we will use results from Kato, Mitrea, Ponce, and Taylor's [13]. For this, we need to explore the Hodge decomposition slightly further than we did in Section 2.

Let $\mathcal{O}_j$ be the component of $\mathbb{R}^3 \setminus \overline{\Omega}$ whose boundary is Γ_j, $j = 1, \dots, b+1$. Let $\Sigma_1, \dots, \Sigma_M$ be pairwise disjoint $C^{N,\alpha}$-regular surfaces ("admissible cuts") which, when removed from Ω render it simply connected. Let $\mathbf{v}$ lie in the space H of (1.8), so $\mathbf{v} \cdot \boldsymbol{n} \in H^{-\frac{1}{2}}(\Sigma)$. The internal flux Φ_i of $\mathbf{v}$ across Σ_i is defined to be the value of

$$\Phi_i(\mathbf{v}) := \int_{\Sigma_i} \mathbf{v} \cdot \boldsymbol{n}, \tag{B.1}$$

where the direction of the unit normal vector $\boldsymbol{n}$ to Σ_i is fixed by an arbitrarily chosen orientation to Σ_i. Because $\mathbf{v}$ is divergence-free and tangential to the boundary, it is easy to see that the internal fluxes do not depend upon the specific choices of the Σ_i. It is classical (going back in some form to Helmholtz) that

$$H_0 = \{\mathbf{v} \in H \colon \text{all internal fluxes are zero}\}.$$

Fix, arbitrarily, points $y_j \in \mathcal{O}_j$ for each $j = 1, \dots, b+1$ and define

$$g_j(x) := \nabla G(\cdot - y_j),$$

where $G(x) := -1/(4\pi|x|)$ is the fundamental solution of the Laplacian. Note, then, that $\operatorname{div} g_j$ and $\operatorname{curl} g_j$ both vanish away from y_j.

Theorem B.1. *Assume that Γ is $C^{k+1,\alpha}$-regular, $k \geqslant 0$, and let $\boldsymbol{\omega} \in C^{k,\alpha}(\Omega)$ (or $\boldsymbol{\omega} \in H^{k,p}(\Omega)$, $p \in (1,\infty)$). There exists an antisymmetric matrix-valued function $M \in C^{k+1,\alpha}(\Omega)$ (or $M \in H^{k+1,p}(\Omega)$) such that*

$$\boldsymbol{\omega} = \operatorname{div} M + \sum_j \lambda_j g_j,$$

where $\operatorname{div} M$ is the row-by-row divergence of the matrix M (observe that $\operatorname{div}\operatorname{div} M = 0$). We have the estimates,

$$\|M\|_{C^{k+1,\alpha}(\Omega)} \leqslant C\|\boldsymbol{\omega}\|_{C^{k,\alpha}(\Omega)}, \quad \|M\|_{H^{k+1,p}(\Omega)} \leqslant C\|\boldsymbol{\omega}\|_{H^{k,p}(\Omega)},$$

and $\sum_{j\geqslant 1}|\lambda_j| \leqslant C\|u\|_{L^2(\Omega)}$. Moreover, if $\boldsymbol{\omega}$ is in the range of the curl then $\lambda_j = 0$ for all j.

Proof. All these observations follow from [13], the explicit bound on M holding by the continuity of the solution operator G^R defined in the proof of Corollary 3.2 of [13] and the comments in Section 5 of [13]. If $\boldsymbol{\omega}$ is in the range of the curl then the external fluxes vanish, which gives each $\lambda_j = 0$ as we can see in (2.2) of [13] (and see the comment immediately following the proof of Proposition 3.1 of [13]). □

Corollary B.2. *Assume that* Γ *is* $C^{k+1,\alpha}$*-regular,* $k \geqslant 0$*, and let* $\boldsymbol{\omega} \in C^{k,\alpha}(\Omega)$ *(or* $\boldsymbol{\omega} \in H^{k,p}(\Omega)$*,* $p \in (1,\infty)$*) be in the range of the curl. Then there exists a unique* $\mathbf{u} \in H_0 \cap C^{k+1,\alpha}(\Omega)$ *(or* $\mathbf{u} \in H_0 \cap H^{k+1,p}(\Omega)$*) for which* $\operatorname{curl} \mathbf{u} = \boldsymbol{\omega}$*, and we have*

$$\|\mathbf{u}\|_{C^{k+1,\alpha}(\Omega)} \leqslant C\|\boldsymbol{\omega}\|_{C^{k,\alpha}(\Omega)}, \quad \|\mathbf{u}\|_{H^{k+1,p}(\Omega)} \leqslant C\|\boldsymbol{\omega}\|_{H^{k,p}(\Omega)}.$$

Proof. Let M be as in Theorem B.1 and observe that $\operatorname{div} M = \operatorname{curl} \mathbf{v}$ for $\mathbf{v} = (M_2^3, M_3^1, M_1^2)$. Solve

$$\begin{cases} \Delta p = \operatorname{div} \mathbf{v} & \text{in } \Omega, \\ \nabla p \cdot \boldsymbol{n} = \mathbf{v} \cdot \boldsymbol{n} & \text{on } \Gamma, \end{cases}$$

and let $\widetilde{\mathbf{u}} = \mathbf{v} - \nabla p$. Then $\operatorname{curl} \widetilde{\mathbf{u}} = \boldsymbol{\omega}$, $\operatorname{div} \widetilde{\mathbf{u}} = 0$, and $\widetilde{\mathbf{u}} \cdot \boldsymbol{n} = 0$ on Γ. Moreover, $\operatorname{div} \mathbf{v} \in C^{k,\alpha}(\Omega)$ and $\mathbf{v} \cdot \boldsymbol{n} \in C^{k+1,\alpha}(\Gamma)$, so elliptic estimates (as in item 3 of Lemma 2 in [14]) give $p \in C^{k+2,\alpha}(\Omega)$. Letting $\mathbf{u} = P_{H_0} \widetilde{\mathbf{u}}$, and noting that P_{H_0} is continuous in $C^{k+1,\alpha}(\Omega)$ by Lemma 2.1, we have

$$\|\mathbf{u}\|_{C^{k+1,\alpha}(\Omega)} \leqslant C\|\mathbf{v}\|_{C^{k+1,\alpha}(\Omega)} + C\|\nabla p\|_{C^{k+1,\alpha}(\Omega)} \leqslant C\|\mathbf{v}\|_{C^{k+1,\alpha}(\Omega)} \leqslant C\|\boldsymbol{\omega}\|_{C^{k,\alpha}(\Omega)}$$

by Theorem B.1. Similar estimates hold for Sobolev spaces. ☐

References

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault. Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998. 8

[2] S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov. *Boundary value problems in mechanics of nonhomogeneous fluids*, volume 22 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1990. Translated from the Russian. 2, 3, 4, 6, 15

[3] Nicolas Besse and Uriel Frisch. Geometric formulation of the Cauchy invariants for incompressible Euler flow in flat and curved spaces. *J. Fluid Mech.*, 825:412–478, 2017. 12, 13

[4] Franck Boyer and Pierre Fabrie. *Éléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, volume 52 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2006. 6

[5] Franck Boyer and Pierre Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013. 6

[6] Marco Bravin and Franck Sueur. Existence of weak solutions to the two-dimensional incompressible Euler equations in the presence of sources and sinks. *arXiv:2103.13912 [math.AP]*, 2021. 6

[7] Jason Cantarella, Dennis DeTurck, and Herman Gluck. Vector calculus and the topology of domains in 3-space. *Amer. Math. Monthly*, 109(5):409–442, 2002. 7

[8] Qingshan Chen, Ming-Cheng Shiue, and Roger Temam. The barotropic mode for the primitive equations. *J. Sci. Comput.*, 45(1-3):167–199, 2010. 6

[9] Gung-Min Gie, Makram Hamouda, Chang-Yeol Jung, and Roger M. Temam. *Singular perturbations and boundary layers*, volume 200 of *Applied Mathematical Sciences*. Springer, Cham, 2018. 6

[10] Gung-Min Gie, Makram Hamouda, and Roger Temam. Asymptotic analysis of the Stokes problem on general bounded domains: the case of a characteristic boundary. *Appl. Anal.*, 89(1):49–66, 2010. 6

[11] Gung-Min Gie, James P. Kelliher, and Anna L. Mazzucato. The 3D Euler equations with inflow, outflow and vorticity boundary conditions. *Preprint*, 2022. 1, 2, 3, 4, 5, 6, 22

[12] Makram Hamouda and Roger Temam. Some singular perturbation problems related to the Navier-Stokes equations. In *Advances in deterministic and stochastic analysis*, pages 197–227. World Sci. Publ., Hackensack, NJ, 2007. 6

[13] Tosio Kato, Marius Mitrea, Gustavo Ponce, and Michael Taylor. Extension and representation of divergence-free vector fields on bounded domains. *Math. Res. Lett.*, 7(5-6):643–650, 2000. 27

[14] Herbert Koch. Transport and instability for perfect fluids. *Math. Ann.*, 323(3):491–523, 2002. 28

[15] Alexander E. Mamontov. On the uniqueness of solutions to boundary value problems for non-stationary Euler equations. In *New directions in mathematical fluid mechanics*, Adv. Math. Fluid Mech., pages 281–299. Birkhäuser Verlag, Basel, 2010. 6

[16] Florent Noisette and Franck Sueur. Uniqueness of Yudovich's solutions to the 2D incompressible Euler equation despite the presence of sources and sinks. *arXiv:2106.11556*, 2021. 6

[17] Madalina Petcu. Euler equation in a 3D channel with a noncharacteristic boundary. *Differential Integral Equations*, 19(3):297–326, 2006. 6

[18] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970. 9

[19] Michael E. Taylor. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. 22

[20] R. Temam and X. Wang. Boundary layers associated with incompressible Navier-Stokes equations: the noncharacteristic boundary case. *J. Differential Equations*, 179(2):647–686, 2002. 6

[21] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition. 7

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