THE 3D EULER EQUATIONS WITH INFLOW, OUTFLOW AND VORTICITY BOUNDARY CONDITIONS

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Abstract. The 3D incompressible Euler equations in a bounded domain are most often supplemented with impermeable boundary conditions, which constrain the fluid to neither enter nor leave the domain. We establish well-posedness with inflow, outflow of velocity when either the full value of the velocity is specified on inflow, or only the normal component is specified along with the vorticity (and an additional constraint). We derive compatibility conditions to obtain regularity in in a Hölder space with prescribed arbitrary index, and allow multiply connected domains. Our results apply as well to impermeable boundaries, establishing higher regularity of solutions in Hölder spaces.

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Part I: Overview

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$, possibly multiply connected, having a boundary that is at least C^2 regular. We define $\boldsymbol{n}$ to be the outward unit normal vector to the boundary, $\Gamma := \partial\Omega$, and follow the convention that for any vector field $\mathbf{v}$,

$$v^{\boldsymbol{n}} := \mathbf{v} \cdot \boldsymbol{n}, \quad \mathbf{v}^{\boldsymbol{n}} := v^{\boldsymbol{n}} \boldsymbol{n}, \quad \mathbf{v}^{\boldsymbol{\tau}} := \mathbf{v} - \mathbf{v}^{\boldsymbol{n}} \text{ on } \Gamma. \tag{1.1}$$

Fixing $T > 0$, the Euler equations on $Q := (0, T) \times \Omega$ can be written,

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{on } \Omega. \end{cases} \tag{1.2}$$

Here, $\mathbf{u}$ is the velocity field of a constant-density incompressible fluid, p its scalar pressure, $\mathbf{f}$ the divergence-free external force tangential to the boundary, and $\mathbf{u}_0$ the initial velocity.

To complete the system of equations in (1.2) we impose inflow, outflow boundary conditions in the spirit of [2]. We partition the boundary Γ into three portions, Γ_+, Γ_-, and Γ_0, corresponding to inflow, outflow, and impermeability, respectively. Each portion consists of a finite number of components (with $\Gamma_0 = \emptyset$ or $\Gamma_0 = \Gamma$ allowed—see Remark 13.1). We fix a vector field $\mathbf{U}$ on $[0, T] \times \Omega$ and assume that

$$U^{\boldsymbol{n}} < 0 \text{ on } \Gamma_+, \qquad U^{\boldsymbol{n}} > 0 \text{ on } \Gamma_-, \qquad U^{\boldsymbol{n}} = 0 \text{ on } \Gamma_0. \tag{1.3}$$

We then define inflow, outflow boundary conditions as

$$\begin{cases} u^{\boldsymbol{n}} = U^{\boldsymbol{n}} & \text{on } [0, T] \times \Gamma, \\ \mathbf{u} = \mathbf{U} & \text{on } [0, T] \times \Gamma_+. \end{cases} \tag{1.4}$$

We also impose on $\mathbf{U}$ the constraint that $\int_{\Gamma_+} U^{\boldsymbol{n}} = -\int_{\Gamma_-} U^{\boldsymbol{n}}$, required to allow $\operatorname{div} \mathbf{u} = 0$.

We choose to impose inflow, outflow boundary conditions in terms of a vector field $\mathbf{U}$ defined on all of Ω because it will be productive for us to view $\mathbf{U}$ as a background flow as done in [9, 28, 32]. We will also choose $\mathbf{U}$ to be divergence-free as, shown can be done in [9], as this will be convenient, though not strictly necessary.

Defining the vorticity,

$$\boldsymbol{\omega} := \operatorname{curl} \mathbf{u},$$

applying curl to both sides of $(1.2)_1$ yields the vorticity equation,

$$\partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \mathbf{g} := \operatorname{curl} \mathbf{f}. \tag{1.5}$$

It follows from (1.5) that the vorticity is transported and stretched (pushedforward) by the flow map for $\mathbf{u}$ (when $\mathbf{g} \equiv 0$).

In particular, the vorticity is brought into the domain from the inflow boundary, making inflow, outflow substantially more difficult to treat than impermeable boundaries: the mechanism for generating vorticity on the inflow boundary must be understood and controlled. This is a key reason for using Hölder spaces, as there is no loss of regularity of the trace of the vorticity on the boundary over that in the domain.

Higher regularity solutions for inflow, outflow boundary conditions are employed, for instance, in Prandtl-type boundary layer expansions (such as [9, 32] and work in progress of the authors). The validity of such expansions for inflow, outflow boundary conditions results from a stability mechanism of injection, suction in boundary layers. These applications were the original motivation for this work: because of this, in Appendix C we give the explicit form of the compatibility conditions for those works.

The system of equations we study, then, are (1.2) with (1.4):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{on } \Omega, \\ u^{\boldsymbol{n}} = U^{\boldsymbol{n}} & \text{on } [0,T] \times \Gamma, \\ \mathbf{u} = \mathbf{U} & \text{on } [0,T] \times \Gamma_+. \end{cases} \tag{1.6}$$

We can state the main result of this paper informally as follows, where **throughout, we fix $\boldsymbol{\alpha} \in (\mathbf{0}, \mathbf{1})$**:

Theorem (Informal statement of main result)**.** *Assume that for some integer $N \geqslant 0$, $\mathbf{u}_0$ is a divergence-free vector field in the classical Hölder space $C^{N+1,\alpha}(\Omega)$, satisfies (1.4), and satisfies a compatibility condition to be described below. There is a $T > 0$ such that there exists a unique solution to (1.6) with $\operatorname{curl} \mathbf{u}(t) \in C^{N,\alpha}(\Omega)$ for all $t \in [0,T]$.*

We state our main result rigorously in Theorem 1.2, but to do so, we must define the function spaces in which we will work, determine proper conditions on the forcing, and determine the required compatibility conditions. It will be helpful, however, to first explain how the boundary conditions in $(1.6)_{4,5}$ arise.

Possible boundary conditions. Being motivational, we will argue somewhat heuristically. Some of what we observe will echo observations in [27]—in particular, the comments on an "open boundary" in Section 2 of [27] on the linearized compressible Euler equations and in Section 3 of [27] on the linearized incompressible Euler equations.

By taking the divergence of $(1.2)_1$, the pressure can be recovered from the velocity field by

$$\begin{cases} \Delta p = -\nabla \mathbf{u} \cdot (\nabla \mathbf{u})^T & \text{in } \Omega, \\ \nabla p \cdot \boldsymbol{n} = \partial_t \mathbf{u} \cdot \boldsymbol{n} - (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{1.7}$$

But also, starting from the Gromeka-Lamb form of the Euler equations, one can easily show (see Proposition 3.1) that any $(\mathbf{u}, p)$ that satisfies (1.2) must satisfy, on Γ, the identity,

$$u^{\boldsymbol{n}} \boldsymbol{\omega}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) + \mathbf{f} \right]^\perp + \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} \mathbf{u}^{\boldsymbol{\tau}}, \qquad \omega^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}. \tag{1.8}$$

Here, $\mathbf{v}^\perp = \boldsymbol{n} \times \mathbf{v}$ is the tangential vector field $\mathbf{v}$ on Γ rotated 90 degrees counterclockwise around the normal vector $\boldsymbol{n}$ when viewed from outside Ω, ∇_Γ is the tangential derivative, and $\operatorname{curl}_\Gamma$ is the curl operator on the boundary. (Appendix B gives details.)

If we impose impermeable boundary conditions, $u^{\boldsymbol{n}} \equiv 0$ on Γ, then the vorticity term disappears in (1.8) and there is no constraint on the vorticity. But on portions of the boundary where $\mathbf{u} \cdot \boldsymbol{n}$ does not vanish, (1.8) gives a relation among $\boldsymbol{\omega}$, $\nabla_\Gamma p$, $\mathbf{u} \cdot \boldsymbol{n}$, and $\mathbf{u}^{\boldsymbol{\tau}}$ on the boundary. At the same time, (1.7) gives a (global) relation between $\mathbf{u} \cdot \boldsymbol{n}$ (via its time derivative) and p. At the risk of oversimplifying, together, (1.7) and (1.8) give two relations among four quantities, so we must have an independent means of determining two of them so as to obtain the value of the other two.

To better understand the consequences of (1.8), we turn to the vorticity equation, (1.5). For impermeable boundary conditions, one can express a Lagrangian solution to the vorticity equation by introducing the flow map, $\eta(t_1, t_2; \mathbf{x})$, for $\mathbf{u}$. This flow map gives the position that a particle at $\mathbf{x} \in \overline{\Omega}$ at time t_1 will be as it moves, forward or backward, along the flow line to time t_2. Given the flow map, $\boldsymbol{\omega}(t, \mathbf{x}) := \nabla\eta(0, t; \eta(t, 0; \mathbf{x}))\boldsymbol{\omega}_0(\eta(t, 0; \mathbf{x}))$ is a Lagrangian solution. (In 2D, it would be $\omega(t, \mathbf{x}) := \omega_0(\eta(t, 0; \mathbf{x}))$.) This works, because η maps any point in $\overline{\Omega}$ to another point in $\overline{\Omega}$, so one can always evaluate $\boldsymbol{\omega}_0(\eta(t, 0; \mathbf{x}))$.

At points on the boundary at which $\mathbf{u}\cdot\boldsymbol{n} < 0$, however, the flow lines enter the domain, and we must have a way of determining, or *generating*, the vorticity so that it can be transported into the domain. From (1.8), we have at such inflow points,

$$\begin{aligned} \boldsymbol{\omega}^{\boldsymbol{\tau}} &:= \frac{1}{u^{\boldsymbol{n}}}\left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma\left(p + \frac{1}{2}|\mathbf{u}|^2\right) + \mathbf{f}\right]^\perp + \frac{1}{u^{\boldsymbol{n}}}\operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}\mathbf{u}^{\boldsymbol{\tau}}, \\ \omega^{\boldsymbol{n}} &:= \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}. \end{aligned} \tag{1.9}$$

There is, however, another constraint: As vorticity is generated on the boundary and pushed forward into the domain, the resulting vorticity must lie in the range of the curl; that is, the vector field that results must actually itself be the vorticity of some divergence-free vector field. In 2D, this is automatic, because vorticity is simply a scalar field. But in 3D, vorticity is in the range of the curl only if it is divergence-free and has vanishing fluxes across each boundary component.

Taking the divergence of (1.5) leads, after some calculation, to the conclusion that $\partial_t \operatorname{div} \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \operatorname{div} \boldsymbol{\omega} = 0$; that is, the divergence of the vorticity is transported by the flow. Since $\operatorname{div} \boldsymbol{\omega}_0 = \operatorname{div} \operatorname{curl} \mathbf{u}_0 = 0$, we need only show that $\operatorname{div} \boldsymbol{\omega} = 0$ at inflow points on the boundary. But another calculation gives that on Γ,

$$u^{\boldsymbol{n}} \operatorname{div} \boldsymbol{\omega} = \mathbf{g} \cdot \boldsymbol{n} - \partial_t \boldsymbol{\omega} \cdot \boldsymbol{n} - \operatorname{div}_\Gamma[\omega^{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{u}^{\boldsymbol{n}}\boldsymbol{\omega}^{\boldsymbol{\tau}}],$$

where $\operatorname{div}_\Gamma$ is the divergence operator on the boundary (see Appendix B). This leads to the constraint,

$$\partial_t \boldsymbol{\omega} \cdot \boldsymbol{n} + \operatorname{div}_\Gamma[\omega^{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{u}^{\boldsymbol{n}}\boldsymbol{\omega}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} = 0 \tag{1.10}$$

at inflow points. These calculations are all formal, but are worked out rigorously in detail in Section 6 of [11].

Insuring that (1.10) holds at inflow points is an issue that must be addressed regardless of the manner in which vorticity is generated on the boundary. (Dealing with the external fluxes vanishing is relatively straightforward, and is also treated in Section 6 of [11].)

Now, without prescribing at least the sign of $\mathbf{u} \cdot \boldsymbol{n}$ on the boundary, we would have to determine the regions of inflow dynamically. To avoid this considerable difficulty, we impose Dirichlet conditions for $\mathbf{u} \cdot \boldsymbol{n}$ on all of Γ. It remains to select a second condition that allows the constraints in (1.7) and (1.8) to be met. Using the value of $\mathbf{u}^{\boldsymbol{\tau}}$ at inflow points as the second condition leads to $(1.6)_{4,5}$.

It is not obvious, but specifying the full velocity field at inflow points on the boundary and generating the vorticity at inflow points via (1.9) automatically gives (1.10), as we show in Proposition 3.3. Hence, (1.10) does represent an additional constraint.

Another possibility, which we also treat in Theorem 1.4, is to specify the value of $\mathbf{u}\cdot\boldsymbol{n}$ and, at inflow points, the value of $\boldsymbol{\omega}$—so-called *vorticity boundary conditions*. This, however, does not lead to the constraint in (1.10) being automatically satisfied; rather, we must impose a restriction on our choice of $\boldsymbol{\omega}$. It is not clear how to do this in greatest generality, but by requiring that the prescribed vorticity be tangential to the inflow boundary, we obtain well-posedness nearly for free from the technology we develop to handle inflow, outflow boundary conditions.

(One could also choose to use an independent relation between $\boldsymbol{\omega}$ and $\mathbf{u}^{\boldsymbol{\tau}}$ for the second condition. This was done by Chemetov and Antontsev [7] for 2D weak solutions in vorticity form, without uniqueness, for Navier friction boundary conditions.)

Once we have points on the boundary at which $\mathbf{u} \cdot \boldsymbol{n} < 0$, we must have other points at which $\mathbf{u} \cdot \boldsymbol{n} > 0$ else the fluid could not be incompressible. Hence, we must have $0 = \int_\Gamma \mathbf{u} \cdot \boldsymbol{n} = \int_\Omega \operatorname{div} \mathbf{u}$. Reflecting upon (1.9), it would be very difficult to handle $\mathbf{u} \cdot \boldsymbol{n}$ vanishing at a point or, even worse, changing sign, especially to obtain classical solutions with higher regularity, which is our intent. To avoid this, each boundary component must have $\mathbf{u} \cdot \boldsymbol{n}$ strictly negative (inflow), strictly positive (outflow), or vanish identically. If a component has inflow, then at least one other component must have outflow.

Contrast with the analytic setting. Motivated in part by the results in [11], the authors of [18] have recently established well-posedness of the Euler equations with inflow and outflow in the analytic category. More precisely, they show existence and uniqueness of solutions in certain spaces of functions that are analytic in the tangential direction near the boundary and otherwise belong to a Sobolev space with sufficiently high index of regularity.

By working with analytic norms they can absorb the loss of a derivative at the boundary in the weak formulation of the Euler equation, this loss arising from the non-homogeneous inflow and outflow boundary conditions. Because of this, they can employ a sequence of approximating solutions based on velocity and pressure with no need to treat the behavior of the vorticity. This also allows them to directly recover the pressure from the velocity in their analytic spaces. As a consequence, they need to prescribe only the normal component of the velocity at the boundary: in their setting, no compatibility conditions are necessary, as the relation in (1.9), while it must hold, never enters into the analysis.

At the same time, working with Sobolev norms they can derive suitable *a priori* estimates by higher energy estimates. Then, the unique solution is obtained directly from the velocity-pressure formulation of the Euler equations via a Picard iteration.

By contrast, we use the transport of vorticity to establish existence as in [2], bypassing the loss of derivative at the boundary. The vorticity generated at the inflow boundary, however, requires both knowledge of the pressure and of the tangential component of the velocity field, as we can see in (1.9). Hence, we must impose an additional constraint, satisfied by imposing full inflow boundary conditions as well as compatibility conditions among the data.

Function spaces. Returning to stating our main result rigorously, we define the function spaces for our solutions. For any $N \geqslant 0$ we define the affine spaces,

$$\begin{aligned} C^{N+1,\alpha}_\sigma(\Omega) &:= \{\mathbf{u} \in C^{N+1,\alpha}(\Omega) \colon \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \boldsymbol{n} = U^{\boldsymbol{n}}(0) \text{ on } \Gamma\}, \\ S^{N+1,\alpha} &:= \{\mathbf{u} \in C^{N,\alpha}(Q) \colon \operatorname{curl} \mathbf{u} \in C^{N,\alpha}(Q), \partial_t^{N+1} \mathbf{u} \in L^\infty([0,T]; C^\alpha(\Omega)), \\ &\qquad \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \boldsymbol{n} = U^{\boldsymbol{n}} \text{ on } [0,T] \times \Gamma\}, \\ \|\mathbf{u}\|_{S^{N+1,\alpha}} &:= \|\mathbf{u}\|_{C^{N,\alpha}(Q)} + \|\operatorname{curl} \mathbf{u}\|_{C^{N,\alpha}(Q)} + \|\partial_t^{N+1} \mathbf{u}\|_{L^\infty([0,T]; C^\alpha(\Omega))}. \end{aligned} \tag{1.11}$$

Although these spaces depend on $\mathbf{U}$, for notational simplicity, we drop the $\mathbf{U}$, as it is fixed.

We also define

$$H := \{\mathbf{u} \in L^2(\Omega)^3 \colon \operatorname{div} \mathbf{u} = 0, \ \mathbf{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\} = H_0 \oplus H_c, \tag{1.12}$$

where

$$H_c := \{\mathbf{v} \in H : \operatorname{curl} \mathbf{v} = 0\}, \quad H_0 := H_c^\perp. \tag{1.13}$$

Let P_{H_c} be the projection operator from H to H_c. We call $P_{H_c}\mathbf{u}$ the *harmonic* component of $\mathbf{u}$.

We define the boundary values (via $\mathbf{U}$) and the forcing $\mathbf{f}$ for all time on $Q_\infty := [0,\infty)\times\Omega)$. We will prove existence only for short time.

Definition 1.1. *We say that the data has regularity N for an integer $N \geqslant 0$ if*

- Γ *is* $C^{N+2,\alpha}$, $\mathbf{f} \in C^{N+1,\alpha}(Q_\infty) \cap C([0,\infty); H_0)$*;*
- $\mathbf{U} \in C^{N+2,\alpha}(Q_\infty)$, $\operatorname{div}\mathbf{U} = 0$, *and (1.3) holds;*
- $U_{min} := \min\{|U^{\boldsymbol{n}}(t,\mathbf{x})| \colon (t,\mathbf{x}) \in [0,\infty)\times\Gamma_+\} > 0$*;*
- $\mathbf{u}_0 \in C^{N+1,\alpha}_\sigma(\Omega)$, $\mathbf{u}_0^{\boldsymbol{\tau}} = \mathbf{U}_0^{\boldsymbol{\tau}}$ *on* Γ_+.

We assumed that $\mathbf{U}$ has one more derivative than $\mathbf{u}$, as explained in Remark 3.2.

Compatibility conditions. The vorticity generated at the inflow boundary is carried by the flow into the interior; at the same time, the flow pushes the initial vorticity forward in time. The interaction between these two sources of vorticity may potentially lead to a singularity. The main thrust of this work is to show that it is possible to avoid such singularities, at least for short time, by imposing suitable conditions on the data. We refer to these conditions as *compatibility conditions*, satisfying two primary principles:

(1) They depend only upon the initial data, $\mathbf{U}$, and $\mathbf{f}$.
(2) They are compatible with being a solution to (1.6); that is, a solution to (1.6) could, in principle, satisfy them.

The conditions we develop will ensure regularity of the solution for short time. It remains an open question whether a regular solution persists for all time even in 2D.

Given $\mathbf{u}$ with data regularity N for some $N \geqslant 0$, we define the N^{th} compatibility condition,

$$\begin{aligned} &\text{cond}_{-1} : \mathbf{u}_0^{\boldsymbol{\tau}} = \mathbf{U}_0^{\boldsymbol{\tau}} \text{ on } \Gamma_+, \\ &\text{cond}_N : \text{cond}_{N-1} \text{ and } \partial_t^{N+1}\mathbf{U}^{\boldsymbol{\tau}}|_{t=0} = \widetilde{\partial}_t^{N+1}\mathbf{u}_0^{\boldsymbol{\tau}} \text{ on } \Gamma_+. \end{aligned} \tag{1.14}$$

For integers $n \geqslant 0$, we define $\widetilde{\partial}_t^n\mathbf{u}_0$ inductively by setting $\widetilde{\partial}_t^0\mathbf{u}_0 = \mathbf{u}_0$, while for $n \geqslant 1$, we take the time derivative of $\widetilde{\partial}_t^{n-1}\mathbf{u}$ at time zero and replace each instance of $\partial_t\mathbf{u}$ in the resulting expression by $-\mathbf{u}_0\cdot\nabla\mathbf{u}_0 - \nabla p^0 + \mathbf{f}(0)$. Here, p^0 is the value the pressure would have at time zero if $\mathbf{u}$ actually solved (1.6); that is, p^0 is the solution to the system in (1.7) at time zero:

$$\begin{cases} \Delta p^0 = -\operatorname{div}(\mathbf{u}_0\cdot\nabla\mathbf{u}_0) & \text{in } \Omega, \\ \nabla p^0\cdot\boldsymbol{n} = -\partial_t U^{\boldsymbol{n}}(0) - \mathbf{u}_0\cdot\nabla\mathbf{u}_0 & \text{on } \Gamma. \end{cases} \tag{1.15}$$

(see (1.15)). We give a more complete account of the compatibility conditions in Section 4.

For $N = 0$, (1.14) is the compatibility condition in (1.10), (1.11) of Chapter 4 of [2]:

$$\text{cond}_0 : \partial_t\mathbf{U}^{\boldsymbol{\tau}}|_{t=0} = [-\mathbf{u}_0\cdot\nabla\mathbf{u}_0 - \nabla p^0 + \mathbf{f}(0)]^{\boldsymbol{\tau}} \text{ on } \Gamma_+.$$

Main result. We can now rigorously state the main result of this paper as follows:

Theorem 1.2. *Assume the data has regularity N for some integer $N \geqslant 0$ as in Definition 1.1 and satisfies* cond_N *of (1.14). There is a $T > 0$ such that there exists a solution* $(\mathbf{u}, p)$ *to (1.6) with* $\mathbf{u} \in S^{N+1,\alpha}$ *and* ∇p *in* $L^\infty([0,T]; C^{N,\alpha}(\Omega))$*, which is unique up to an additive constant for the pressure. If $N \geqslant 1$, ∇p is also in $C^{N-1,\alpha}(Q)$.*

Remark 1.3. *It follows from the proof of Theorem 1.2 that T is bounded below by a continuous, increasing function of norms of* $(U^{\boldsymbol{n}})^{-1}$, $\mathbf{U}$, $\mathbf{f}$, *and* $\mathbf{u}_0$ *in the spaces appearing in*

Definition 1.1. The explicit form of the estimate is, however, involved and may not be optimal. It also follows from the proof that the higher regularity of $\mathbf{U}$ *and so* $\mathbf{u}$ *on the inflow boundary leads to higher time regularity of the pressure near the inflow boundary that need not, however, extend through the whole domain.*

Vorticity boundary condition. We also consider solutions $(\mathbf{u}, p, \mathbf{z})$ to the Euler equations with vorticity boundary conditions, where the value of the vorticity on the inflow boundary is given by a function $\mathbf{H}$ on $[0, T] \times \Gamma_+$:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} + \mathbf{z} & \text{in } Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{on } \Omega, \\ u^{\boldsymbol{n}} = U^{\boldsymbol{n}} & \text{on } [0, T] \times \Gamma, \\ \operatorname{curl} \mathbf{u} = \mathbf{H} & \text{on } [0, T] \times \Gamma_+. \end{cases} \tag{1.16}$$

Here, $\mathbf{z} \in H_c$ is an harmonic vector field.

As stated, this system is not yet complete, as we must have a means of determining the harmonic component P_{H_c} of $\mathbf{u}$. We can do this either (1) directly, by setting $P_{H_c}\mathbf{u} = \mathbf{u}_c$ for some harmonic vector field $\mathbf{u}_c$ and letting $\mathbf{z}$ be obtained as part of the solution or (2) indirectly, by prescribing the value of $\mathbf{z}$ and obtaining $P_{H_c}\mathbf{u}$ as part of the solution. In Theorem 1.4 we choose (1), as it allows for the uniqueness of solutions.

Theorem 1.4. *Fix* $\mathbf{u}_c \in C^{N+1,\alpha}(Q) \cap C([0, T]; H_c)$. *Assume that the data has regularity* N *for some integer* $N \geqslant 0$ *as in Definition 1.1, that* cond_N *holds, and that* $\mathbf{u}_c(0) = P_{H_c} P_H \mathbf{u}_0$. *Also assume that* $\mathbf{H} \in C^{\max\{N,1\},\alpha}([0, T] \times \Gamma_+)$ *and*

$$H^{\boldsymbol{n}} = 0, \quad \operatorname{div}_\Gamma[U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] + \operatorname{curl} \mathbf{f} \cdot \boldsymbol{n} = 0 \text{ on } [0, T] \times \Gamma_+. \tag{1.17}$$

There is a $T > 0$ *such that there exists a solution* $(\mathbf{u}, p, \mathbf{z})$ *in* $S^{N+1,\alpha} \times L^\infty([0, T]; C^{N,\alpha}(\Omega)) \times (C^{N+1,\alpha}(Q) \cap C([0, T]; H_c)$ *to (1.16) supplemented with the condition that*

$$P_{H_c}\mathbf{u} = \mathbf{u}_c \text{ on } [0, t] \times \Omega.$$

If $N \geqslant 1$, ∇p *is also in* $L^\infty([0, T]; C^{N,\alpha}(\Omega))$ *and the solution is unique up to an additive constant for the pressure. In addition,* $\mathbf{z}(0) = 0$.

The condition in Remark 2.4 reflects the constraint in (1.10).

Approaches to well-posedness of Euler equations. There are many proofs of well-posedness of the Euler equations taking different approaches. Most such proofs in Hölder spaces in a 3D domain with boundary, including this paper, and many in the whole space or a periodic domain, follow in the tradition of McGrath [23, 24] and Kato [13], in which the solution is obtained as a fixed point of an operator A derived from a linearization of the Euler equations, employing Schauder's fixed point theorem. A notable exception is [3], which establishes well-posedness in $C^{N+1,\alpha}(\Omega)$ for impermeable boundary conditions (finite time for 3D) using a more direct iteration scheme, obtaining a contraction mapping.

For inflow, outflow boundary conditions, the Schauder's fixed point theorem approach was taken in Chapter 4 of [2], which establishes Theorem 1.2 for $N = 0$ and simply connected domains. The operator A is derived from a linearization of the vorticity equation (1.5) with prescribed values on the inflow boundary (see Definition 3.6). This leads to linear compatibility conditions based on vorticity, whereas the nonlinear boundary conditions are based on the velocity. In fact, one challenge is to ensure that the nonlinear compatibility conditions at the level of the velocity imply the linear ones at the level of the vorticity.

To handle inflow, outflow boundary conditions, the authors of [2] make many adaptations to the Kato-McGrath approach, but we would identify their two key innovations as the following:

- They obtain estimates on the operator A under the simple linear compatibility condition that on the inflow boundary, the vorticity matches the prescribed inflow vorticity at time zero (akin to the Rankine-Hugoniot condition).
- They show how to achieve the needed regularity of the inflow vorticity from the pressure.

What is novel in our approach. For $N \geqslant 1$, several complications arise. We can still use the same operator A as in [2], but now the linear compatibility conditions becomes more involved (see (2.2) and (2.3)). This linear issue was resolved in [11], but deriving and relating the nonlinear compatibility conditions to the linear ones remained a significant challenge, which we address here. (In Remark 8.5, we say a few words about the analysis in [11], after we have introduced a number of the constructs involved.)

Moreover, while for $N = 0$, the linear compatibility condition implies the nonlinear compatibility condition, this is no longer the case for $N \geqslant 1$. To address this, we must restrict the domain of the operator A by imposing an additional condition on the time derivative of the initial velocity (as in (3.8)) and show that, in fact, the resulting domain is nonempty.

The estimates on the operator A that result become much more complex for the higher regularity we treat here. This is in contrast to proving existence in the full space or a periodic domain, where one can bootstrap as in Section 4.4 of [19], which takes advantage of the simple form of Biot-Savart kernel for the full space. And in 2D, where the vorticity equation has no stretching term, one can bootstrap as Marchioro and Pulvirenti do in [22] (which originates in their earlier text [21]). Instead, we must obtain existence directly in the higher-regularity spaces: the resulting estimates are therefore much more intricate than the $N = 0$ case.

Other Prior work. In addition to [2], we also drew ideas from [17], which proves well-posedness of the 3D Euler equations for impermeable boundary conditions in Hölder spaces (the equivalent of our $N = 0$ regularity). We mention also the work of Petcu [28], who presents a version of the argument in Chapter 4 of [2], specializing it to a 3D channel with a constant $\mathbf{U}$, which simplifies and makes clearer some of the arguments in [2].

Section 1.4 of [20] contains an extensive survey of results, both 2D and 3D, related to the problem we are studying here. We also mention the 2D work of Boyer and Fabrie [4, 5] and the recent works [6, 26].

Vorticity boundary conditions were studied in 2D by Yudovich in [12]. We refer in addition to the historical comments in Section 1.4 of [20] concerning partial results in 3D.

Finally, we mention the works [14, 33], which give well-posedness of solutions to the Euler equations with impermeable boundaries in Sobolev spaces.

Structure of this paper. This paper consists of three parts, along with three appendices.

In Part I, following this introduction, we begin in Section 2 by summarizing results from [11] on the linearization of (1.6), a key tool at the heart of all of our arguments. In Section 3 we detail how vorticity is generated on the inflow boundary and then define the operator A. In Section 4, we explore in-depth the nonlinear compatibility conditions cond_N as they apply to (1.6) and their counterparts for the linearized equations. We then give the proof of our main result, Theorem 1.2, in Section 5. This proof relies upon three propositions, Propositions 5.3 to 5.5: the rest of the paper is devoted to proving these propositions.

In Part II, we develop some properties related to the function space $S^{N+1,\alpha}$, then present identities and estimates on the flow map, on the vorticity generated on the boundary, and on the pressure.

In Part III, we use results primarily from the second part to prove Proposition 5.3, then leverage it to obtain Proposition 5.4. We also give the proof of Proposition 5.5. In the final section of this part, we describe how Theorem 1.4 follows from the estimates obtained in Part II.

Appendix A contains a number of estimates in Hölder spaces, some very standard, some specific to this paper. In Appendix B we construct a convenient coordinate system in an ε-neighborhood of Γ_+. We use this system to develop properties of the operators ∇_Γ, $\operatorname{div}_\Gamma$, and $\operatorname{curl}_\Gamma$ we use in the body of the paper. This allows us to treat the various calculations on the boundary in a coordinate-free manner, which makes the calculations more transparent. Finally, in Appendix C, we discuss the compatibility conditions in the special case in which $\mathbf{U}^{\boldsymbol{\tau}} \equiv 0$ and $U^{\boldsymbol{n}}$ is constant along Γ_+ (as occurs in [9, 32]).

We have structured this paper so as to allow the reader to grasp the overall structure of the proof of Theorem 1.2 without it being obscured by the many technical details. It is possible to read only Part I and get the gist of the proof. A more in-depth reading would involve at least examining the key pressure estimates in Section 10 and a reading of [11], to understand how the linear compatibility conditions arise.

On notation. Our notation, while fairly standard, has a few subtleties. If M is a matrix, M^i_n refers to the entry in row i of M, column n; v^i refers to the i^{th} entry in the vector $\mathbf{v}$, which we always treat as a column vector for purposes of multiplication. If M and N are matrices of the same dimensions then $M \cdot N := M^i_n N^i_n$, where here, and elsewhere, we use implicit sum notation. If $\mathbf{u}$ and $\mathbf{v}$ are vectors then the matrix $\mathbf{u} \otimes \mathbf{v}$ has components $[\mathbf{u} \otimes \mathbf{v}]^i_n = u^i v^n$.

We define the divergence of a matrix row-by-row, so $\operatorname{div} M$ is the column vector with components $[\operatorname{div} M]^i = \partial_\ell M^i_\ell$. Hence, $[\operatorname{div}[\mathbf{u} \otimes \mathbf{v}]]^i = \operatorname{div}[\mathbf{u} \otimes \mathbf{v}]^i = \partial_\ell(u^i v^\ell)$, where ∂_ℓ is the derivative with respect to the ℓ^{th} spatial variable. The notation ∇ means the gradient with respect to the spatial variables only; by D we mean the gradient with respect to all variables, time and space. When applied to the flow map $\eta(t_1, t_2, \mathbf{x})$, we write $\partial_{t_1}\eta$, $\partial_{t_2}\eta$ to mean the derivative with respect to the first, second time variable. Finally, for vector fields $\mathbf{u}$ and $\mathbf{v}$, we will interchangeably write $\mathbf{u} \cdot \nabla \mathbf{v}$ and $\nabla \mathbf{v}\, \mathbf{u}$, as they both are vectors with i^{th} component $u^m \partial_m v^i$.

For any tangent vector field $\mathbf{v}$ on Γ, $\mathbf{v}^\perp = \boldsymbol{n} \times \mathbf{v}$ is the tangent vector field $\mathbf{v}$ on Γ rotated 90 degrees counterclockwise around the normal vector $\boldsymbol{n}$ when viewed from outside Ω. We write the gradient, divergence, and curl operators on the boundary as ∇_Γ, $\operatorname{div}_\Gamma$, and $\operatorname{curl}_\Gamma$, as defined in Appendix B.

When we write that a function is in a Hölder space $C^{k,\alpha}(U)$ (defined in Section 7) we mean not just that it has the given regularity but that its norm is finite. Since a function in $C^{k,\alpha}(U)$ extends uniquely to a function in $C^{k,\alpha}(\overline{U})$, this will rarely have an impact.

2. The linearized problem

The linearized Euler equations corresponding to the vorticity form of (1.6)$_1$ are

$$\begin{cases} \partial_t \overline{\boldsymbol{\omega}} + \mathbf{u} \cdot \nabla \overline{\boldsymbol{\omega}} - \overline{\boldsymbol{\omega}} \cdot \nabla \mathbf{u} = \mathbf{g} & \text{in } Q, \\ \overline{\boldsymbol{\omega}} = \mathbf{H} & \text{on } [0,T] \times \Gamma_+, \\ \overline{\boldsymbol{\omega}}(0) = \overline{\boldsymbol{\omega}}_0 & \text{on } \Omega. \end{cases} \tag{2.1}$$

Here, $\mathbf{H}$ is given on $[0,T] \times \Gamma_+$, $\overline{\boldsymbol{\omega}}_0$ is given on Ω, $\mathbf{u}$ and $\mathbf{g}$ are **given** on Q, and (2.1) is to be solved for $\overline{\boldsymbol{\omega}}$. In application, we will set $\overline{\boldsymbol{\omega}}_0 = \boldsymbol{\omega}_0 := \operatorname{curl} \mathbf{u}(0)$, though then $\overline{\boldsymbol{\omega}}(t) \neq \operatorname{curl} \mathbf{u}(t)$ in general for $t > 0$.

We employ the following three types of solution to (2.1):

(1) *Classical Eulerian* or simply *classical* solutions to (2.1), by which we mean that (2.1)$_1$ holds pointwise, and each term is continuous on $\overline{Q}$.

(2) *Weak Eulerian solutions*, defined as follows:

Definition 2.1. *We say that $\overline{\boldsymbol{\omega}} \in C(\overline{Q})$ is a weak (Eulerian) solution to (2.1) if $\overline{\boldsymbol{\omega}} = \mathbf{H}$ on $[0,T] \times \Gamma_+$ pointwise, $\overline{\boldsymbol{\omega}}(0) = \overline{\boldsymbol{\omega}}_0$, and $\partial_t \overline{\boldsymbol{\omega}} + \operatorname{div}(\overline{\boldsymbol{\omega}} \otimes \mathbf{u}) - \overline{\boldsymbol{\omega}} \cdot \nabla \mathbf{u} = \mathbf{g}$ in $\mathcal{D}'(Q)$.*

Note that $\overline{\boldsymbol{\omega}}$ has sufficient time and boundary regularity that we do not need to enforce the initial and boundary conditions weakly. Also, $\overline{\boldsymbol{\omega}} \otimes \mathbf{u}$ is a regular distribution, so $\operatorname{div}(\overline{\boldsymbol{\omega}} \otimes \mathbf{u})$ is a distribution even for $N = 0$.

(3) *Lagrangian solutions*, are obtained by pushing forward the vorticity by the flow map, including the vorticity generated on the inflow boundary. Because we must first introduce some concepts related to this inflow, we defer to Definition 8.4.

To allow $C^{N,\alpha}$ solutions to (2.1), we must impose linear compatibility condition, lincond_N, defined for $N = 0, 1, \dots,$ as follows:

$$\begin{aligned} &\text{lincond}_0: \quad \mathbf{H}(0) = \boldsymbol{\omega}_0 \text{ on } \Gamma_+, \\ &\text{lincond}_1: \quad \text{lincond}_0 \text{ and } \partial_t \mathbf{H}|_{t=0} = \boldsymbol{\omega}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 + \mathbf{g}(0) \text{ on } \Gamma_+, \end{aligned} \tag{2.2}$$

where $\mathbf{u}_0 := \mathbf{u}(0)$. In lincond_1, we formally replaced $\partial_t \overline{\boldsymbol{\omega}}(0)$ with the value it would have were $\overline{\boldsymbol{\omega}}$ an actual classical solution to (2.1) at time zero. Continuing this process inductively on higher derivatives defines a formal operator $\widetilde{\partial}_t^n$ (see Definition 4.1 for the details). Then we set, for all $N \geqslant 2$,

$$\text{lincond}_N: \quad \text{lincond}_{N-1} \text{ and } \partial_t^N \mathbf{H}|_{t=0} = \widetilde{\partial}_t^N \boldsymbol{\omega}_0 \text{ on } \Gamma_+. \tag{2.3}$$

Theorem 2.2 ([11]). *Assume that the data has regularity N for some $N \geqslant 0$ and that*

- $\mathbf{g} := \operatorname{curl} \mathbf{f}$,
- $\mathbf{u} \in S^{N+1,\alpha}$,
- $\mathbf{H} \in C^{\max\{N,1\},\alpha}([0,T] \times \Gamma_+)$,
- lincond_N *holds,*
- $\boldsymbol{\omega}_0$ *is in the range of the curl, by which we mean that* $\boldsymbol{\omega}_0 \in \operatorname{curl}(H^1(\Omega)^3)$,
- *The following condition on* $\mathbf{H}$ *holds on* $(0,T] \times \Gamma_+$*:*

$$\partial_t H^{\boldsymbol{n}} + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} = 0. \tag{2.4}$$

There exists a solution $\overline{\boldsymbol{\omega}}$ to (2.1) in $C^{N,\alpha}(Q)$, such that $\overline{\boldsymbol{\omega}}(t)$ is in the range of the curl for all $t \in [0,T]$. When $N \geqslant 1$, the solution is classical Eulerian and unique. When $N = 0$, the *solution is Lagrangian and is also the unique weak Eulerian solution as in Definition 2.1 for which* $\overline{\boldsymbol{\omega}}(t)$ *is in the range of the curl for all* $t \in [0,T]$*.*

Moreover, there exists a unique $\mathbf{v} \in S^{N+1,\alpha}$ *with* $\operatorname{curl}\mathbf{v} = \overline{\boldsymbol{\omega}}$ *and* $\mathbf{v}(0) = \mathbf{u}_0$*, and a unique mean-zero pressure field* π *with* $\nabla\pi$ *in* $L^\infty([0,T]; C^{N,\alpha}(\Omega))$ *and, if* $N \geqslant 1$*, also in* $C^{N-1,\alpha}(Q)$*, satisfying*

$$\partial_t \mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot(\nabla\mathbf{v})^T = \partial_t\mathbf{v} + \mathbf{u}\cdot\overline{\boldsymbol{\Omega}} = -\nabla\pi + \mathbf{f}, \tag{2.5}$$

where the antisymmetric matrix $\overline{\boldsymbol{\Omega}} = \nabla\mathbf{v} - (\nabla\mathbf{v})^T$*, whose nonzero components form the vector* $\overline{\boldsymbol{\omega}}$*.*

Remark 2.3. *In [11], the stronger condition that* $\mathbf{u} \in C^{N+1,\alpha}(Q)$ *was assumed, but* $\mathbf{u} \in S^{N+1,\alpha}$ *suffices. This is because for the linearized problem it is only the regularity of the flow map for* $\mathbf{u}$ *that is needed, and as we will see in Section 8,* $\mathbf{u} \in S^{N+1,\alpha}$ *is sufficient. It was concluded in [11] that* $\mathbf{v} \in C^{N+1,\alpha}(Q)$*, but the argument there only gives* $\mathbf{v} \in S^{N+1,\alpha}$ *(see Corollary 11.4). Also, [11] states the requirement that* $\mathbf{g}$ *have, in effect, one more derivative of regularity in space than in time, but in fact does not use that additional regularity. Hence, we need only assume that* $\mathbf{g} \in C^{N,\alpha}(Q)$*, which follows from Definition 1.1.*

Remark 2.4. *The condition in (2.4) is the linear analog of (1.10), required to insure that* $\overline{\boldsymbol{\omega}}(t)$ *lies in the range of the curl. As applied to the solution of the linearized problem given by Theorem 2.2, (2.4) is a condition on the data, not on the solution, since* $\mathbf{u}$ *is given. Applied to the fully nonlinear problem with* $H = \operatorname{curl}\mathbf{u}$*, however, the appearance of* $\mathbf{u}^{\boldsymbol{\tau}}$ *in (2.4) makes (2.4) a condition on the solution. Eliminating the term involving* $\mathbf{u}^{\boldsymbol{\tau}}$ *by requiring that the normal component of the vorticity on inflow vanish gives (1.17), which is a condition on the data:* $\mathbf{u}_0$*,* $\mathbf{f}$*,* $\mathbf{U}$*, and* $\mathbf{H}$*.*

In what follows, we will use $\overline{\boldsymbol{\omega}}$ as a Lagrangian solution, but we will need to estimate $\mathbf{v}$, which is obtained from the Eulerian solution. Hence, it is crucial that Eulerian and Lagrangian solutions agree.

3. Vorticity on inflow, and the operator A

At the end of this section, we will use a solution to (2.1) to define an operator A whose fixed point is a solution to the fully nonlinear Euler equations, (1.6). To do this, we first show in Proposition 3.1 how vorticity is generated on the inflow boundary if $(\mathbf{u}, p)$ is a classical solution to (1.6).

Proposition 3.1. *Assume that* $(\mathbf{u}, p)$ *satisfies* $(1.6)_1$ *in a classical sense and let* $\boldsymbol{\omega} := \operatorname{curl}\mathbf{u}$*. Then on* $[0,T]\times\Gamma$*,*

$$u^{\boldsymbol{n}}\boldsymbol{\omega}^{\boldsymbol{\tau}} = \left[-\partial_t\mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma\left(p + \frac{1}{2}|\mathbf{u}|^2\right) + \mathbf{f}\right]^\perp + \left(\operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}\right)\mathbf{u}^{\boldsymbol{\tau}}, \qquad \omega^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}.$$

Here, ∇_Γ *is the tangential derivative, and* $\operatorname{curl}_\Gamma$ *is the curl operator on the boundary. (See Appendix B.)*

Proof. As on p. 155 of [2], we start with the Gromeka-Lamb form of the Euler equations,

$$\partial_t\mathbf{u} + \nabla\left(p + \frac{1}{2}|\mathbf{u}|^2\right) - \mathbf{u}\times\boldsymbol{\omega} - \mathbf{f} = 0. \tag{3.1}$$

The equivalence of (3.1) and $(1.6)_1$ follows from the identity,

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\mathbf{u}\times\boldsymbol{\omega} + \frac{1}{2}\nabla|\mathbf{u}|^2, \tag{3.2}$$

which holds as long as $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$.

From Lemma B.2

$$[\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}} = u^{\boldsymbol{n}}[\boldsymbol{\omega}^{\boldsymbol{\tau}}]^{\perp} - \omega^{\boldsymbol{n}}[\mathbf{u}^{\boldsymbol{\tau}}]^{\perp},$$

so restricting (3.1) to $[0, T] \times \Gamma_+$, we have

$$\partial_t \mathbf{u}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) - u^{\boldsymbol{n}}[\boldsymbol{\omega}^{\boldsymbol{\tau}}]^{\perp} + \omega^{\boldsymbol{n}}[\mathbf{u}^{\boldsymbol{\tau}}]^{\perp} - \mathbf{f}^{\boldsymbol{\tau}} = 0.$$

Hence, since $(\mathbf{v}^{\perp})^{\perp} = -\mathbf{v}$ for any tangent vector $\mathbf{v}$,

$$u^{\boldsymbol{n}} \boldsymbol{\omega}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^{\perp} + \omega^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}}.$$

The proof is completed by observing that $\omega^{\boldsymbol{n}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}$ by (B.2). □

We see from Proposition 3.1 that for a solution to $(1.6)_{1\text{-}4}$ with $\boldsymbol{\omega} := \operatorname{curl} \mathbf{u}$, we have

$$\boldsymbol{\omega} = \mathbf{W}[\mathbf{u}, p] \text{ on } [0, T] \times \Gamma_+, \tag{3.3}$$

where, using that $\mathbf{u}^n = \mathbf{U}^n \neq 0$ on Γ_+, $\mathbf{W}[\mathbf{u}, p]$ is defined on $[0, T] \times \Gamma_+$ by

$$\begin{aligned} \mathbf{W}^{\boldsymbol{\tau}}[\mathbf{u}, p] &:= \frac{1}{U^{\boldsymbol{n}}} \left[-\partial_t \mathbf{u}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p + \frac{1}{2}|\mathbf{u}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^{\perp} + \frac{1}{U^{\boldsymbol{n}}} \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} \mathbf{u}^{\boldsymbol{\tau}}, \\ W^{\boldsymbol{n}}[\mathbf{u}, p] &:= \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}. \end{aligned} \tag{3.4}$$

Now let $\mathbf{u}$ be any element of $S^{N+1,\alpha}$, not necessarily a solution of (1.6). We seek to define a function $\mathbf{H}$ in $C^{N,\alpha}([0, T] \times \Gamma_+)$ as a modification of the expression for $\mathbf{W}[\mathbf{u}, p]$ in such a way that when the data has regularity N, at least the following two properties hold:

(P1) $\mathbf{H}$ at time zero can be defined in terms of $\mathbf{u}_0$, $\mathbf{f}$, and $\mathbf{U}$ only.

(P2) If $(\mathbf{u}, p)$ solves $(1.6)_{1\text{-}4}$ and the vorticity $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ satisfies $\boldsymbol{\omega} = \mathbf{H}$ on $[0, T] \times \Gamma_+$ then $(\mathbf{u}, p)$ satisfies $(1.6)_5$ as well—and so solves (1.6).

We define the function $\mathbf{H}$ for all $N \geqslant 0$ as done in [2] for $N = 0$. First construct a "regularized pressure" p_r from $\mathbf{u}$ as the unique mean-zero solution to

$$\begin{cases} \Delta p_r = -\operatorname{div}(\mathbf{u} \cdot \nabla \mathbf{u}) & \text{in } \overline{Q}, \\ \nabla p_r \cdot \boldsymbol{n} = -\partial_t U^{\boldsymbol{n}} - N[\mathbf{u}] & \text{on } [0, T] \times \Gamma, \end{cases} \tag{3.5}$$

where

$$N[\mathbf{u}] := \begin{cases} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} & \text{on } [0, T] \times (\Gamma_- \cup \Gamma_0), \\ (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} + \operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}})) & \text{on } [0, T] \times \Gamma_+. \end{cases} \tag{3.6}$$

Though not so evident here, using $N[\mathbf{u}]$ in place of $(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n}$ regularizes the pressure, as we will explain in Section 9. But it is clear from its definition that if $\mathbf{u}$ satisfies $(1.6)_5$, that is, $\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{U}^{\boldsymbol{\tau}}$ on Γ_+, then $N[\mathbf{u}] = (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n}$ on $[0, T] \times \Gamma$, so that $\nabla p_r = \nabla p$ on $\overline{Q}$, where p, given by the system (1.7), is the "true pressure."

Finally, define $\mathbf{H}$ on $[0, T] \times \Gamma_+$ by replacing $\mathbf{u}^{\boldsymbol{\tau}}$ with $\mathbf{U}^{\boldsymbol{\tau}}$ in all terms in the expression for $\mathbf{W}[\mathbf{u}, p]$ having a derivative on $\mathbf{u}^{\boldsymbol{\tau}}$. This gives

$$\begin{aligned} \mathbf{H}^{\boldsymbol{\tau}} &:= \frac{1}{U^{\boldsymbol{n}}} \left[-\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p_r + \frac{1}{2}|\mathbf{U}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^{\perp} + \frac{1}{U^{\boldsymbol{n}}} \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} \mathbf{u}^{\boldsymbol{\tau}}, \\ H^{\boldsymbol{n}} &:= \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}}, \end{aligned} \tag{3.7}$$

and we see that property $(P1)$ of $\mathbf{H}$ holds. We will show property (P2) in Proposition 5.5.

Remark 3.2. *Because we assumed that* $\mathbf{U}$ *has higher regularity than* $\mathbf{u}$, *the function* $\mathbf{H}$ *has one more derivative than* $\mathbf{W}[\mathbf{u}, p]$ *in (3.4). This higher regularity will be needed to obtain estimates on* p_r *in Section 10; it is needed as well to solve the linearized problem in Theorem 2.2, though only for* $N = 0$.

The next proposition shows that our choice of $\mathbf{H}$ does, in fact, satisfy the constraint in (2.4) that is necessary to ensure $\operatorname{curl} \mathbf{u} = \boldsymbol{\omega}$.

Proposition 3.3. *Assume that the data has regularity* 0 *as in Definition 1.1. For* $\mathbf{u} \in S^{1,\alpha}$, *the condition in (2.4) is satisfied on* $(0, T] \times \Gamma_+$.

Proof. From (3.7) and using that $\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = H^{\boldsymbol{n}}$ we have

$$U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}} - H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p_r + \frac{1}{2} |\mathbf{U}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^\perp.$$

By (B.2), $\operatorname{div}_\Gamma \mathbf{v} = -\operatorname{div}_\Gamma((\mathbf{v}^\perp)^\perp) = \operatorname{curl}_\Gamma \mathbf{v}^\perp$ for any tangent vector $\mathbf{v}$. Hence,

$$\begin{aligned} \partial_t H^{\boldsymbol{n}} + \operatorname{div}_\Gamma [H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}}] - \mathbf{g} \cdot \boldsymbol{n} &= \partial_t H^{\boldsymbol{n}} + \operatorname{curl}_\Gamma [(H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} - U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}})^\perp] - \mathbf{g} \cdot \boldsymbol{n} \\ &= \partial_t \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} - \partial_t \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} + \mathbf{g} \cdot \boldsymbol{n} - \mathbf{g} \cdot \boldsymbol{n} = 0, \end{aligned}$$

where $\operatorname{curl}_\Gamma \mathbf{f}^{\boldsymbol{\tau}} = (\operatorname{curl} \mathbf{f}) \cdot \boldsymbol{n} = \mathbf{g} \cdot \boldsymbol{n}$ by (B.2). This gives (2.4). □

The operator A**.** Our proof of Theorem 1.2 in Section 5 will involve establishing the existence of a fixed point of an operator, which we denote A. We first define its domain:

Definition 3.4 (Domain of A)**.** *Assume that the data has regularity* N *as in Definition 1.1 and fix* $\mathbf{u}_0 \in C^{N+1,\alpha}_\sigma(\Omega)$ *satisfying* cond_N. *Define*

$$\operatorname{Dom}_N(A) := \{\mathbf{u} \in S^{N+1,\alpha} \colon \mathbf{u}(0) = \mathbf{u}_0,\ \partial_t^n \mathbf{u}|_{t=0} = \widetilde{\partial}_t^n \mathbf{u}_0 \text{ on } \overline{\Omega},\ 0 \leqslant n \leqslant N\}, \tag{3.8}$$

where $\widetilde{\partial}_t^n \mathbf{u}_0$, *which appears in* cond_N *of (1.14), will be defined in detail in Definition 4.1.*

Remark 3.5. *The condition in* $\operatorname{Dom}_N(A)$ *that* $\partial_t^n \mathbf{u}(0) = \widetilde{\partial}_t^n \mathbf{u}_0$ *on* $\overline{\Omega}$ *for all* $1 \leqslant n \leqslant N$ *arises from the need to show that the nonlinear compatibility conditions imply the linear compatibility conditions for velocities in* $\operatorname{Dom}_N(A)$*—as we will show in Proposition 4.6.*

Definition 3.6 (Operator A)**.** *Assume that the data has regularity* N *as in Definition 1.1 and fix* $\mathbf{u}_0 \in C^{N+1,\alpha}_\sigma(\Omega)$ *satisfying* cond_N. *Let* $\mathbf{u} \in \operatorname{Dom}_N(A)$ *and define* $\mathbf{H}$ *as in (3.7). Let* $\overline{\boldsymbol{\omega}} \in C^{N,\alpha}(Q)$ *be the unique solution to (2.1) with* $\overline{\boldsymbol{\omega}}_0 = \boldsymbol{\omega}_0 = \operatorname{curl} \mathbf{u}_0$ *given by Theorem 2.2 (see Remark 3.7), with* $\mathbf{v}, \pi$ *the corresponding velocity field* $\mathbf{v} \in S^{N+1,\alpha}$ *and pressure* π *with* $\operatorname{curl} \mathbf{v} = \overline{\boldsymbol{\omega}}$ *satisfying (2.5). Define*

$$A\mathbf{u} := \mathbf{v}, \tag{3.9}$$

and define also

$$\Lambda \mathbf{u} := \operatorname{curl} A\mathbf{u} = \overline{\boldsymbol{\omega}}. \tag{3.10}$$

Remark 3.7. *In Definition 3.6, we use that* $\operatorname{lincond}_N$ *is satisfied for any* $\mathbf{u} \in \operatorname{Dom}_N(A)$ *as in Remark 3.5, and that (2.4) is satisfied by Proposition 3.3. This allows us to apply Theorem 2.2 to obtain* $\overline{\boldsymbol{\omega}}$, $\mathbf{v}$, *and* π *in the given spaces.*

4. Compatibility conditions

If $(\mathbf{u}, p)$ is a classical solution to (1.6)$_{1\text{-}4}$ and $\boldsymbol{\omega} := \operatorname{curl} \mathbf{u}$, then, of course,

$$\begin{aligned} \partial_t \boldsymbol{\omega}(0) &= \boldsymbol{\omega}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 + \mathbf{g}(0), \\ \partial_t \mathbf{u}(0) &= -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla p(0) + \mathbf{f}(0), \end{aligned} \tag{4.1}$$

where $\mathbf{g} := \operatorname{curl} \mathbf{f}$. This simple observation is behind both cond_N of (1.14) and $\operatorname{lincond}_N$ of (2.3), which are based upon applying ∂_t, n times, each time replacing $\partial_t \mathbf{u}$ or $\partial_t \boldsymbol{\omega}$ at time zero with the relation in (4.1), thereby replacing all time derivatives with spatial derivatives. This produces expressions, $\widetilde{\partial}_t^n \mathbf{u}_0$ and $\widetilde{\partial}_t^n \boldsymbol{\omega}_0$, which are equal to $\partial_t^n \mathbf{u}|_{t=0}$ and $\partial^n \boldsymbol{\omega}|_{t=0}$, respectively, for any actual solution to the Euler equations having sufficient regularity.

We make this process of constructing $\widetilde{\partial}_t^n \mathbf{u}_0$ and $\widetilde{\partial}_t^n \boldsymbol{\omega}_0$ precise in Definition 4.1. We stress that in this definition, we assume of $\mathbf{u}$ only that it lies in $S^{N+1,\alpha}$ with $\mathbf{u}(0) = \mathbf{u}_0$.

Definition 4.1. *Assume that the data has regularity $N \geqslant 0$ as in Definition 1.1 and let $\mathbf{u} \in S^{N+1,\alpha}$ with $\mathbf{u}(0) = \mathbf{u}_0$. In accord with (4.1), we define*

$$\widetilde{\partial}_t \mathbf{u}_0 := -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla p^0 + \mathbf{f}(0), \quad \widetilde{\partial}_t \boldsymbol{\omega}_0 := -\mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 + \boldsymbol{\omega}_0 \cdot \nabla \mathbf{u}_0 + \mathbf{g}(0),$$

where p^0 satisfies (1.15). We then define (recall that $\mathbf{g} := \operatorname{curl} \mathbf{f}$)

$$\begin{aligned} \widetilde{\partial}_t^2 \mathbf{u}_0 &:= -\widetilde{\partial}_t (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) - \nabla \widetilde{\partial}_t p^0 + \partial_t \mathbf{f}|_{t=0}, \\ \widetilde{\partial}_t^2 \boldsymbol{\omega}_0 &:= -\widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 - \mathbf{u}_0 \cdot \nabla \widetilde{\partial}_t \boldsymbol{\omega}_0 + \widetilde{\partial}_t \boldsymbol{\omega}_0 \cdot \nabla \mathbf{u}_0 + \boldsymbol{\omega}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0 + \partial_t \mathbf{g}|_{t=0}, \end{aligned} \tag{4.2}$$

where

$$\widetilde{\partial}_t (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) := \widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0, \tag{4.3}$$

and define $\widetilde{\partial}_t p^0$ to be the unique mean-zero solution to (see Remark 4.2, below)

$$\begin{cases} \Delta \widetilde{\partial}_t p^0 = -\operatorname{div} \widetilde{\partial}_t (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) & \text{in } \Omega, \\ \nabla \widetilde{\partial}_t p^0 \cdot \boldsymbol{n} = -\partial_t^2 U^{\boldsymbol{n}}|_{t=0} - \widetilde{\partial}_t (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) \cdot \boldsymbol{n} & \text{on } \Gamma. \end{cases}$$

We note that

$$\begin{aligned} \widetilde{\partial}_t^2 \mathbf{u}_0 = -(-\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla p^0 + \mathbf{f}(0)) \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla (-\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla p^0 + \mathbf{f}(0)) \\ - \nabla \widetilde{\partial}_t p^0 + \partial_t \mathbf{f}|_{t=0}. \end{aligned}$$

For $\widetilde{\partial}_t^n$, we repeat this process inductively, up to order $N+1$ for $\widetilde{\partial}_t \mathbf{u}$ and order N for $\widetilde{\partial}_t \boldsymbol{\omega}_0$.

Remark 4.2. *In the inductive extension of $\widetilde{\partial}_t^n p^0$ in Definition 4.1, $\widetilde{\partial}_t^n p^0$ is the unique mean-zero solution to*

$$\begin{cases} \Delta \widetilde{\partial}_t^n p^0 = -\operatorname{div} \widetilde{\partial}_t^n (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) & \text{in } \Omega, \\ \nabla \widetilde{\partial}_t^n p^0 \cdot \boldsymbol{n} = -\partial_t^{n+1} U^{\boldsymbol{n}}|_{t=0} - \widetilde{\partial}_t^n (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) \cdot \boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{4.4}$$

Then

$$\int_\Gamma \left[\partial_t^n U^{\boldsymbol{n}}(0) + \widetilde{\partial}_t^n (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) \cdot \boldsymbol{n} \right] = \int_\Omega \operatorname{div} \widetilde{\partial}_t^n (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0),$$

since $\operatorname{div} \mathbf{U} = 0$. Hence, (4.4) is solvable. Also, from the manner in which $\widetilde{\partial}_t^n p^0$ was defined, $\widetilde{\partial}_t^n \mathbf{u}_0 \cdot \boldsymbol{n} = \partial_t^n U^{\boldsymbol{n}}(0)$ on Γ.

In Definition 4.1, $\widetilde{\partial}_t^n$ does not represent a derivative. Rather, it is a shorthand notation to properly account for the combinatorial nature of lincond_N and cond_N.

Proposition 4.3 shows that, in effect, $\widetilde{\partial}_t^n \operatorname{curl} \mathbf{u} = \operatorname{curl} \widetilde{\partial}_t^n \mathbf{u}$.

Proposition 4.3. *Let* $\mathbf{u}$ *and* $\widetilde{\partial}_t^n$ *be as in Definition 4.1. Then*

$$\begin{aligned} &\operatorname{div} \widetilde{\partial}_t^n \mathbf{u}_0 = 0 && \text{for all } 0 \leqslant n \leqslant N+1, \\ &\widetilde{\partial}_t^n \boldsymbol{\omega}_0 = \operatorname{curl} \widetilde{\partial}_t^n \mathbf{u}_0 && \text{for all } 0 \leqslant n \leqslant N. \end{aligned} \tag{4.5}$$

Proof. We constructed the pressure p^0 in Definition 4.1 so that $\operatorname{div} \widetilde{\partial}_t^n \mathbf{u}_0 = 0$. Then, for $n = 1$, $(4.5)_2$ follows from the identity, $\operatorname{curl}(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \nabla p^0) = \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 - \boldsymbol{\omega}_0 \cdot \nabla \mathbf{u}_0$.

For $n = 2$, we will use two vector identities that hold for vector fields $\mathbf{a}, \mathbf{b} \in C^2(\overline{\Omega})$ and follow from direct calculation:

$$\begin{aligned} &\mathbf{a} \cdot (\nabla \mathbf{b})^T + \mathbf{b} \cdot (\nabla \mathbf{a})^T = \nabla(\mathbf{a} \cdot \mathbf{b}), \\ &\operatorname{curl}(\mathbf{a} \cdot \nabla \mathbf{b} + \mathbf{b} \cdot (\nabla \mathbf{a})^T) = \mathbf{a} \cdot \nabla \operatorname{curl} \mathbf{b} - \operatorname{curl} \mathbf{a} \cdot \nabla \mathbf{b} \quad (\text{if } \operatorname{div} \mathbf{a} = 0). \end{aligned} \tag{4.6}$$

Then,

$$\begin{aligned} \widetilde{\partial}_t^2 \boldsymbol{\omega}_0 &= -\widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 - \mathbf{u}_0 \cdot \nabla \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}_0) + \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}_0) \cdot \nabla \mathbf{u}_0 + \boldsymbol{\omega}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0 + \mathbf{g}(0) \\ &= \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}_0) \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}_0) + \boldsymbol{\omega}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0 - \widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \boldsymbol{\omega}_0 + \mathbf{g}(0) \\ &= -\operatorname{curl}(\widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \widetilde{\partial}_t \mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T) - \operatorname{curl}(\mathbf{u}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0 + \mathbf{u}_0 \cdot (\nabla \widetilde{\partial}_t \mathbf{u}_0)^T) + \mathbf{g}(0) \\ &= \operatorname{curl}(-\widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0 + \partial_t \mathbf{f}(0)) - \operatorname{curl}(\widetilde{\partial}_t \mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T + \mathbf{u}_0 \cdot (\nabla \widetilde{\partial}_t \mathbf{u}_0)^T) \\ &= \operatorname{curl}(-\widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{u}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0 - \nabla \widetilde{\partial}_t p^0 + \partial_t \mathbf{f}(0)) = \operatorname{curl} \widetilde{\partial}_t^2 \mathbf{u}_0. \end{aligned}$$

The first equality is $(4.2)_2$, the second equality is a rearrangement of it. The third equality used $(4.6)_2$, the fourth equality rearranges terms. The fifth equality uses $(4.6)_2$ and $\operatorname{curl} \nabla = 0$. The final equality uses $(4.6)_1$.

Equality in $(4.5)_2$ follows inductively for higher values of n. □

For the linearized problem in (2.1), $\mathbf{u}$ is given, so lincond_N is a condition on the data, $\mathbf{H}$, along with the forcing $\mathbf{g}$. For the nonlinear problem in (1.6), however, $\mathbf{H}$ must be generated on the inflow boundary from the solution itself. This means that our nonlinear compatibility condition cond_N must be such that the function $\mathbf{H}$ given by (3.7) satisfies the linear compatibility condition lincond_N. We begin with the $N = 0$ case.

Using Lemma B.2 along with $\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = H^{\boldsymbol{n}}$, on all of $[0, T] \times \Gamma_+$, we have

$$-[[\mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}}]^\perp = U^{\boldsymbol{n}} \mathbf{H}^{\boldsymbol{\tau}} - H^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} = \left[-\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma \left(p_r + \frac{1}{2} |\mathbf{U}|^2 \right) + \mathbf{f}^{\boldsymbol{\tau}} \right]^\perp,$$

so,

$$\begin{aligned} [\mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}} &= \partial_t \mathbf{U}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(p_r + \frac{1}{2} |\mathbf{U}|^2 \right) - \mathbf{f}^{\boldsymbol{\tau}} \\ &= \partial_t \mathbf{U}^{\boldsymbol{\tau}} + \nabla_\Gamma \left(p_r + \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{f}^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right). \end{aligned} \tag{4.7}$$

Then using the vector identity in (3.2),

$$\nabla_\Gamma \left(p_r + \frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{f}^{\boldsymbol{\tau}} = [\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p_r - \mathbf{f}]^{\boldsymbol{\tau}} + [\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}}. \tag{4.8}$$

At $t = 0$, $p_r = p^0$ so $[\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p_r - \mathbf{f}]^{\boldsymbol{\tau}} = -\widetilde{\partial}_t \mathbf{u}_0^{\boldsymbol{\tau}}$, and on Γ_+,

$$[\mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}}_{t=0} = \partial_t \mathbf{U}^{\boldsymbol{\tau}}|_{t=0} - \widetilde{\partial}_t \mathbf{u}_0^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \left(|\mathbf{U}(0)|^2 - |\mathbf{u}_0|^2 \right) + [\mathbf{u}_0 \times \boldsymbol{\omega}_0]^{\boldsymbol{\tau}}. \tag{4.9}$$

Proposition 4.4. *Assume the data has regularity* 0, $\mathbf{u} \in S^{1,\alpha}$, *and* cond_0 *in (1.14) holds. Then* $\mathrm{lincond}_0$ *in (2.3) holds.*

Proof. **All the calculations in this proof apply at time zero on Γ_+.**

We have $\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \widetilde{\partial}_t \mathbf{u}_0^{\boldsymbol{\tau}} = 0$ by cond_0. Since also $\mathbf{u}(0) = \mathbf{U}(0)$ on Γ_+, we know that $\nabla_\Gamma |\mathbf{U}|^2 = \nabla_\Gamma |\mathbf{u}|^2$, and (4.9) reduces to $[\mathbf{U} \times \mathbf{H}]^{\boldsymbol{\tau}} = [\mathbf{U} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}}$, or,

$$[\mathbf{U} \times (\mathbf{H} - \boldsymbol{\omega})]^{\boldsymbol{\tau}} = 0.$$

Also from $(3.7)_2$, $H^{\boldsymbol{n}} = \mathrm{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = \mathrm{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} = \omega^{\boldsymbol{n}}$. Then, since $H^{\boldsymbol{n}} = \omega^{\boldsymbol{n}}$ and only $(\mathbf{H} - \boldsymbol{\omega})^{\boldsymbol{\tau}}$ contributes to $\boldsymbol{n} \times (\mathbf{H} - \boldsymbol{\omega})$, we can apply the vector identity, $A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$ to give

$$\begin{aligned} 0 &= \boldsymbol{n} \times [\mathbf{U} \times (\mathbf{H} - \boldsymbol{\omega})]^{\boldsymbol{\tau}} = \boldsymbol{n} \times [\mathbf{U} \times (\mathbf{H} - \boldsymbol{\omega})] \\ &= [\boldsymbol{n} \cdot (\mathbf{H} - \boldsymbol{\omega})]\mathbf{U} - [\boldsymbol{n} \cdot \mathbf{U}](\mathbf{H} - \boldsymbol{\omega}) = -U^{\boldsymbol{n}}(\mathbf{H} - \boldsymbol{\omega}). \end{aligned}$$

Since $U^{\boldsymbol{n}}$ never vanishes on Γ_+, we conclude that $\mathbf{H} = \boldsymbol{\omega}$ on $\{0\} \times \Gamma_+$, meaning that $\mathrm{lincond}_0$ is satisfied. □

If $\mathbf{u} \in S^{N+1,\alpha}$ for $N \geqslant 1$, however, $\mathrm{cond}_N \not\Longrightarrow \mathrm{lincond}_N$ unless we restrict $\mathbf{u}$ to the subspace $\mathrm{Dom}_N(A)$ of $S^{N+1,\alpha}$. To show this, we will find it convenient to extend our definition of the $\widetilde{\partial}_t^n$ "operator" to apply to all of $[0,T] \times \overline{\Omega}$ by replacing p^0 with p_r in Definition 4.1 and not restricting the calculations to $t = 0$. Since $p_r|_{t=0} = p^0$, the definitions of $\widetilde{\partial}_t \mathbf{u}_0$ and $\widetilde{\partial}_t \boldsymbol{\omega}_0$ are unchanged in the sense that

$$\widetilde{\partial}_t \mathbf{u}_0 = \widetilde{\partial}_t \mathbf{u}|_{t=0}, \quad \widetilde{\partial}_t \boldsymbol{\omega}_0 = \widetilde{\partial}_t \boldsymbol{\omega}|_{t=0},$$

where we have used the same symbol $\widetilde{\partial}_t$ for both versions of the "operator." We then define

$$\widetilde{\partial}_t^2 \mathbf{u} = -(-\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p_r + \mathbf{f}) \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla(-\mathbf{u} \cdot \nabla \mathbf{u} - \nabla p_r + \mathbf{f}) - \nabla \widetilde{\partial}_t p_r + \partial_t \mathbf{f} \tag{4.10}$$

on $[0,T] \times \overline{\Omega}$, where

$$\begin{cases} \Delta \widetilde{\partial}_t p_r = -\operatorname{div} \widetilde{\partial}_t (\mathbf{u} \cdot \nabla \mathbf{u}) & \text{in } \overline{Q}, \\ \nabla \widetilde{\partial}_t p_r \cdot \boldsymbol{n} = -\partial_t U^{\boldsymbol{n}} - \widetilde{\partial}_t N[\mathbf{u}] & \text{on } [0,T] \times \Gamma, \end{cases}$$

with

$$\widetilde{\partial}_t N[\mathbf{u}] := \begin{cases} \widetilde{\partial}_t (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} & \text{on } [0,T] \times (\Gamma_- \cup \Gamma_0), \\ \widetilde{\partial}_t (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} + \operatorname{div}_\Gamma (U^{\boldsymbol{n}} (\widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} - \partial_t \mathbf{U}^{\boldsymbol{\tau}})) & \text{on } [0,T] \times \Gamma_+. \end{cases}$$

Then $\widetilde{\partial}_t^n \mathbf{u}$ is defined inductively for all $n > 2$, and $\widetilde{\partial}_t^n \boldsymbol{\omega}$ is defined similarly. An additional assumption on u is required, however, to have the two definitions of $\widetilde{\partial}_t^n \mathbf{u}$ at time zero to agree for $n \geqslant 2$, as we see in Proposition 4.5.

Proposition 4.5. *Assume that the data has regularity* $N \geqslant 0$, cond_N *holds, and* $\partial_t \mathbf{u}|_{t=0} = \widetilde{\partial}_t \mathbf{u}_0$ *on* $\overline{\Omega}$. *Then*

(1) $\widetilde{\partial}_t^n p_r|_{t=0} = \widetilde{\partial}_t^n p^0$ *on* $\overline{\Omega}$ *for all* $0 \leqslant n \leqslant N$;

(2) $\widetilde{\partial}_t^n \mathbf{u}|_{t=0} = \widetilde{\partial}_t^n \mathbf{u}_0$ *on* $\overline{\Omega}$ *for all* $0 \leqslant n \leqslant N+1$;

(3) $\partial_t \widetilde{\partial}_t^n \mathbf{u}|_{t=0} = \widetilde{\partial}_t^{n+1} \mathbf{u}_0$ *on* $\overline{\Omega}$ *for all* $0 \leqslant n \leqslant N$*.*

Proof. By cond_N, we have $\partial_t^n \mathbf{U}^{\boldsymbol{\tau}} = \widetilde{\partial}_t^n \mathbf{u}^{\boldsymbol{\tau}}$ on Γ_+, so $\widetilde{\partial}_t^n N[\mathbf{u}] = \widetilde{\partial}_t^n(\mathbf{u} \cdot \nabla \mathbf{u})$ and hence $\widetilde{\partial}_t^n p_r|_{t=0} = \widetilde{\partial}_t^n p^0$. This gives (1) and then (2) follows directly from (1). For (3), observe that $\widetilde{\partial}_t^n \mathbf{u}$ fully expanded (as in (4.10) for $n = 2$) contains no time dervatives of $\mathbf{u}$. Using the product rule to obtain $\partial_t \widetilde{\partial}_t^n \mathbf{u}|_{t=0}$ and using that $\partial_t \mathbf{u}|_{t=0} = \widetilde{\partial}_t \mathbf{u}_0$ we obtain, using also (1), the same expression as $\widetilde{\partial}_t^{n+1} \mathbf{u}_0$. □

Proposition 4.6. *Assume that the data has regularity* $N \geqslant 0$, cond_N *holds, and* $\mathbf{u} \in \text{Dom}_N(A)$*. Then* lincond_N *in (2.3) holds.*

Proof. Let $N = 1$. With our extended definition of $\widetilde{\partial}_t$, (4.8) holds beyond time zero; that is,

$$[\mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}} = \partial_t \mathbf{U}^{\boldsymbol{\tau}} - \widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) + [\mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}} \text{ on}$$

on all of $[0, T] \times \Gamma_+$ (this does not require any compatibility conditions). Differentiating both sides in time gives

$$\begin{aligned} [\partial_t \mathbf{u} \times \mathbf{H}]^{\boldsymbol{\tau}} + [\mathbf{u} \times \partial_t \mathbf{H}]^{\boldsymbol{\tau}} = \partial_{tt} \mathbf{U}^{\boldsymbol{\tau}} - \partial_t \widetilde{\partial}_t \mathbf{u}^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \partial_t \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) \\ + [\partial_t \mathbf{u} \times \boldsymbol{\omega}]^{\boldsymbol{\tau}} + [\mathbf{u} \times \partial_t \boldsymbol{\omega}]^{\boldsymbol{\tau}} \end{aligned} \tag{4.11}$$

on $[0, T] \times \Gamma_+$. We know from Proposition 4.4 that if cond_0 holds then $\mathbf{H} = \boldsymbol{\omega}$ on $\{0\} \times \Gamma_+$, so two terms above cancel, leaving, at time zero,

$$[\mathbf{u} \times \partial_t \mathbf{H}]^{\boldsymbol{\tau}} = \left[\partial_{tt} \mathbf{U}^{\boldsymbol{\tau}} - \partial_t \widetilde{\partial}_t \mathbf{u}_0^{\boldsymbol{\tau}} + \frac{1}{2} \nabla_\Gamma \partial_t \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) \right] + [\mathbf{u} \times \partial_t \boldsymbol{\omega}]^{\boldsymbol{\tau}} \text{ on } \{0\} \times \Gamma_+. \tag{4.12}$$

From Proposition 4.3, $\widetilde{\partial}_t \boldsymbol{\omega}_0 = \operatorname{curl} \widetilde{\partial}_t \mathbf{u}_0 = \operatorname{curl} \partial_t \mathbf{u}(0) = \partial_t \operatorname{curl} \mathbf{u}(0) = \partial_t \boldsymbol{\omega}(0)$, and from Proposition 4.5 we know that $\partial_t \widetilde{\partial}_t \mathbf{u}|_{t=0} = \widetilde{\partial}_t^2 \mathbf{u}_0$. Also,

$$\begin{aligned} \partial_t \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) |_{t=0} &= 2(\mathbf{U} \cdot \partial_t \mathbf{U} - \mathbf{u} \cdot \partial_t \mathbf{u})|_{t=0} = 2(\mathbf{U}(0) \cdot \partial_t \mathbf{U}|_{t=0} - \mathbf{U}(0) \cdot \widetilde{\partial}_t \mathbf{u}_0) \\ &= 2(\mathbf{U} \cdot \partial_t \mathbf{U} - \mathbf{U} \cdot \partial_t \mathbf{U})|_{t=0} = 0, \end{aligned} \tag{4.13}$$

where we used that $\mathbf{u} \in \text{Dom}_N(A)$ in the second equality and cond_1 with $\widetilde{\partial}_t \mathbf{u}_0 \cdot \boldsymbol{n} = \partial_t U^{\boldsymbol{n}}(0)$ on Γ as in Remark 4.2 in the third equality.

Thus, the term in the brackets in (4.12) vanishes because of cond_1, and we are left with

$$[\mathbf{u}_0 \times \partial_t \mathbf{H}|_{t=0}]^{\boldsymbol{\tau}} = [\mathbf{u}_0 \times \widetilde{\partial}_t \boldsymbol{\omega}_0]^{\boldsymbol{\tau}} \text{ on } \Gamma_+.$$

Also, $\partial_t H^{\boldsymbol{n}}|_{t=0} = \operatorname{curl}_\Gamma \partial_t \mathbf{U}^{\boldsymbol{\tau}}|_{t=0} = \operatorname{curl}_\Gamma \widetilde{\partial}_t \mathbf{u}_0^{\boldsymbol{\tau}} = \widetilde{\partial}_t \omega_0^{\boldsymbol{n}}$, so arguing as in the proof of Proposition 4.4, we see that $\partial_t \mathbf{H}|_{t=0} = \widetilde{\partial}_t \boldsymbol{\omega}_0$, which is lincond_1.

The result for $N \geqslant 2$ follows inductively, where we note that, as in (4.13), showing that $\partial_t^N \left(|\mathbf{U}|^2 - |\mathbf{u}|^2 \right) |_{t=0} = 0$ uses that $\partial_t^n \mathbf{u}|_{t=0} = \widetilde{\partial}_t^n \mathbf{u}_0$ on Γ_+ for all $0 \leqslant n \leqslant N$, since $\mathbf{u} \in \text{Dom}_N(A)$. □

Remark 4.7. *In the proof of Proposition 4.6, we only required of* $\mathbf{u}$ *that* $\mathbf{u}(0) = \mathbf{u}_0$ *satisfy* cond_N*,* $\partial_t \mathbf{u}|_{t=0} = \widetilde{\partial}_t \mathbf{u}_0$ *on* $\overline{\Omega}$ *(because we applied Proposition 4.5), and* $\partial_t^n \mathbf{u}_{t=0} = \widetilde{\partial}_t^n \mathbf{u}_0$ *on* Γ_+*. The full conditions on* $\text{Dom}_N(A)$ *will be required shortly, however, in Proposition 5.1.*

5. Proof of well-posedness with inflow, outflow

In this section, we present the three key propositions on which the proof of Theorem 1.2 relies, then give the proof of Theorem 1.2 itself.

Proposition 5.1. *A maps $\mathrm{Dom}_N(A)$ to itself.*

Proof. Let $\mathbf{u} \in \mathrm{Dom}_N(A)$ and let $\mathbf{v} = A\mathbf{u}$. Theorem 2.2 shows that $\mathbf{v} \in S^{N+1,\alpha}$ and $\mathbf{v}(0) = \mathbf{u}_0$, so it remains only to show that $\partial_t^n \mathbf{v}|_{t=0} = \widetilde{\partial}_t^n \mathbf{u}_0$ for $1 \leqslant n \leqslant N$.

Suppose $N = 1$. Then since $\mathbf{v}(0) = \mathbf{u}(0)$, (2.5) gives

$$\partial_t \mathbf{v}|_{t=0} = -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T - \nabla \pi(0) + \mathbf{f}(0).$$

But $\mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T = (1/2)\nabla |\mathbf{u}_0|^2$, so we have

$$\partial_t \mathbf{v}|_{t=0} = -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla r + \mathbf{f}(0)$$

for some "pressure" r. But r is recovered in the same manner as p, which is the same as p_r at time zero. We see, then, that $\partial_t \mathbf{v}|_{t=0} = \widetilde{\partial}_t \mathbf{u}_0$.

For $N = 2$, a time derivative of (2.5) yields,

$$\partial_t^2 \mathbf{v} + \partial_t \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{u} \cdot \nabla \partial_t \mathbf{v} - [\partial_t \mathbf{u} \cdot (\nabla \mathbf{v})^T + \mathbf{u} \cdot (\nabla \partial_t \mathbf{v})^T] = -\nabla \partial_t \pi + \partial_t \mathbf{f}.$$

From the $N = 1$ result, $\partial_t \mathbf{u}|_{t=0} = \partial_t \mathbf{v}|_{t=0} = \widetilde{\partial}_t \mathbf{u}_0$ and we have

$$\partial_t^2 \mathbf{v}|_{t=0} + \widetilde{\partial}_t \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla \widetilde{\partial}_t \mathbf{u}_0 - [\widetilde{\partial}_t \mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T + \mathbf{u}_0 \cdot (\nabla \widetilde{\partial}_t \mathbf{u}_0)^T] = -\nabla \partial_t \pi|_{t=0} + \partial_t \mathbf{f}|_{t=0}.$$

By $(4.6)_1$, $\widetilde{\partial}_t \mathbf{u}_0 \cdot (\nabla \mathbf{u}_0)^T + \mathbf{u}_0 \cdot (\nabla \widetilde{\partial}_t \mathbf{u}_0)^T$ is a gradient, so we see, also using (4.3), that

$$\partial_t^2 \mathbf{v}|_{t=0} + \widetilde{\partial}_t (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) = \nabla q + \partial_t \mathbf{f}|_{t=0}$$

for some q. But from $(4.2)_1$,

$$\widetilde{\partial}_t^2 \mathbf{u}_0 + \widetilde{\partial}_t (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0) = \nabla \widetilde{\partial}_t p^0 + \partial_t \mathbf{f}|_{t=0}.$$

Hence, $\mathbf{w} := \partial_t^2 \mathbf{v}|_{t=0} - \widetilde{\partial}_t^2 \mathbf{u}_0$ is a gradient in the space H, since $\operatorname{div} \mathbf{w} = 0$ with $\mathbf{w} \cdot \boldsymbol{n} = 0$ on Γ; hence, $\mathbf{w} = 0$, giving $\partial_t^2 \mathbf{v}|_{t=0} = \widetilde{\partial}_t^2 \mathbf{u}_0$.

The result for $N > 2$ follows inductively. We note that because it involves differentiating in time (2.5) $N-1$ times, it requires that $\partial_t^n \mathbf{u}|_{t=0} = \widetilde{\partial}_t^n \mathbf{u}_0$ on Ω for all $1 \leqslant n \leqslant N$, so the full condition on $\mathbf{u}$ in $\mathrm{Dom}_N(A)$ is required. □

We will also show in Lemma 7.9 that $\mathrm{Dom}_N(A)$ is a nonempty, convex subset of $S^{N+1,\alpha}$.

We will apply Schauder's fixed point theorem to obtain the existence of a fixed point of A, but this requires that A be continuous. Results in [11] would give that A is bounded as a map from $\mathrm{Dom}_N(A)$ to $\mathrm{Dom}_N(A)$ in the $S^{N+1,\alpha}$ norm, as long as we can obtain sufficient control of the pressure so as to control $\mathbf{H}$. But A, which is nonlinear, need not be continuous from $S^{N+1,\alpha}$ to itself. To ensure continuity, we need to work with a weaker topology, which we introduce next.

Definition 5.2. *Fixing $\beta \in (0, \alpha]$, we define the (affine) space $\mathring{S}^{N+1,\beta}$ to be all vector fields in $S^{N+1,\alpha}$ endowed with the norm,*

$$\|\mathbf{u}\|_{\mathring{S}^{N+1,\beta}} = \|\mathbf{u}\|_{C^{N,\beta}(Q)} + \|\operatorname{curl} \mathbf{u}\|_{C^{N,\beta}(Q)}.$$

We note that $\mathring{S}^{N+1,\beta}$ is a locally convex topological affine space. Because $C^{N,\alpha}(Q)$ is compactly embedded in $C^{N,\beta}(Q)$ for $\beta < \alpha$, we see that $\mathring{S}^{N+1,\alpha}$ is compactly embedded in $\mathring{S}^{N+1,\beta}$ for $\beta < \alpha$. Like $S^{N+1,\alpha}$, which is also an affine space, we will often apply the $\mathring{S}^{N+1,\beta}$ norm to the difference of two elements in $\mathring{S}^{N+1,\beta}$, even though that difference does not lie in the space. In particular, we do this in Proposition 5.4. Finally, observe that $\|\mathbf{u}\|_{S^{N+1,\alpha}} = \|\mathbf{u}\|_{\mathring{S}^{N+1,\alpha}} + \|\partial_t^{N+1}\mathbf{u}\|_{L^\infty([0,T];C^\alpha(Q))}$.

In outline, our proof of Theorem 1.2 is as follows: We show that A maps a nonempty convex set $K \subseteq \mathrm{Dom}_N(A)$ into itself (an A-invariant set), that K is compact in the $\mathring{S}^{N+1,\beta}$ norm, and that A is continuous on K in the $\mathring{S}^{N+1,\beta}$ norm. Applying Schauder's fixed point theorem gives the existence of a fixed point. We show a posteriori that the full inflow, outflow boundary conditions in $(1.6)_{4,5}$ are satisfied, and, finally, prove uniqueness.

These steps are detailed in Propositions 5.3 to 5.5, followed by the proof itself. To streamline the presentation, we defer the proofs of these technical propositions to later sections.

Proposition 5.3. *For all M larger than a value that depends only upon the data, there exists $T > 0$ for which the set*

$$\mathcal{K} = \mathcal{K}_{M,T} := \{\mathbf{u} \in \mathrm{Dom}_N(A) \colon \|\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant M\} \tag{5.1}$$

is invariant under A. That is, $\mathbf{u} \in \mathrm{Dom}_N(A)$ with $\|\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant M$ implies that $A\mathbf{u} \in \mathrm{Dom}_N(A)$ with $\|A\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant M$.

Proof. Given in Section 11. We note here only that $\mathcal{K}$ depends on T because each $\mathbf{u} \in \mathrm{Dom}_N(A)$ is defined on $Q = Q_T$. When M and T are fixed, we will generally refer to the set simply as $\mathcal{K}$. □

Proposition 5.4. *With $\mathcal{K}$ as in (5.1), for any $\beta \in (0, \alpha)$, $A\colon \mathcal{K} \to \mathcal{K}$ is continuous in the $\mathring{S}^{N+1,\beta}$ norm.*

Proof. Given in Section 12. □

Proposition 5.5. *Assume that $(\mathbf{u}, \nabla p_r) \in S^{1,\alpha} \times C^\alpha(Q)$ and (u, p_r) solves $(1.6)_{1\text{-}4}$ (with p_r in place of p) and that $\operatorname{curl} \mathbf{u} = \mathbf{H}$ on $[0,T] \times \Gamma_+$, with $\mathbf{H}$ given in (3.7). Then $(1.6)_5$ also holds.*

Proof. Given in Section 13. □

Proof of well-posedness. Theorem 1.2 we now see is a consequence of Propositions 5.3 to 5.5:

Proof of Theorem 1.2. Let $M > 0$ depending on the initial data, $T > 0$, and $\mathcal{K} = \mathcal{K}_{M,T}$ be given by Proposition 5.3.

Choose any $\beta \in (0, \alpha)$. Because $C^{N,\alpha}$ is compactly embedded in $C^{N,\beta}$, and also using Lemma 7.9, below, we see that $\mathcal{K}$ is a nonempty convex compact subset of $\mathring{S}^{N+1,\beta}$, and $A\colon \mathcal{K} \to \mathcal{K}$ by Proposition 5.3. By Proposition 5.4, A is continuous as a map from $\mathcal{K}$ to $\mathcal{K}$ in the $\mathring{S}^{N+1,\beta}$ norm, and so has a fixed point $\mathbf{u}$ by Schauder's Fixed Point Theorem. It follows that $A\mathbf{u} = \mathbf{u}$ with $\mathbf{u} \in \mathring{S}^{N+1,\beta}$. Since $\mathbf{u} \in \mathrm{Dom}_N(A)$ it follows that, in fact, $\mathbf{u} \in S^{N+1,\alpha}$.

Since $\mathbf{v} := A\mathbf{u} = \mathbf{u}$, Theorem 2.2 implies that $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}$ for some pressure p. Hence, $(\mathbf{u}, p)$ is a solution to $(1.6)_{1\text{-}4}$. But since $\mathbf{u} = A\mathbf{u}$, we have $\boldsymbol{\omega} := \operatorname{curl} \mathbf{u} = \mathbf{H}$ on $[0,T] \times \Gamma_+$. Thus, Proposition 5.5 gives that $(1.6)_5$ holds, so $(\mathbf{u}, p)$ is a solution to (1.6).

To prove uniqueness, let $(\mathbf{u}_1, p_1)$, $(\mathbf{u}_2, p_2)$ be two solutions to (1.6) with the same initial velocity in $C^{1,\alpha}$ (so we prove uniqueness for $N = 0$ and it then follows for all $N \geqslant 0$). Letting

$\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, subtracting $(1.6)_1$ for $(\mathbf{u}_2, p_2)$ from $(1.6)_1$ for $(\mathbf{u}_1, p_1)$,

$$\partial_t \mathbf{w} + \mathbf{u}_1 \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}_2 + \nabla(p_1 - p_2) = 0.$$

Multiplying by $\mathbf{w}$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|^2 = -\int_\Omega (\mathbf{w} \cdot \nabla \mathbf{u}_2) \cdot \mathbf{w} - \frac{1}{2}\int_\Omega \mathbf{u}_1 \cdot \nabla |\mathbf{w}|^2 \leqslant \|\nabla \mathbf{u}_2\|_{L^\infty(Q)} \|\mathbf{w}\|^2 - \frac{1}{2}\int_\Omega \mathbf{u}_1 \cdot \nabla |\mathbf{w}|^2.$$

But,

$$-\int_\Omega \mathbf{u}_1 \cdot \nabla |\mathbf{w}|^2 = -\int_\Gamma (\mathbf{u}_1 \cdot \boldsymbol{n}) |\mathbf{w}|^2 = -\int_{\Gamma_-} (\mathbf{u}_1 \cdot \boldsymbol{n}) |\mathbf{w}|^2 \leqslant 0,$$

since $\mathbf{w} = 0$ on Γ_+, $\mathbf{u}_1 \cdot \boldsymbol{n} = 0$ on Γ_0, and $\mathbf{u}_1 \cdot \boldsymbol{n} > 0$ on Γ_-. Hence,

$$\frac{d}{dt}\|\mathbf{w}\|^2 \leqslant 2\|\nabla \mathbf{u}_2\|_{L^\infty(Q)} \|\mathbf{w}\|^2,$$

and we conclude that $\mathbf{w} = 0$ by Grönwall's Lemma, giving the uniqueness in Theorem 1.2. □

When $\Gamma_0 = \Gamma$—that is, when classical impermeable boundary conditions are imposed on the entire boundary—Theorem 1.2 gives well-posedness of the 3D Euler equations in $S^{N+1,\alpha}$ for any $N \geqslant 0$. The proof simplifies in this case, as we discuss briefly in Remark 13.1.

Part II: Preliminary Estimates

Organization of Part II. We introduce in Section 6 some conventions that we will use throughout the remainder of this paper to streamline the presentation. In Section 7 we develop some properties related to the function space $S^{N+1,\alpha}$, and in Section 8 derive some properties of the flow map. We describe the generation of vorticity on the boundary in Section 9, and obtain critical estimates on the pressure in Section 10.

6. Some conventions

Constants. To simplify notation, we write M as a universal but unspecified bound on $\|\mathbf{u}\|_{S^{N+1,\alpha}}$. Thus, we assume that

$$\|\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant M \text{ for some } M \geqslant 1 \tag{6.1}$$

in what follows. (Having $M \geqslant 1$ simplifies the form of some estimates.)

Definition 6.1. *We define the following three types of positive "constant":*

$$\begin{aligned} c_0 &= c_0(\|\mathbf{u}_0\|_{C_\sigma^{N+1,\alpha}(\Omega)}, U_{min}^{-1}, \|\mathbf{U}\|_{C^{N+2,\alpha}(Q_\infty)}, \|\operatorname{curl} \mathbf{f}\|_{C^{N,\alpha}(Q_\infty)}), \\ c_X &= c_X(c_0, M), \\ c_{\mathcal{N}} &= c_{\mathcal{N}}(c_0, M, T), \end{aligned}$$

where U_{min} is as in Definition 1.1. Each of c_0, c_X, and $c_{\mathcal{N}}$ are continuous, increasing functions of each of their arguments, and each appearance of c_0, c_X, and $c_{\mathcal{N}}$ may have different values, even within the same expression. Moreover, we require of $c_{\mathcal{N}}$ that for all $M > 0$,

$$c_{\mathcal{N}}(c_0, M, 0) = c_0. \tag{6.2}$$

The property in (6.2) is critical in establishing the existence of an invariant set for the operator A in the proof of Proposition 5.3 in Section 11. All the necessary quantities will be bounded by a $c_{\mathcal{N}}$ constant, and all those bounds ultimately derive from the key bound on $\|\mathbf{u}\|_{C^{N,\alpha}(Q)}$ in Proposition 7.5 for $\mathbf{u} \in \mathrm{Dom}_N(A)$, which we establish in the next section.

In the process of obtaining constants c_0, c_X, or $c_{\mathcal{N}}$ it will be clear that they increase with their arguments. It is sometimes clearer to write the expression for a constant $c_{\mathcal{N}}$ more explicitly; typical examples are $c_0 + c_X T^\alpha$ and e^{MT}.

Because we imposed the restrictions on $\mathbf{U}$ and $\mathbf{f}$ in Definition 1.1 for all time, a c_0 constant has no dependence on T.

Remark 6.2. *Many of our estimates contain factors of the form $C_1 T^{e_1} + C_2 T^{e_2} + C_3 T^{e_3}$, $0 < e_1 < e_2 < e_3$, where C_1, C_2, and C_3 may depend upon the norms of the data or the solution, but have no explicit dependence on time. To simplify matters, we will assume that $T \leqslant T_0$ for some fixed but arbitrarily large $T_0 > 0$. Then*

$$\begin{aligned} &C_1 T^{e_1} + C_2 T^{e_2} + C_3 T^{e_3} \leqslant C_1 T^{e_1} + C_2 T^{e_1} T_0^{e_2 - e_1} + C_3 T^{e_1} T_0^{e_3 - e_1} \leqslant C' T^{e_1}, \\ &C' := (1 + T_0^{e_2 - e_1} + T_0^{e_3 - e_1}) \max\{C_1, C_2, C_3\}. \end{aligned}$$

Hence, in the final forms of estimates, we will only keep the lowest exponents of T and, similarly, of $|t_1 - t_2|$ for $t_1, t_2 \in [0, T]$.

7. Function spaces and the Biot-Savart law

In this section we give some basic properties of Hölder spaces that we will use throughout most of this paper and summarize those properties of the Biot-Savart law that we will need. We use these properties to establish the key estimate on velocity fields in Proposition 7.5, and then to prove Lemma 7.9, showing that $\mathrm{Dom}_N(A)$ is nonempty and convex.

Hölder spaces. Let $k \geqslant 0$ be an integer and U be an open subset U of $\mathbb{R}^d$, $d \geqslant 1$. We define $C^k(U)$ to be the space of all k-times continuously differentiable functions with the norm

$$\|f\|_{C^k(U)} := \sum_{|\gamma| \leqslant k} \|D^\gamma f\|_{L^\infty(U)}.$$

Letting $r \in (0,1)$ we define the Hölder space, $C^{k,r}(U)$, to be the space of all $f \in C^k(U)$ for which

$$\begin{aligned} \|f\|_{C^{k,r}(U)} &:= \|f\|_{C^k(U)} + \sum_{|\gamma| = k} \|D^\gamma f\|_{\dot{C}^r(U)} < \infty, \\ \|g\|_{\dot{C}^r(U)} &:= \sup_{x \neq y \in U} \frac{|g(x) - g(y)|}{|x - y|^r}. \end{aligned} \tag{7.1}$$

Now consider a time-space domain of the form $Q = [0,T] \times \Omega$. For any f continuous on Q and $r \in (0,1]$, define

$$\begin{aligned} \|f\|_{\dot{C}_t^r(Q)} &:= \|f\|_{\dot{C}^r([0,T]; L^\infty(\Omega))} = \sup_{\mathbf{x} \in \Omega} \|f(\cdot, \mathbf{x})\|_{\dot{C}^r([0,T])}, \\ \|f\|_{\dot{C}_x^r(Q)} &:= \|f\|_{L^\infty([0,T]; \dot{C}^r(\Omega))} = \sup_{t \in [0,T]} \|f(t, \cdot)\|_{\dot{C}^r(\Omega)}. \end{aligned} \tag{7.2}$$

Lemma 7.1. *For any integer $k \geqslant 0$ and $r \in (0,1)$,*

$$\|f\|_{C^N(Q)} + \|f\|_{\dot{C}_t^r(Q)} + \|f\|_{\dot{C}_t^r(Q)}$$

is equivalent to the $C^{N,r}(Q)$ norm.

Biot-Savart law. We need a few facts from [11] related to the Biot-Savart law, which we present now. We use the spaces H, H_c, and H_0 of (1.12) and (1.13).

Lemma 7.2. *Assume that Γ is $C^{n,\alpha}$-regular and let X be any function space contained in $C^{n,\alpha}(\Omega)^3$. For any $\mathbf{v} \in H$, $\|P_{H_c}\mathbf{v}\|_X \leqslant C(X)\|\mathbf{v}\|_H$.*

Lemma 7.3. *If $\mathbf{u} \in H$ with $\operatorname{curl} \mathbf{u} = 0$ and $P_H \mathbf{u} = 0$ then $\mathbf{u} = 0$.*

For any $\boldsymbol{\omega}$ in the range of the curl, $\operatorname{curl}(H^1(\Omega)^3)$, there exists a unique $\mathbf{u} = K[\boldsymbol{\omega}] \in H_0 \cap H^1(\Omega)^3$ for which $\operatorname{curl} \mathbf{u} = \boldsymbol{\omega}$. The operator K, which recovers the unique divergence-free vector field in H_0 having a given curl, encodes the Biot-Savart law.

There exists a vector field $\boldsymbol{\mathcal{V}}$ as regular as $\mathbf{U}$ with $\operatorname{div} \boldsymbol{\mathcal{V}} = 0$, $\operatorname{curl} \boldsymbol{\mathcal{V}} = 0$, and $\boldsymbol{\mathcal{V}} \cdot \boldsymbol{n} = U^{\boldsymbol{n}}$ on $[0, T] \times \Gamma$. We define

$$K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}] := K[\boldsymbol{\omega}] + \boldsymbol{\mathcal{V}}. \tag{7.3}$$

Lemma 7.4. *Assume $\mathbf{U} \in S^{N+1,\alpha}$. Let $\boldsymbol{\omega}$ be a divergence-free vector field on Ω having vanishing external fluxes. Let $\mathbf{u}_c \in H_c$ and set $\mathbf{u} = K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}] + \mathbf{u}_c$. For all $t \in [0, T]$ and all integers k with $0 \leqslant k \leqslant N$,*

$$\begin{aligned}
\|\mathbf{u}(t)\|_{W^{k+1,p}(\Omega)} &\leqslant C\|\boldsymbol{\omega}(t)\|_{W^{k,p}(\Omega)} + \|\mathbf{U}(t)\|_{W^{k+1,p}(\Omega)} + \|\mathbf{u}_c(t)\|_{W^{k+1,p}(\Omega)}, \\
\|\mathbf{u}(t)\|_{C_\sigma^{N+1,\alpha}(\Omega)} &\leqslant C\|\boldsymbol{\omega}(t)\|_{C^{N,\alpha}(\Omega)} + \|\mathbf{U}(t)\|_{C^{N+1,\alpha}(\Omega)} + \|\mathbf{u}_c(t)\|_{C^{N+1,\alpha}(\Omega)}
\end{aligned}$$

for all $p \in (1, \infty)$, whenever the norms on the right-hand side are finite. In each case, $\mathbf{U}$ can be replaced by $\boldsymbol{\mathcal{V}}$ and the final term can be replaced by $C\|\mathbf{u}\|_H$.

Proof. For the two inequalities see, for instance, [11]. Lemma 7.2 allows us to replace each of the final terms by $C\|\mathbf{u}\|_H$. □

A key property of $S^{N+1,\alpha}$. The purpose of this subsection is to prove the following:

Proposition 7.5. *Assume that $\mathbf{u} \in S^{N+1,\alpha}$. Then*

$$\|\mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant C \sum_{j=0}^{N} \|\partial_t^j \mathbf{u}|_{t=0}\|_{C^{N-j,\alpha}(\Omega))} + C\left[\|\mathbf{u}\|_{S^{N+1,\alpha}} + \|\mathbf{U}\|_{S^{N+1,\alpha}}\right] \max\{T^\alpha, T^{1-\alpha}\}.$$

Moreover, if $\mathbf{u} \in \operatorname{Dom}_N(A)$ then

$$\|\mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant c_0 + c_X \max\{T^\alpha, T^{1-\alpha}\} \leqslant c_{\mathcal{N}},$$

where c_0, c_X, and $c_{\mathcal{N}}$ are as in Definition 6.1.

To prove Proposition 7.5, we will make use of the following space:

Definition 7.6. *For an integer $k \geqslant 0$ and $\alpha \in (0, 1)$, define the space,*

$$\begin{aligned}
R^{k,\alpha} &:= \{f \in C^{k-1}(Q) : \partial_t^j f \in L^\infty([0, T]; C^{k-j,\alpha}(\Omega)), 0 \leqslant j \leqslant k\}, \\
\|f\|_{R^{k,\alpha}} &= \sum_{j=0}^{k} \|\partial_t^j f\|_{L^\infty([0,T];C^{k-j,\alpha}(\Omega))}.
\end{aligned}$$

We allow $R^{k,\alpha}$ to apply to scalar-, vector-, or matrix-valued functions.

Lemma 7.7. *Let $f \in R^{k,\alpha}$, $k \geqslant 1$. Then*

$$\|f\|_{R^{k-1,\alpha}} \leqslant \sum_{j=0}^{k-1} \|\partial_t^j f|_{t=0}\|_{C^{k-1-j,\alpha}(\Omega)} + \|f\|_{R^{k,\alpha}} T.$$

Proof. Let j be an integer with $0 \leqslant j \leqslant k-1$. Because

$$\partial_t^j f(t,\mathbf{x}) = \partial_t^j f(t,\mathbf{x})|_{t=0} + \int_0^t \partial_t^{j+1} f(s,\mathbf{x})\, ds,$$

we have

$$\|\partial_t^j D^\beta f\|_{L^\infty([0,T];C^{k-1-j,\alpha}(\Omega))} \leqslant \|\partial_t^j D^\beta f|_{t=0}\|_{C^{k-1-j,\alpha}(\Omega)} + \|\partial_t^{j+1} D^\beta f\|_{L^\infty([0,T];C^{k-j,\alpha}(\Omega))} T.$$

Summing over j from 0 to $k-1$ gives the result. □

Lemma 7.8. *If $f \in R^{k,\alpha}$ for $k \geqslant 1$ then $f \in C^{k-1,\alpha}(Q)$ with*

$$\|f\|_{C^{k-1,\alpha}(Q)} \leqslant \sum_{j=0}^{k-1} \|\partial_t^j f|_{t=0}\|_{C^{k-1-j,\alpha}(\Omega)} + 4\|f\|_{R^{k,\alpha}} \max\{T^{1-\alpha}, T\}. \tag{7.4}$$

Proof. First suppose that $k=1$. Since $\partial_t f \in L^\infty([0,T];C^\alpha(\Omega))$ and $f(t,\cdot) \in C(\overline{\Omega})$ for all $t \in [0,T]$, we have that for all $(t,\mathbf{x}) \in [0,T] \times \Omega$,

$$f(t,\mathbf{x}) = f(0,\mathbf{x}) + \int_0^t \partial_s f(s,\mathbf{x}) ds.$$

Then for all $t_1, t_2 \in [0,T]$ and $x_1, x_2 \in \Omega$,

$$\begin{aligned} |f(t_1,\mathbf{x}_1) - f(t_2,\mathbf{x}_2)| &\leqslant |f(t_1,\mathbf{x}_1) - f(t_1,\mathbf{x}_2)| + |f(t_1,\mathbf{x}_2) - f(t_2,\mathbf{x}_2)| \\ &\leqslant \|f(t_1,\cdot)\|_{\dot{C}^\alpha} |\mathbf{x}_1 - \mathbf{x}_2|^\alpha + \|\partial_t f\|_{L^\infty([0,T];C^\alpha(\Omega))} |t_1 - t_2|. \end{aligned} \tag{7.5}$$

Dividing both sides by $|(t_1,\mathbf{x}_1) - (t_2,\mathbf{x}_2)|^\alpha$, which we note is greater than both $|\mathbf{x}_1 - \mathbf{x}_2|^\alpha$ and $|t_1 - t_2|$, we see that, in fact, $f \in C^\alpha(Q)$ with $\|f\|_{C^\alpha(Q)} \leqslant \|f\|_{R^{1,\alpha}}$.

Moreover, we can estimate the term $|f(t_1,\mathbf{x}_1) - f(t_1,\mathbf{x}_2)|$ in two other ways. First, because $f \in L^\infty([0,T];C^{1,\alpha}(\Omega))$, we have

$$|f(t_1,\mathbf{x}_1) - f(t_1,\mathbf{x}_2)| \leqslant a_1 := \|f\|_{L^\infty([0,T];C^{1,\alpha}(\Omega))} |\mathbf{x}_1 - \mathbf{x}_2|.$$

Second, we have

$$|f(t,\mathbf{x}_1) - f(t,\mathbf{x}_2)| = \left| \int_0^t \partial_s f(s,\mathbf{x}_1)\, ds - \int_0^t \partial_s f(s,\mathbf{x}_2)\, ds \right| \leqslant a_2 := 2\|\partial_t f\|_{L^\infty([0,T]\times\Omega)} T.$$

Hence,

$$\begin{aligned} |f(t,\mathbf{x}_1) - f(t,\mathbf{x}_2)| &= |f(t,\mathbf{x}_1) - f(t,\mathbf{x}_2)|^\alpha |f(t,\mathbf{x}_1) - f(t,\mathbf{x}_2)|^{1-\alpha} \leqslant a_1^\alpha a_2^{1-\alpha} \\ &\leqslant \|f\|_{L^\infty([0,T];C^{1,\alpha}(\Omega)}^\alpha 2^{1-\alpha} \|\partial_t f\|_{L^\infty([0,T]\times\Omega)}^{1-\alpha} |\mathbf{x}_1 - \mathbf{x}_2|^\alpha T^{1-\alpha} \\ &\leqslant 2\|f\|_{R^{1,\alpha}} |\mathbf{x}_1 - \mathbf{x}_2|^\alpha T^{1-\alpha}. \end{aligned}$$

Then, as in (7.5), and using that $|t_1 - t_2| \leqslant |t_1 - t_2|^\alpha T^{1-\alpha}$,

$$|f(t_1,\mathbf{x}_1) - f(t_2,\mathbf{x}_2)| \leqslant 2\|f\|_{R^{1,\alpha}} |\mathbf{x}_1 - \mathbf{x}_2|^\alpha T^{1-\alpha} + \|\partial_t f\|_{L^\infty([0,T];C^\alpha(\Omega))} |t_1 - t_2|^\alpha T^{1-\alpha}.$$

Dividing both sides by $|(t_1,\mathbf{x}_1) - (t_2,\mathbf{x}_2)|^\alpha$ and taking the supremum over all $(t_1,\mathbf{x}_1) \neq (t_2,\mathbf{x}_2)$ yields

$$\|f\|_{\dot{C}^\alpha} \leqslant 3\|f\|_{R^{1,\alpha}} T^{1-\alpha}.$$

Also, for any $(t,\mathbf{x}) \in Q$,

$$|f(t,\mathbf{x})| \leqslant |f(0,\mathbf{x})| + |f(t,\mathbf{x}) - f(0,\mathbf{x})| \leqslant |f(0,\mathbf{x})| + \|\partial_t f\|_{L^\infty([0,T];C^\alpha(\Omega))} |t|$$

so

$$\|f\|_{L^\infty([0,T]\times\Omega)} \leqslant \|f(0)\|_{L^\infty(\Omega)} + \|\partial_t f\|_{L^\infty([0,T];C^\alpha(\Omega))}T.$$

Combined, these estimates yield (7.4) for $k = 1$.

The result for $k > 1$ follows from applying the above argument to $D^\beta f$ for any $|\beta| = k$, and controlling all the lower-order derivatives via Lemma 7.7. □

Proof of Proposition 7.5. First we prove that $\mathbf{u} \in R^{N+1,\alpha}$ with

$$\|\mathbf{u}\|_{R^{N+1,\alpha}} \leqslant C\|\mathbf{u}\|_{S^{N+1,\alpha}} + \|\mathbf{U}\|_{S^{N+1,\alpha}}. \tag{7.6}$$

We have,

$$\partial_t^j \mathbf{u} = \partial_t^j K_{U^n}[\boldsymbol{\omega}] + \partial_t^j \mathbf{u}_c = K_{U^n}[\partial_t^j \boldsymbol{\omega}] + \partial_t^j \mathbf{u}_c,$$

so by Lemma 7.4, for $j \leqslant N$ and all $t \in [0,T]$,

$$\begin{aligned}
\|\partial_t^j \mathbf{u}(t)\|_{C^{N+1-j,\alpha}(\Omega)} &\leqslant C\|\partial_t^j \boldsymbol{\omega}(t)\|_{C^{N-j,\alpha}(\Omega)} + \|\mathbf{U}(t)\|_{C^{N+1-j,\alpha}(\Omega)} + C\|\partial_t^j \mathbf{u}(t)\|_H \\
&\leqslant C\|\boldsymbol{\omega}\|_{C^{N,\alpha}(Q)} + C\|\mathbf{u}\|_{C^{N,\alpha}(Q)} + \|\mathbf{U}(t)\|_{C^{N,\alpha}(Q)} \\
&\leqslant C\|\mathbf{u}\|_{S^{N+1,\alpha}} + \|\mathbf{U}(t)\|_{C^{N,\alpha}(Q)},
\end{aligned}$$

where we used Lemma 7.2. For $j = N+1$, the $L^\infty([0,T]; C^\alpha(\Omega))$ norm is included as part of both the $R^{N+1,\alpha}$ and $S^{N+1,\alpha}$ norms, and we see that (7.6) follows.

From Lemma 7.8 and (7.6), then,

$$\begin{aligned}
\|\mathbf{u}\|_{C^{N,\alpha}(Q)} &\leqslant \sum_{j=0}^N \|\partial_t^j \mathbf{u}|_{t=0}\|_{C^{N-j,\alpha}(\Omega)} + 4\|\mathbf{u}\|_{R^{N+1,\alpha}} \max\{T^{1-\alpha}, T\}. \\
&\leqslant \sum_{j=0}^N \|\partial_t^j \mathbf{u}|_{t=0}\|_{C^{N-j,\alpha}(\Omega)} + C\left[\|\mathbf{u}\|_{S^{N+1,\alpha}} + \|\mathbf{U}\|_{S^{N+1,\alpha}}\right] \max\{T^\alpha, T^{1-\alpha}\},
\end{aligned}$$

giving the first bound on $\|\mathbf{u}\|_{C^{N,\alpha}(Q)}$. If $\mathbf{u} \in \mathrm{Dom}_N(A)$, then

$$\sum_{j=0}^N \|\partial_t^j \mathbf{u}|_{t=0}\|_{C^{N-j,\alpha}(\Omega)} = \sum_{j=0}^N \|\widetilde{\partial}_t^j \mathbf{u}_0\|_{C^{N-j,\alpha}(\Omega)} = c_0,$$

giving the second bound on $\|\mathbf{u}\|_{C^{N,\alpha}(Q)}$. □

We now have the tools needed to prove Lemma 7.9:

Lemma 7.9. *Assuming* cond_N *holds,* $\mathrm{Dom}_N(A)$ *is a nonempty, convex subset of* $S^{N+1,\alpha}$*.*

Proof. We first show that $\mathrm{Dom}_N(A)$ is convex. Let $a, b \in [0,1]$ with $a + b = 1$, let $\mathbf{v}$, $\mathbf{w}$ be in $\mathrm{Dom}_N(A)$, and let $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$. Then $\mathbf{u}(0) = a\mathbf{u}_0 + b\mathbf{u}_0 = \mathbf{u}_0$, and so also cond_N is satisfied. Similarly, $\partial_t^n \mathbf{u}|_{t=0} = a\partial_t^n \mathbf{v}|_{t=0} + b\partial_t^n \mathbf{w}|_{t=0} = a\widetilde{\partial}_t^n \mathbf{u}_0 + b\widetilde{\partial}_t^n \mathbf{u}_0 = \widetilde{\partial}_t^n \mathbf{u}_0$. It follows that $\mathrm{Dom}_N(A)$ is convex.

To show that $\mathrm{Dom}_N(A)$ is nonempty, let $\boldsymbol{\omega}_0 := \operatorname{curl} \mathbf{u}_0$ and define

$$\boldsymbol{\omega}(t) := \boldsymbol{\omega}_0 + \sum_{n=1}^N \frac{t^n}{n!} \widetilde{\partial}_t^n \boldsymbol{\omega}_0,$$

so that for all $0 \leqslant n \leqslant N$, $\partial_t^n \boldsymbol{\omega}(0) = \widetilde{\partial}_t^n \boldsymbol{\omega}_0$. Because $\boldsymbol{\omega}(t)$ is in the range of the curl for all $t \in [0,T]$ by Proposition 4.3, we can define

$$\mathbf{u}(t) := K_{U^n}[\boldsymbol{\omega}] + \sum_{n=0}^{N} \frac{t^n}{n!} P_{H_c} \widetilde{\partial}_t^n \mathbf{u}_0,$$

which we note lies in $S^{N+1,\alpha}$. Then $\mathbf{u}(0)$ and $\mathbf{u}_0$ have the same curl and same harmonic component, and $\mathbf{u}(0) - \mathbf{u}_0 \in H$, so $\mathbf{u}(0) = \mathbf{u}_0$ by Lemma 7.3. Moreover, for $1 \leqslant n \leqslant N$,

$$\operatorname{curl} \partial_t^n \mathbf{u}(0) = \partial_t^n \boldsymbol{\omega}(0) = \widetilde{\partial}_t^n \boldsymbol{\omega}_0 = \operatorname{curl} \widetilde{\partial}_t^n \mathbf{u}_0$$

by Proposition 4.3. Also, $P_{H_c} \partial_t^n \mathbf{u}(0) = P_{H_c} \widetilde{\partial}_t^n \mathbf{u}_0$. That is, $\partial_t^n \mathbf{u}(0)$ and $\widetilde{\partial}_t^n \mathbf{u}_0$ have the same curl and same harmonic component, while $\partial_t^n \mathbf{u}(0) \cdot \boldsymbol{n} = \widetilde{\partial}_t^n \mathbf{u}_0 \cdot \boldsymbol{n}$ on Γ. Hence, it follows from Lemma 7.3 that $\partial_t^n \mathbf{u}(0) = \widetilde{\partial}_t^n \mathbf{u}_0$, and we see that $\mathbf{u} \in \operatorname{Dom}_N(A)$, demonstrating that $\operatorname{Dom}_N(A)$ is nonempty. □

8. Flow map estimates

The pushforward of the initial vorticity by the flow map meets, along a hypersurface $\mathcal{S}$ in Q, the pushforward of the vorticity generated on the inflow boundary. This requires some analysis at the level of the flow map. For the most part, the analysis in [11], which we summarize here, suffices. The coarse bounds developed on the flow map in [11], however, would only be sufficient for us to obtain small data existence of solutions: for the short time result for general data that we desire, we will require more explicit and refined bounds, which we develop in Lemma 8.2.

We assume throughout this section that $\mathbf{U} \in C^{N+2,\alpha}(Q), \mathbf{u} \in S^{N+1,\alpha}$ for some $N \geqslant 0$. As in [11], we extend $\mathbf{u}$ to be defined on all of $\mathbb{R} \times \mathbb{R}^3$ using an extension operator like that in Theorem 5′, chapter VI of [29]. This extension need not be divergence-free, and is used only as a matter of convenience in stating results; it is only the value of $\mathbf{u}$ on $\overline{Q}$ that ultimately concerns us.

We define $\eta\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ to be the unique flow map for $\mathbf{u}$, so that $\partial_{t_2} \eta(t_1, t_2; \mathbf{x}) = \mathbf{u}(t_2, \eta(t_1, t_2; \mathbf{x}))$. Then $\eta(t_1, t_2; \mathbf{x})$ is the position that a particle starting at time t_1 at position $\mathbf{x} \in \mathbb{R}^3$ will be at time t_2 as it moves under the action of the velocity field $\mathbf{u}$.

For any $(t, \mathbf{x}) \in \overline{Q}$ let

- $\boldsymbol{\gamma}(t, \mathbf{x})$ be the point on Γ_+ at which the flow line through x at time t intersects Γ_+;
- $\tau(t, \mathbf{x})$ be the time at which that intersection occurs.

For our purposes, we can leave τ and $\boldsymbol{\gamma}$ undefined if the flow line never intersects with Γ_+.

Remark 8.1. *We will often drop the $(t, \mathbf{x})$ arguments on τ and $\boldsymbol{\gamma}$ for brevity.*

We define the hypersurface,

$$\mathcal{S} := \{(t, \mathbf{x}) \in \overline{Q} \colon \tau(t, \mathbf{x}) = 0\},$$

which is nonempty since it contains at least $\Gamma_+ \times \{0\}$, and the open sets $U_\pm \subseteq Q$,

$$\begin{aligned} U_- &:= \{(t, \mathbf{x}) \in Q \colon (t, \mathbf{x}) \notin \text{ domain of } \tau, \boldsymbol{\gamma}\}, \\ U_+ &:= \{(t, \mathbf{x}) \in Q \colon \tau(t, \mathbf{x}) > 0\}. \end{aligned}$$

Then $\mathcal{S}$ is of class $C^{N+1,\alpha}$ as a hypersurface in Q and $\mathcal{S}(t) := \{\mathbf{x} \in \Omega \colon (t, \mathbf{x}) \in \mathcal{S}\}$ is of class $C^{N+1,\alpha}$ as a surface in Ω.

The estimates on the flow map in Lemma 8.2 are more explicit than in [11], where we required only coarse estimates. We note that η has one more derivative in both time variables than has $\mathbf{u}$, which we can see in the explicit estimates.

Lemma 8.2. *The flow map $\eta \in C^{N+1,\alpha}([0,T]^2 \times \mathbb{R}^3)$. Define $\mu\colon U_+ \to [0,T] \times \Gamma_+$ by*

$$\mu(t,\mathbf{x}) = (\tau(t,\mathbf{x}), \boldsymbol{\gamma}(t,\mathbf{x}))$$

and let $M := \|\mathbf{u}\|_{S^{1,\alpha}}$. The functions τ, $\boldsymbol{\gamma}$, μ lie in $C^{N+1,\alpha}(\overline{U}_+ \setminus \{0\} \times \Gamma_+)$. Moreover,

$$\begin{aligned}
\|\partial_{t_1}\eta(t_1,t_2;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} &\leqslant \|\mathbf{u}\|_{L^\infty(Q)} h(t_1,t_2),\\
\|\nabla\eta(t_1,t_2;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} &\leqslant h(t_1,t_2),\\
\|\nabla\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{t_2}(Q)} &\leqslant \|\nabla\mathbf{u}\|_{L^\infty(Q)} h(0,T) T^{1-\alpha},\\
\|\nabla\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}(Q)} &\leqslant h(0,T)^{1+2\alpha} \int_0^T \|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds,\\
\|\nabla\eta(0,T;\mathbf{x})\|_{\dot{C}^\alpha(Q)} &\leqslant e^{(1+2\alpha)MT} M T^{1-\alpha},
\end{aligned} \tag{8.1}$$

where

$$h(t_1,t_2) := \exp\left|\int_{t_1}^{t_2} \|\nabla\mathbf{u}(s)\|_{L^\infty}\, ds\right| \leqslant e^{MT}.$$

Also,

$$\|D\mu\|_{L^\infty(U_+)} \leqslant C U_{min}^{-1}[1 + \|\mathbf{u}\|^2_{L^\infty(Q)}] h(0,T), \tag{8.2}$$

where U_{min} is as in Definition 1.1.

More generally, for any $N \geqslant 0$, defining $\exp^n$ to be $\exp$ composed with itself n times,

$$\begin{aligned}
\|\partial_{t_1}^{N+1}\eta(t_1,t_2;\mathbf{x})\|_{L^\infty([0,T]^2\times\Omega)} &\leqslant C\|\mathbf{u}\|_{C^N(Q)} \exp^{N+1}(MT),\\
\|\nabla^{N+1}\eta(t_1,t_2;\mathbf{x})\|_{L^\infty([0,T]^2\times\Omega)} &\leqslant \exp^{N+1}(MT),\\
\|\nabla^{N+1}\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{t_2}(Q)} &\leqslant \|\nabla^{N+1}\mathbf{u}\|_{L^\infty(Q)} \exp^{N+1}(MT) T^{1-\alpha},\\
\|\nabla^{N+1}\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}(Q)} &\leqslant \exp^{N+1}(CMT) \int_0^T \|\nabla^{N+1}\mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds,\\
\|\nabla^{N+1}\eta(0,T;\mathbf{x})\|_{\dot{C}^\alpha(Q)} &\leqslant \exp^{N+1}(CMT) M T^{1-\alpha},\\
\|D^{N+1}\mu\|_{L^\infty(U_+)} &\leqslant c_0[1 + \|\mathbf{u}\|^{2(N+1)}_{C^N(Q)}] \exp^{N+1}(MT).
\end{aligned} \tag{8.3}$$

Proof. We will apply Lemma A.2 multiple times without explicit reference.

Taking the gradient of the integral expression in (3.1) of [11],

$$\nabla\eta(t_1,t_2;\mathbf{x}) = I + \int_{t_1}^{t_2} \nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\nabla\eta(t_1,s;\mathbf{x})\, ds. \tag{8.4}$$

Thus,

$$\|\nabla\eta(t_1,t_2;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} \leqslant 1 + \left|\int_{t_1}^{t_2} \|\nabla\mathbf{u}(s)\|_{L^\infty} \|\nabla\eta(t_1,s;\mathbf{x})\|_{L^\infty_{\mathbf{x}}}\, ds\right|.$$

Grönwall's Lemma, applied with fixed t_1, gives $(8.1)_2$. Lemma 3.1 of [11] gives $\partial_{t_1}\eta(t_1,t_2;\mathbf{x}) = -\mathbf{u}(t_1,\mathbf{x}) \cdot \nabla\eta(t_1,t_2;\mathbf{x})$, from which $(8.1)_1$ follows.

It also follows from (8.4) that

$$\begin{aligned}\|\nabla\eta(0,t_2;\mathbf{x})\|_{\dot{C}(Q)^\alpha_{t_2}} &\leqslant \sup_{t_2\neq t_2'} \frac{\|\nabla\mathbf{u}\|_{L^\infty(Q)}\|\nabla\eta\|_{L^\infty(Q)}}{|t_2-t_2'|^\alpha}|t_2-t_2'| \\ &\leqslant \|\nabla\mathbf{u}\|_{L^\infty(Q)} h(0,T)T^{1-\alpha},\end{aligned}$$

giving $(8.1)_3$.

Returning once more to (8.4),

$$\|\nabla\eta(t_1,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}} \leqslant \int_0^{t_2} \|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}\, ds.$$

But, using Lemma A.1,

$$\begin{aligned}&\|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}} \\ &\qquad\leqslant \|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\|_{\dot{C}^\alpha_{\mathbf{x}}}\|\nabla\eta(t_1,s;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} + \|\nabla\mathbf{u}(s,\eta(t_1,s;\mathbf{x}))\|_{L^\infty_{\mathbf{x}}}\|\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}} \\ &\qquad\leqslant \|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}\|\eta(t_1,s;\mathbf{x})\|^\alpha_{Lip_{\mathbf{x}}}\|\nabla\eta(t_1,s;\mathbf{x})\|_{L^\infty_{\mathbf{x}}} + \|\nabla\mathbf{u}(s)\|_{L^\infty}\|\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}} \\ &\qquad\leqslant \|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha} h(t_1,s)^{2\alpha} + \|\nabla\mathbf{u}(s)\|_{L^\infty}\|\nabla\eta(t_1,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}},\end{aligned}$$

so

$$\begin{aligned}&\|\nabla\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}} \\ &\qquad\leqslant \int_0^{t_2}\|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha} h(0,s)^{2\alpha}\, ds + \int_0^{t_2}\|\nabla\mathbf{u}(s)\|_{L^\infty(\Omega)}\|\nabla\eta(0,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}\, ds \\ &\qquad\leqslant h(0,t_2)^{2\alpha}\int_0^{t_2}\|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds + \int_0^{t_2}\|\nabla\mathbf{u}(s)\|_{L^\infty(\Omega)}\|\nabla\eta(0,s;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}}\, ds.\end{aligned}$$

Applying Grönwall's Lemma gives

$$\begin{aligned}\|\nabla\eta(0,t_2;\mathbf{x})\|_{\dot{C}^\alpha_{\mathbf{x}}} &\leqslant \left[h(0,t_2)^{2\alpha}\int_0^{t_2}\|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds\right]\exp\int_0^{t_2}\|\nabla\mathbf{u}(s)\|_{L^\infty(\Omega)}\, ds \\ &= h(0,t_2)^{1+2\alpha}\int_0^{t_2}\|\nabla\mathbf{u}(s)\|_{\dot{C}^\alpha}\, ds,\end{aligned}$$

which is $(8.1)_4$.

From Lemma 3.5 of [11],

$$\begin{aligned}&\partial_t\tau = -U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma})^{-1}\partial_{t_1}\eta(t,\tau;\mathbf{x})\cdot\boldsymbol{n}(\boldsymbol{\gamma}), && \nabla\tau = -U^{\boldsymbol{n}}(\tau,\boldsymbol{\gamma})^{-1}(\nabla\eta(t,\tau;\mathbf{x}))^T\boldsymbol{n}(\boldsymbol{\gamma}), \\ &\partial_t\boldsymbol{\gamma} = \partial_{t_1}\eta(t,\tau;\mathbf{x}) + \partial_t\tau\mathbf{u}(\tau,\boldsymbol{\gamma}), && \nabla\boldsymbol{\gamma} = \mathbf{u}(\tau,\boldsymbol{\gamma})\otimes\nabla\tau + \nabla\eta(t,\tau;\mathbf{x}).\end{aligned} \tag{8.5}$$

We use these expressions to calculate,

$$\begin{aligned}\|\partial_t\tau\|_{L^\infty(U_+)} &\leqslant CU_{min}^{-1}\|\partial_{t_1}\eta\|_{L^\infty(Q)} \leqslant CU_{min}^{-1}\|\mathbf{u}\|_{L^\infty(Q)}h(0,T), \\ \|\nabla\tau\|_{L^\infty} &\leqslant CU_{min}^{-1}\|\nabla\eta\|_{L^\infty(Q)} \leqslant CU_{min}^{-1}h(0,T), \\ \|\partial_t\boldsymbol{\gamma}\|_{L^\infty} &\leqslant CU_{min}^{-1}\|\partial_{t_1}\eta\|_{L^\infty(Q)} + \|\mathbf{u}\|_{L^\infty(Q)}\|\partial_t\tau\|_{L^\infty} \\ &\leqslant CU_{min}^{-1}[\|\mathbf{u}\|_{L^\infty(Q)} + \|\mathbf{u}\|^2_{L^\infty(Q)}]h(0,T), \\ \|\nabla\boldsymbol{\gamma}\|_{L^\infty} &\leqslant \|\mathbf{u}\|_{L^\infty(Q)}\|\nabla\tau\|_{L^\infty} + \|\nabla\eta\|_{L^\infty(Q)} \leqslant [1 + CU_{min}^{-1}\|\mathbf{u}\|_{L^\infty(Q)}]h(0,T).\end{aligned}$$

Summing these estimates gives the bound on $D\mu = (\partial_t\mu, \nabla\mu)$.

The bounds for higher N follow from inductive extension of these arguments. □

Remark 8.3. *The exact bounds in Lemma 8.2 are not so important, but it is important that M only appear in the exponentials. Because of that and Proposition 7.5, we see that for $\mathbf{u} \in \mathrm{Dom}_N(A)$, each of the bounds in (8.1) through (8.3) is of the form $c_{\mathcal{N}}$ of Definition 6.1. Similarly, τ and $\boldsymbol{\gamma}$ can be bounded in $C^{N,\alpha}(U_+)$ by $c_{\mathcal{N}}$.*

We are now in a position to give the definition of a Lagrangian solution to (2.1), as it appears in [11]. For this purpose, define

$$\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0(t, \mathbf{x}) := \eta(t, 0; \mathbf{x}). \tag{8.6}$$

As with τ and $\boldsymbol{\gamma}$ (see Remark 8.1) we will often drop the $(t, \mathbf{x})$ arguments on $\boldsymbol{\gamma}_0$.

Definition 8.4 (Lagrangian solution to (2.1))**.** *Define $\overline{\boldsymbol{\omega}}_\pm$ and $\mathbf{G}_\pm$ on $U_\pm$ by*

$$\begin{aligned}
\overline{\boldsymbol{\omega}}_-(t, \mathbf{x}) &= \nabla\eta(0, t; \boldsymbol{\gamma}_0)\overline{\boldsymbol{\omega}}_0(\boldsymbol{\gamma}_0) + \mathbf{G}_+(t, \mathbf{x}), \\
\overline{\boldsymbol{\omega}}_+(t, \mathbf{x}) &= \nabla\eta(\tau, t; \boldsymbol{\gamma})\mathbf{H}(\tau, \boldsymbol{\gamma}) + \mathbf{G}_-(t, \mathbf{x}), \\
\mathbf{G}_-(t, \mathbf{x}) &:= \int_0^t \nabla\eta(s, t; \eta(t, s; \mathbf{x}))\mathbf{g}(s, \eta(t, s; \mathbf{x}))\, ds, \\
\mathbf{G}_+(t, \mathbf{x}) &:= \int_{\tau(t,\mathbf{x})}^t \nabla\eta(s, t; \eta(t, s; \mathbf{x}))\mathbf{g}(s, \eta(t, s; \mathbf{x}))\, ds.
\end{aligned} \tag{8.7}$$

Then $\overline{\boldsymbol{\omega}}$ defined by $\overline{\boldsymbol{\omega}}|_{U_\pm} = \overline{\boldsymbol{\omega}}_\pm$ is called a Lagrangian solution to (2.1).

Remark 8.5. *A few words are appropriate here about the treatment of (8.7) in [11].*

The inflow vorticity, $\overline{\boldsymbol{\omega}}_-$, behaves and can be analyzed much like the full vorticity in the classical setting of an impermeable boundary: the initial vorticity is pushed forward from time zero by the flow map η for the given velocity field, $\mathbf{u}$, and Duhamel's principle is used to treat the forcing term, $\mathbf{G}_-$. Such an analysis yields $\overline{\boldsymbol{\omega}}_- \in C^{N,\alpha}(\Omega_-)$.

The outflow vorticity, $\overline{\boldsymbol{\omega}}_+$, is somewhat more complicated since the vorticity is pushed off the 2D inflow boundary Γ_+ into the 3D domain Ω, producing an inflow component $\Omega_+(t)$ expanding in time. Also, the regularity of $\tau(t, \mathbf{x})$ and $\boldsymbol{\gamma}(t, \mathbf{x})$ must be accounted for, and the time t enters into the Duhamel integral in both limits. Nevertheless, $\overline{\boldsymbol{\omega}}_+ \in C^{N,\alpha}(\Omega_+)$ holds.

The key difficulty, however, lies not with the inflow or outflow vorticity individually, but rather with insuring that they meet across the hypersurface $\mathcal{S}$ in a manner that allows the full vorticity $\overline{\boldsymbol{\omega}}$ to be regular enough to lie in $C^{N,\alpha}(\Omega)$. That $\mathrm{lincond}_N$ *is the right condition to insure this is natural and is easy to show for $N = 0$, primarily because no derivatives are involved to obtain $C^\alpha(Q)$ regularity. The situation for $N > 0$, is much more involved.*

To obtain regularity across $\mathcal{S}$, the $N = 1$ case formally reduces to the $N = 0$ case. But for $N \geqslant 2$, such a reduction to the $N - 1$ case can be obtained, and allows an induction argument to be made to reduce the problem to the $N = 1$ case. This leaves the $N = 1$ case, which requires a delicate analysis.

Complicating the argument slightly is that for $N \geqslant 1$, the two terms making up $\overline{\boldsymbol{\omega}}_\pm$ in (8.7) need not be $C^{N,\alpha}$-continuous across $\mathcal{S}$, though their sum is. But as long as $\mathrm{lincond}_N$ *holds, $C^{N,\alpha}$ estimates on each of the four terms making up ω^+ and ω^- can be combined to give estimates on $\overline{\boldsymbol{\omega}}$ in $C^{N,\alpha}(Q)$.*

9. The nonlinear term on the boundary

Proposition 9.2 gives coordinate-free expressions for $(\mathbf{u} \cdot \nabla\mathbf{u}) \cdot \boldsymbol{n}$ on Γ. The proof of Proposition 9.2 is most readily obtained using the boundary coordinates introduced in Appendix B, so we defer it to that appendix.

Definition 9.1. *For any tangent vector field* $\mathbf{v}$ *on* Γ*, define* $\mathbf{v}^\perp$ *to be* $\mathbf{v}$ *rotated 90 degrees counterclockwise around the normal vector when viewed from outside* Ω *(so* $\mathbf{v}^\perp = \boldsymbol{n} \times \mathbf{v}$*).*

We write the gradient and divergence on the boundary as ∇_Γ and $\operatorname{div}_\Gamma$, as in Appendix B.

Proposition 9.2. *Assume that* Γ *is* C^2*. Let* $\mathbf{u}$ *be a divergence-free differentiable vector field, let* $u^{\boldsymbol{n}} = \mathbf{u} \cdot \boldsymbol{n}$*, and, as in (1.1), let* $\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{u} - u^{\boldsymbol{n}}\boldsymbol{n}$*. Let* κ_1, κ_2 *be the principal curvatures on* Γ*. On* $[0, T] \times \Gamma$*, we have*

$$(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} = -u^{\boldsymbol{n}} \operatorname{div}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma u^{\boldsymbol{n}} - (\kappa_1 + \kappa_2)(u^{\boldsymbol{n}})^2 - \mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}. \tag{9.1}$$

Here, $\mathcal{A}$ *is the shape operator on the boundary: for any tangential vector field,* $\mathcal{A}\mathbf{v}$ *is the directional derivative of* $\boldsymbol{n}$ *in the direction of* $\mathbf{v}$*, which is also a tangential vector field.*

The nonlinear term on the boundary is key to recovering the pressure, as we will see in the next section. It was for these purposes that we used $N[\mathbf{u}]$ given in (3.6) to define the regularized pressure in (3.5). Using that $\mathbf{u}^{\boldsymbol{n}} = \mathbf{U}^{\boldsymbol{n}}$, substituting the expression in (9.1) for $(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n}$, and using (B.1), we see that on Γ_+,

$$N[\mathbf{u}] = -U^{\boldsymbol{n}} \operatorname{div}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma U^{\boldsymbol{n}} - (\kappa_1 + \kappa_2)(U^{\boldsymbol{n}})^2 - \mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}, \tag{9.2}$$

so $N[\mathbf{u}]$ has no derivatives on $\mathbf{u}^{\boldsymbol{\tau}}$. Nonetheless, integrating $(3.6)_2$ by parts along each boundary component using Lemma B.1, we see that

$$\int_\Gamma N[\mathbf{u}] = \int_\Gamma (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n}. \tag{9.3}$$

Hence, replacing $(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n}$ with $N[\mathbf{u}]$ does not alter the compatibility condition for recovering the pressure, as in Section 10.

10. Pressure Estimates

We can determine the pressure from the velocity as in (1.7). On Γ_0, as we can see from (9.1), $\nabla p \cdot \boldsymbol{n} = -\mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}$ $(= -\kappa|\mathbf{u}|^2$ in 2D, where κ is the scalar curvature). Hence, when $\Gamma = \Gamma_0$, standard Schauder estimates imply that ∇p and $\mathbf{u}$ have the same spatial regularity. This is the impermeable boundary case. But for inflow, outflow boundary conditions, the expression for $\nabla p \cdot \boldsymbol{n}$ contains spatial derivatives of $\mathbf{u}$, as we can see from (9.1), on which we have no a priori control. (Because $\mathbf{u} \cdot \boldsymbol{n} = U^{\boldsymbol{n}}$ on all of Γ, the time derivative in $(1.7)_2$ does not impact the regularity of p.)

We circumvent this difficulty using the simple but clever technique in [2]: we replace the boundary condition in $(1.7)_2$ using $N[\mathbf{u}]$ of (3.6), solving instead, (3.5) for the pressure p_r. We see from (9.3) that the required compatibility condition coming from $\int_\Gamma \nabla p_r \cdot \boldsymbol{n} = \int_\Omega \Delta p_r = \int_\Omega \operatorname{div}(-\partial_t \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u})$ remains satisfied when using $-\partial_t U^{\boldsymbol{n}} - N[\mathbf{u}]$ in place of $-\partial_t u^{\boldsymbol{n}} - (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n}$ on Γ. For $\mathbf{u}(t) \in C^{N+1,\alpha}(\Omega)$, classical elliptic regularity theory gives a solution to (3.5) with $p_r(t)$ in $C^{N+1,\alpha}(\Omega)$, unique up to an additive constant. Ultimately, we show, in Proposition 5.5, that $p_r = p$ for the fixed point of the operator A.

Since we are seeking solutions to (1.6) in Hölder spaces, it would seem natural to use elliptic estimates in Hölder spaces. To obtain the needed control on our pressure estimates in time, however, we will find it necessary to use, instead, elliptic estimates in Sobolev spaces, as given in Lemma 10.1. The reason for this is explained in Remark 10.6 following the proof of Proposition 10.4.

Lemma 10.1. *Let $n \geqslant 0$ and $f \in W^{n,q}(\Omega)$, where $q \in (1, \infty)$. If f has mean zero then*

$$\|f\|_{W^{n+2,q}(\Omega)} \leqslant C \left[\|\Delta f\|_{W^{n,q}(\Omega)} + \|\nabla f \cdot \boldsymbol{n}\|_{W^{n+1-\frac{1}{q},q}(\Gamma)} \right]$$

and for any nonempty compact subset of Ω' of $\Omega \cup \Gamma_+$ (without assuming f has mean zero)

$$\|f\|_{W^{n+2,q}(\Omega')} \leqslant C \left[\|\Delta f\|_{W^{n,q}(\Omega)} + \|\nabla f \cdot \boldsymbol{n}\|_{W^{n+1-\frac{1}{q},q}(\Gamma_+)} + \|f\|_{L^q(\Omega)} \right]. \tag{10.1}$$

Proof. The bounds in (10.1) for $n = 0$ are stated near the bottom of page 174 of [2], but let us say a few words about them. First, they are derived from combining an interior estimate away from all boundaries with an estimate that includes only Γ_+. Second, [2] treats the $N = 0$ case, and we use (15.1.5) of [1] for the $N \geqslant 1$ case. □

In what follows, we will use L^q-based Sobolev spaces on Ω with $q > 3/(1-\alpha)$. This will give us some useful properties, which we summarize in Lemma 10.2.

Lemma 10.2. *Let $q > 3/(1-\alpha)$. Then for any integer $k \geqslant 1$,*

$$W^{k+1,q}(\Omega) \subseteq C^{k,\alpha}(\Omega) \tag{10.2}$$

and for any $r \in [1, \infty]$ and any $f \in W^{2,q}(\Omega)$,

$$\begin{aligned} \|f\|_{L^r(\Gamma)} &\leqslant \|f\|_{C^\alpha(\Gamma)} \leqslant \|f\|_{C^\alpha(\Omega)} \leqslant C\|f\|_{W^{1,q}(\Omega)}, \\ \|\nabla f\|_{L^q(\Gamma)} &\leqslant \|\nabla f\|_{W^{1-1/q,q}(\Gamma)} \leqslant C\|\nabla f\|_{W^{1,q}(\Omega)} \leqslant C\|f\|_{W^{2,q}(\Omega)}. \end{aligned} \tag{10.3}$$

For $k \geqslant 1$, $W^{k,q}(\Omega)$ is an algebra, while for $k \geqslant 0$ its trace space $W^{k-1/q,q}(\Gamma)$ is an algebra.

Proof. Sobolev embedding gives (10.2). The inequality in $(10.3)_1$ follows from (10.2) for any $f \in C^\infty(\Omega) \cap W^{2,q}(\Omega)$, which is dense in $W^{1,q}(\Omega)$; $(10.3)_2$ follows from the trace inequality.

For $k \geqslant 1$, $kq > 3k \geqslant 3$ so $W^{k,q}(\Omega)$ is an algebra, and this same condition gives that $W^{k-1/q,q}(\Gamma)$ is an algebra. □

Not only will we need estimates on p_r, but, letting $\mathbf{u}_1, \mathbf{u}_2 \in S^{1,\alpha}$, where $p_{r,1}, p_{r,2}$ solve (3.5) for $\mathbf{u}_1$, $\mathbf{u}_2$, respectively, we will need estimates for $N = 0$ on $P := p_{1,r} - p_{2,r}$. Fixing $t_1, t_2 \in [0, T]$, we will also need to estimate

$$\overline{p}_r := p_r(t_1) - p_r(t_2), \quad \overline{P} := P(t_1) - P(t_2).$$

We start in Propositions 10.3 and 10.4 by controlling only the spatial derivatives of q.

Proposition 10.3. *Let $q > 3/(1-\alpha)$, $t_1, t_2 \in [0, T]$, and let p_r be the unique solution to (3.5) for some $\mathbf{u} \in S^{N+1,\alpha}$ normalized so that $M_q(p_r(t)) := \int_\Omega p_r |p_r|^{q-2} = 0$. Then*

$$\|p_r(t)\|_{L^q(\Omega)} \leqslant C_1, \tag{10.4}$$

where

$$C_1 := \|\mathbf{U}\|^2_{L^\infty(Q)} + \|\partial_t \mathbf{U}\|_{L^\infty(Q)} + \|\mathbf{u}\|^2_{L^\infty(Q)}.$$

Fixing $t_1, t_2 \in [0, T]$, normalize p_r so that $M_q(p_r(t_1) - p_r(t_2)) = 0$. Then

$$\|p_r(t_1) - p_r(t_2)\|_{L^q(\Omega)} \leqslant C_2 |t_1 - t_2|, \tag{10.5}$$

where

$$C_2 := C \left[\|\mathbf{U}\|_{S^{2,\alpha}} + \|\mathbf{U}\|^2_{S^{1,\alpha}} + \|\mathbf{u}\|_{L^\infty(Q)} \|\mathbf{u}\|_{S^{1,\alpha}} \right],$$

the constant C depending only upon Ω and q.

Proof. We adapt the argument on pages 175-176 of [2]. For now we suppress the time variable.

Let β be the unique mean-zero solution to

$$\begin{cases} \Delta\beta = p_r|p_r|^{q-2} & \text{in } \Omega, \\ \nabla\beta\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases}$$

where the normalization of p_r gives solvability. Letting $q' = q/(q-1)$, which we note is Hölder conjugate to q, Lemma 10.1 gives

$$\|\beta\|_{W^{2,q'}(\Omega)} \leqslant C\||p_r|^{q-1}\|_{L^{q'}(\Omega)} = C\|p_r\|^{q-1}_{L^q(\Omega)}.$$

From (3.5) and (3.6), the elliptic problem for p_r can be written, for a fixed time, as

$$\begin{cases} \Delta p_r = -\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u}) & \text{in } \Omega, \\ \nabla p_r\cdot\boldsymbol{n} = -\partial_t U^{\boldsymbol{n}} - (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n} - \mathbb{1}_{\Gamma_+}\operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}})) & \text{on } \Gamma, \end{cases} \tag{10.6}$$

where $\mathbb{1}_{\Gamma_+}$ is the characteristic function for Γ_+. Then, using (10.6),

$$\begin{aligned} \|p_r\|^q_{L^q(\Omega)} &= (\Delta\beta, p_r) = -(\nabla\beta,\nabla p_r) + \int_\Gamma(\nabla\beta\cdot\boldsymbol{n})p_r = (\Delta p_r,\beta) - \int_\Gamma(\nabla p_r\cdot\boldsymbol{n})\beta \\ &= -(\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u}),\beta) - \int_\Gamma(\nabla p_r\cdot\boldsymbol{n})\beta \\ &= (\mathbf{u}\cdot\nabla\mathbf{u},\nabla\beta) - \int_\Gamma((\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{n})\beta - \int_\Gamma(\nabla p_r\cdot\boldsymbol{n})\beta \\ &= (\mathbf{u}\cdot\nabla\mathbf{u},\nabla\beta) - \int_{\Gamma_+}\left(\partial_t U^n + \operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}}))\right)\beta. \end{aligned}$$

But,

$$\begin{aligned} (\mathbf{u}\cdot\nabla\mathbf{u},\nabla\beta) &= \int_\Omega u^i\partial_i u^j\partial_j\beta = \int_\Omega u^i\partial_i(u^j\partial_j\beta) - \int_\Omega u^iu^j\partial_i\partial_j\beta \\ &= (\mathbf{u},\nabla(\mathbf{u}\cdot\nabla\beta)) - (\mathbf{u}\otimes\mathbf{u},\nabla\nabla\beta) = -(\mathbf{u}\otimes\mathbf{u},\nabla\nabla\beta) - \int_\Gamma U^{\boldsymbol{n}}(\mathbf{u}\cdot\nabla\beta) \end{aligned}$$

and, applying Lemma B.1,

$$-\int_{\Gamma_+}\operatorname{div}_\Gamma(U^{\boldsymbol{n}}(\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}}))\beta = \int_{\Gamma_+}U^{\boldsymbol{n}}(\mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}})\cdot\nabla_\Gamma\beta = \int_{\Gamma_+}U^{\boldsymbol{n}}(\mathbf{u}-\mathbf{U})\cdot\nabla\beta.$$

Here, we used that $\nabla\beta\cdot\boldsymbol{n} = 0$, so $\mathbf{v}^{\boldsymbol{\tau}}\cdot\nabla_\Gamma\beta = \mathbf{v}\cdot\nabla\beta$. Hence,

$$\begin{aligned} \|p_r\|^q_{L^q(\Omega)} &= -(\mathbf{u}\otimes\mathbf{u},\nabla\nabla\beta) - \int_\Gamma U^{\boldsymbol{n}}(\mathbf{u}\cdot\nabla\beta) - \int_{\Gamma_+}\partial_t U^n\beta + \int_{\Gamma_+}U^{\boldsymbol{n}}(\mathbf{u}-\mathbf{U})\cdot\nabla\beta \\ &= -(\mathbf{u}\otimes\mathbf{u},\nabla\nabla\beta) - \int_{\Gamma_-}U^{\boldsymbol{n}}(\mathbf{u}\cdot\nabla\beta) - \int_{\Gamma_+}U^{\boldsymbol{n}}\mathbf{U}\cdot\nabla\beta - \int_{\Gamma_+}\partial_t U^n\beta. \end{aligned}$$

Exploiting (10.3), we have the bound,

$$\begin{aligned} \|p_r\|^q_{L^q(\Omega)} &\leqslant \|\mathbf{u}\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^q(\Omega)}\|\beta\|_{W^{2,q'}(\Omega)} + \|\partial_t U^{\boldsymbol{n}}\|_{L^q(\Gamma)}\|\beta\|_{L^{q'}(\Gamma)} \\ &\qquad + \|U^n\|_{L^{q'}(\Gamma)}\left[\|\mathbf{U}\|_{L^\infty(\Gamma)} + \|\mathbf{u}\|_{L^\infty(\Gamma)}\right]\|\nabla\beta\|_{L^q(\Gamma)} \\ &\leqslant C_1\|\beta\|_{W^{2,q'}(\Omega)} \leqslant C_1\|p_r\|^{q-1}_{L^q(\Omega)}, \end{aligned}$$

from which (10.4) follows.

To obtain (10.5) we argue the same way, bounding $\overline{p}_r := p_r(t_1) - p_r(t_2)$, where now β is the unique mean-zero solution to

$$\begin{cases} \Delta\beta = \overline{p}_r|\overline{p}_r|^{q-2} & \text{in } \Omega, \\ \nabla\beta\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases}$$

Letting $\overline{\mathbf{v}} := \mathbf{u}(t_1) - \mathbf{u}(t_2)$, $\overline{\mathbf{U}} := \mathbf{U}(t_1) - \mathbf{U}(t_2)$ we see from (10.6) that

$$\begin{cases} \Delta\overline{p}_r = -\operatorname{div}(\mathbf{u}(t_1)\cdot\nabla\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\mathbf{u}(t_2)) & \text{in } \Omega, \\ \nabla\overline{p}_r\cdot\boldsymbol{n} = -\partial_t\overline{U}^n - (\mathbf{u}(t_1)\cdot\nabla\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\mathbf{u}(t_2))\cdot\boldsymbol{n} & \\ \qquad -\mathbb{1}_{\Gamma_+}\operatorname{div}_\Gamma(U^n(t_1)\overline{\mathbf{v}}^{\boldsymbol{\tau}} + \overline{U}^n\cdot\mathbf{u}^{\boldsymbol{\tau}}(t_2)) & \\ \qquad +\mathbb{1}_{\Gamma_+}\operatorname{div}_\Gamma(U^n(t_1)\overline{\mathbf{U}}^{\boldsymbol{\tau}} + \overline{U}^n\cdot\mathbf{U}^{\boldsymbol{\tau}}(t_2)) & \text{on } \Gamma, \end{cases}$$

Then, using (10.6),

$$\begin{aligned} \|\overline{p}_r\|^q_{L^q(\Omega)} &= (\Delta\beta, \overline{p}_r) = -(\nabla\beta, \nabla\overline{p}_r) + \int_\Gamma(\nabla\beta\cdot\boldsymbol{n})\overline{p}_r = (\Delta\overline{p}_r, \beta) - \int_\Gamma(\nabla\overline{p}_r\cdot\boldsymbol{n})\beta \\ &= -(\operatorname{div}(\mathbf{u}(t_1)\cdot\nabla\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\mathbf{u}(t_2)), \beta) - \int_\Gamma(\nabla\overline{p}_r\cdot\boldsymbol{n})\beta \\ &= (\mathbf{u}(t_1)\cdot\nabla\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\mathbf{u}(t_2), \nabla\beta) - \int_\Gamma((\mathbf{u}(t_1)\cdot\nabla\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\mathbf{u}(t_2))\cdot\boldsymbol{n})\beta \\ &\qquad - \int_\Gamma(\nabla\overline{p}_r\cdot\boldsymbol{n})\beta \\ &= (\mathbf{u}(t_1)\cdot\nabla\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\mathbf{u}(t_2), \nabla\beta) + \int_{\Gamma_+}(-\partial_t\overline{U}^n \\ &\qquad - \operatorname{div}_\Gamma(U^n(t_1)\overline{\mathbf{v}}^{\boldsymbol{\tau}} + \overline{U}^n\cdot\mathbf{u}^{\boldsymbol{\tau}}(t_2) + \operatorname{div}_\Gamma(U^n(t_1)\overline{\mathbf{U}}^{\boldsymbol{\tau}} + \overline{U}^n\cdot\mathbf{U}^{\boldsymbol{\tau}}(t_2))\beta. \end{aligned}$$

But, for vector fields $\mathbf{v}$, $\mathbf{w}$ with $\operatorname{div}\mathbf{v} = 0$,

$$\begin{aligned} (\mathbf{v}\cdot\nabla\mathbf{w}, \nabla\beta) &= \int_\Omega v^i\partial_i w^j\partial_j\beta = \int_\Omega v^i\partial_i(w^j\partial_j\beta) - \int_\Omega v^i w^j\partial_i\partial_j\beta \\ &= (\mathbf{v}, \nabla(\mathbf{w}\cdot\nabla\beta)) - (\mathbf{v}\otimes\mathbf{w}, \nabla\nabla\beta) = -(\mathbf{v}\otimes\mathbf{w}, \nabla\nabla\beta) - \int_\Gamma v^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla\beta), \end{aligned}$$

so

$$\begin{aligned} &(\mathbf{u}(t_1)\cdot\nabla\overline{\mathbf{v}} + \overline{\mathbf{v}}\cdot\nabla\mathbf{u}(t_2), \nabla\beta) \\ &\quad = -(\mathbf{u}(t_1)\otimes\overline{\mathbf{v}} + \overline{\mathbf{v}}\otimes\mathbf{u}(t_2), \nabla\nabla\beta) - \int_\Gamma\left(U^n(t_1)(\overline{\mathbf{v}}\cdot\nabla\beta) + \overline{U}^n(\mathbf{u}(t_2)\cdot\nabla\beta)\right). \end{aligned}$$

Also, applying Lemma B.1,

$$\begin{aligned} -\int_{\Gamma_+}\operatorname{div}_\Gamma(U^n(t_1)\overline{\mathbf{v}}^{\boldsymbol{\tau}} + \overline{U}^n\cdot\mathbf{u}^{\boldsymbol{\tau}}(t_2))\beta &= \int_{\Gamma_+}(U^n(t_1)\overline{\mathbf{v}}^{\boldsymbol{\tau}} + \overline{U}^n\cdot\mathbf{u}^{\boldsymbol{\tau}}(t_2))\cdot\nabla_\Gamma\beta \\ &= \int_{\Gamma_+}(U^n(t_1)\overline{\mathbf{v}} + \overline{U}^n\cdot\mathbf{u}(t_2))\cdot\nabla\beta. \end{aligned}$$

Hence,

$$\begin{aligned}\|\overline{p}_r\|^q_{L^q(\Omega)} &= -(\mathbf{u}(t_1)\otimes\overline{\mathbf{v}}+\overline{\mathbf{v}}\otimes\mathbf{u}(t_2),\nabla\nabla\beta) - \int_\Gamma \left(U^n(t_1)(\overline{\mathbf{v}}\cdot\nabla\beta)+\overline{U}^n(\mathbf{u}(t_2)\cdot\nabla\beta)\right)\\ &\quad - \int_\Gamma \partial_t\overline{U}^n\beta + \int_{\Gamma_+}(U^n(t_1)\overline{\mathbf{v}}+\overline{U}^n\cdot\mathbf{u}(t_2))\cdot\nabla\beta\\ &\quad + \int_{\Gamma_+}\operatorname{div}_\Gamma(U^n(t_1)\overline{\mathbf{U}}^{\boldsymbol{\tau}}+\overline{U}^n\cdot\mathbf{U}^{\boldsymbol{\tau}}(t_2))\,\beta\\ &= -(\mathbf{u}(t_1)\otimes\overline{\mathbf{v}}+\overline{\mathbf{v}}\otimes\mathbf{u}(t_2),\nabla\nabla\beta) - \int_\Gamma \partial_t\overline{U}^n\beta\\ &\quad - \int_{\Gamma_-}(U^n(t_1)\overline{\mathbf{v}}+\overline{U}^n\cdot\mathbf{u}(t_2))\cdot\nabla\beta\\ &\quad - \int_{\Gamma_+}(U^n(t_1)\overline{\mathbf{U}}^{\boldsymbol{\tau}}+\overline{U}^n\cdot\mathbf{U}^{\boldsymbol{\tau}}(t_2))\cdot\nabla\beta\end{aligned}$$

Thus,

$$\begin{aligned}\|\overline{p}_r\|^q_{L^q(\Omega)} \leqslant{}& 2\|\mathbf{u}\|_{L^\infty(Q)}\|\overline{\mathbf{v}}\|_{L^q(\Omega)}\|\beta\|_{W^{2,q'}(\Omega)} + \|\partial_t\overline{U}^{\boldsymbol{n}}\|_{L^q(\Gamma)}\|\beta\|_{L^{q'}(\Gamma)}\\ &+ \|U^n(t_1)\|_{L^{q'}(\Gamma)}\left[\|\overline{\mathbf{v}}\|_{L^\infty(\Gamma)}+\|\overline{\mathbf{U}}\|_{L^\infty(\Gamma)}\right]\|\nabla\beta\|_{L^q(\Gamma)}\\ &+ \|\overline{\mathbf{U}}\|_{L^{q'}(\Gamma)}\left[\|\mathbf{U}\|_{L^\infty(\Gamma)}+\|\mathbf{u}(t_2)\|_{L^\infty(\Gamma)}\right]\|\nabla\beta\|_{L^q(\Gamma)}.\end{aligned}$$

But by Lemma A.6,

$$\begin{aligned}\|\overline{\mathbf{v}}\|_{L^q(\Omega)} &\leqslant \|\overline{\mathbf{v}}\|_{L^\infty(\Omega)} \leqslant \|\mathbf{u}\|_{\dot{C}^{0,1}_t(Q)}|t_1-t_2| \leqslant \|\mathbf{u}\|_{S^{1,\alpha}}|t_1-t_2|\\ \|\partial_t\overline{U}^{\boldsymbol{n}}\|_{L^q(\Gamma)} &\leqslant \|\partial_t\overline{U}^{\boldsymbol{n}}\|_{L^\infty(\Gamma)} \leqslant \|\partial_t U\|_{\dot{C}^{0,1}_t(Q)} \leqslant \|\mathbf{U}\|_{S^{2,\alpha}}|t_1-t_2|,\\ \|\overline{\mathbf{U}}\|_{L^q(\Gamma)} &\leqslant \|\mathbf{U}\|_{L^\infty(\Gamma)} \leqslant \|U\|_{\dot{C}^{0,1}_t(Q)} \leqslant \|\mathbf{U}\|_{S^{1,\alpha}}|t_1-t_2|,\end{aligned}$$

so, exploiting (10.3), we have

$$\|\overline{p}_r\|^q_{L^q(\Omega)} \leqslant C_2\|\beta\|_{W^{2,q'}(\Omega)}|t_1-t_2| \leqslant C_2\|\overline{p}_r\|^{q-1}_{L^q(\Omega)}|t_1-t_2|,$$

giving (10.5). □

Proposition 10.4. *Assume that the data has regularity N and let Ω' be as in Lemma 10.1. Let $\mathbf{u}\in S^{N+1,\alpha}$ and let p_r be the unique mean-zero solution to (3.5) with $q>3/(1-\alpha)$. Then for any integer k, $0\leqslant k\leqslant N$,*

$$\begin{aligned}\|\nabla p_r(t_1)-\nabla p_r(t_2)\|_{W^{k+1,q}(\Omega')} &\leqslant C_3(k)^2|t_1-t_2|^\alpha,\\ \|\nabla p_r(t_1)-\nabla p_r(t_2)\|_{C^{k,\alpha}(\Omega')} &\leqslant C_3(k)^2|t_1-t_2|^\alpha\end{aligned}\tag{10.7}$$

for all $t_1,t_2\in[0,T]$, where

$$C_3(k) := C\left[\|\boldsymbol{\omega}\|_{C^{k,\alpha}(Q)}+\|\mathbf{U}\|_{C^{k+2,\alpha}(Q)}+\|\mathbf{u}\|_{L^\infty(0,T;H)}\right].\tag{10.8}$$

Proof. We first prove $(10.7)_1$. Defining $\overline{p}_r := p_r(t_1)-p_r(t_2)$ and applying Lemma 10.1, we have

$$\|\overline{p}_r\|_{W^{k+2,q}(\Omega')} \leqslant C\left[\|\Delta\overline{p}_r\|_{W^{k,q}(\Omega)}+\|\nabla\overline{p}_r\cdot\boldsymbol{n}\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)}+\|\overline{p}_r\|_{L^q(\Omega)}\right].$$

This estimate is based upon the specific normalization of $\overline{p}_r$ given in Lemma 10.1, but $(10.7)_1$ itself is independent of that normalization, since the gradient eliminates any normalization constant. (But see Remark 10.5.)

Now,

$$\begin{aligned}\Delta \overline{p}_r &= \nabla \mathbf{u}(t_2) \cdot (\nabla \mathbf{u}(t_2))^T - \nabla \mathbf{u}(t_1) \cdot (\nabla \mathbf{u}(t_1))^T \\ &= \nabla(\mathbf{u}(t_2) - \mathbf{u}(t_1)) \cdot (\nabla \mathbf{u}(t_2))^T + \nabla \mathbf{u}(t_1) \cdot (\nabla(\mathbf{u}(t_2) - \mathbf{u}(t_1)))^T.\end{aligned}$$

Thus, for $k = 0$,

$$\|\Delta \overline{p}_r\|_{L^q(\Omega)} \leqslant 2\|\nabla(\mathbf{u}(t_1) - \mathbf{u}(t_2))\|_{L^q(\Omega)} \left[\|\nabla \mathbf{u}(t_1)\|_{L^\infty(\Omega)} + \|\nabla \mathbf{u}(t_2)\|_{L^\infty(\Omega)}\right].$$

For $k \geqslant 1$, $W^{k,q}(\Omega)$ is an algebra, as noted above, so

$$\|\Delta \overline{p}_r\|_{W^{k,q}(\Omega)} \leqslant C\|\nabla(\mathbf{u}(t_1) - \mathbf{u}(t_2))\|_{W^{k,q}(\Omega)} \left[\|\nabla \mathbf{u}(t_1)\|_{W^{k,q}(\Omega)} + \|\nabla \mathbf{u}(t_2)\|_{W^{k,q}(\Omega)}\right].$$

In either case, we have

$$\|\Delta \overline{p}_r\|_{W^{k,q}(\Omega)} \leqslant C\|\nabla \mathbf{u}\|_{W^{k,\alpha}(\Omega)}\|\nabla(\mathbf{u}(t_1) - \mathbf{u}(t_2))\|_{W^{k,q}(\Omega)}.$$

But, setting $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$,

$$\mathbf{u}(t_1) - \mathbf{u}(t_2) = K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}(t_1)] - K_{U^{\boldsymbol{n}}}[\boldsymbol{\omega}(t_2)] = K[\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)] + \mathbf{w}, \tag{10.9}$$

where

$$\mathbf{w} = \boldsymbol{\mathcal{V}}(t_1) - \boldsymbol{\mathcal{V}}(t_2) + \mathbf{u}_c(t_1) - \mathbf{u}_c(t_2).$$

Hence, applying Lemma 7.4,

$$\|\nabla \mathbf{u}(t_1) - \nabla \mathbf{u}(t_2)\|_{W^{k,q}(\Omega)} \leqslant C\|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{W^{k,q}(\Omega)} + C\|\nabla \mathbf{w}\|_{W^{k,q}(\Omega)}. \tag{10.10}$$

Applying Lemma A.7,

$$\|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{W^{k,q}(\Omega)} \leqslant \|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{C^k(\Omega)} \leqslant \|\boldsymbol{\omega}\|_{C^{k,\alpha}(Q)}|t_1 - t_2|^\alpha. \tag{10.11}$$

Using Lemma A.7 again,

$$\begin{aligned}\|\nabla \mathbf{w}\|_{W^{k,q}(\Omega)} &\leqslant C\|\nabla \mathbf{w}\|_{C^k(\Omega)} \leqslant \|\nabla \mathbf{w}\|_{C^{k,\alpha}(Q)}|t_1 - t_2|^\alpha \\ &\leqslant \|\mathbf{U}\|_{C^{k+1,\alpha}(Q)}|t_1 - t_2|^\alpha + \|\mathbf{u}\|_{L^\infty(0,T;H)}|t_1 - t_2|^\alpha.\end{aligned}$$

where we also used Lemma 7.2. Hence,

$$\|\nabla \mathbf{u}(t_1) - \nabla \mathbf{u}(t_2)\|_{W^{k,q}(\Omega)} \leqslant C_3(k)|t_1 - t_2|^\alpha. \tag{10.12}$$

Similarly, $\|\nabla \mathbf{u}\|_{W^{k,\alpha}(\Omega)} \leqslant C_3(k)$, so

$$\|\Delta \overline{p}_r\|_{L^q(\Omega)} \leqslant C_3(k)^2|t_1 - t_2|^\alpha.$$

On Γ_+, we use (9.2):

$$\nabla p_r \cdot \boldsymbol{n} = -\partial_t U_n + U^{\boldsymbol{n}} \operatorname{div}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} + (\kappa_1 + \kappa_2)(U^{\boldsymbol{n}})^2 - \mathbf{u}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma U^{\boldsymbol{n}} + \mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}.$$

Let us focus on the term $\mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}$, the other terms being similarly, though more simply, bounded.

By Lemma 10.2, $W^{k+1-\frac{1}{q},q}(\Gamma_+)$ is an algebra. Hence, starting with the trace inequality,

$$\begin{aligned}&\|(\mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}})(t_1) - \mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}})(t_2)\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)} \\ &\qquad\leqslant C\|\mathbf{u}^{\boldsymbol{\tau}}(t_1) \cdot \mathcal{A}(\mathbf{u}^{\boldsymbol{\tau}}(t_1) - \mathbf{u}^{\boldsymbol{\tau}}(t_2))\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)} \\ &\qquad\qquad + C\|(\mathbf{u}^{\boldsymbol{\tau}}(t_1) - \mathbf{u}^{\boldsymbol{\tau}}(t_2)) \cdot \mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}(t_2)\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|\mathbf{u}^{\boldsymbol{\tau}}(t_1)\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)}\|\mathcal{A}(\mathbf{u}^{\boldsymbol{\tau}}(t_1)-\mathbf{u}^{\boldsymbol{\tau}}(t_2))\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)}\\
&\qquad + C\|(\mathbf{u}^{\boldsymbol{\tau}}(t_1)-\mathbf{u}^{\boldsymbol{\tau}}(t_2))\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)}\|\mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}(t_2)\|_{W^{k+1-\frac{1}{q},q}(\Gamma_+)}\\
&\leqslant C\|\mathbf{u}^{\boldsymbol{\tau}}(t_1)\|_{W^{k+1,q}(\Omega)}\|\mathcal{A}(\mathbf{u}^{\boldsymbol{\tau}}(t_1)-\mathbf{u}^{\boldsymbol{\tau}}(t_2))\|_{W^{k+1,q}(\Omega)}\\
&\qquad + C\|(\mathbf{u}^{\boldsymbol{\tau}}(t_1)-\mathbf{u}^{\boldsymbol{\tau}}(t_2))\|_{W^{k+1,q}(\Omega)}\|\mathcal{A}\mathbf{u}^{\boldsymbol{\tau}}(t_2)\|_{W^{k+1,q}(\Omega)}\\
&\leqslant C\|\mathbf{u}\|_{W^{k+1,q}(\Omega)}\|\mathbf{u}(t_1)-\mathbf{u}(t_2)\|_{W^{k+1,q}(\Omega)}\\
&\leqslant C_3(k)\|\mathbf{u}\|_{S^{k+1,\alpha}}|t_1-t_2|^\alpha.
\end{aligned}
$$

In the last inequality we used that $\|\mathbf{u}\|_{W^{k+1,q}(\Omega)} \leqslant \|\mathbf{u}\|_{L^\infty([0,T];C^{k+1,\alpha}(\Omega))} \leqslant \|\mathbf{u}\|_{S^{k+1,\alpha}}$.

Along with similar bounds on the other terms coming from $\nabla p \cdot \boldsymbol{n}$, and using (10.5) of Proposition 10.3, noting that the constant C_2 can be absorbed into $C_3(k)$, these bounds give $(10.7)_1$. Then (10.2) with $(10.7)_1$ gives $(10.7)_2$. □

Remark 10.5. *Suppose that, instead of normalizing p_r so it has mean-zero, we were, for a fixed $t_1, t_2 \in [0,T]$, to normalize it so that $M_q(p_r(t_1) - p_r(t_2)) = 0$. Then we would have*

$$
\|p_r(t_1) - p_r(t_2)\|_{C^{k+1,\alpha}(\Omega')} \leqslant C_3(k)^2|t_1-t_2|^\alpha.
$$

Remark 10.6. *In the proof of Proposition 10.4 we used both the embedding of $W^{k+1,q}(\Omega)$ in $C^{k,\alpha}(\Omega)$ of (10.2) and, in (10.11), the simple embedding of $C^k(\Omega)$ in $W^{k,q}(\Omega)$ (using that the domain Ω is bounded). In each of these inequalities we lost, in a sense, information. It would seem, then, that a more direct estimate using the Hölder space analog of the elliptic estimates in Lemma 10.1 would be cleaner. Were we to do that, however, (10.10) would become*

$$
\|\nabla \mathbf{u}(t_1) - \nabla \mathbf{u}(t_2)\|_{C^{k,\alpha}(\Omega)} \leqslant C\|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{C^{k,\alpha}(\Omega)} + C\|\nabla \mathbf{w}\|_{C^{k,\alpha}(\Omega)},
$$

and there would be no way to obtain the needed factor of $|t_1 - t_2|^\alpha$ in $(10.7)_2$ as we obtained in (10.11).

To account for time derivatives $\partial_t^j p_r$, $j \leqslant N$, we note that (3.5) becomes

$$
\begin{cases}
\Delta \partial_t^j p_r = -\partial_t^j(\nabla \mathbf{u} \cdot (\nabla \mathbf{u})^T) & \text{in } \Omega,\\
\nabla \partial_t^j p_r \cdot \boldsymbol{n} = -\partial_t^{j+1} U^{\boldsymbol{n}} - \partial_t^j N[\mathbf{u}] & \text{on } \Gamma,
\end{cases}
$$

and the same analysis in Propositions 10.3 and 10.4 applies to $\partial_t^j p_r$. This yields the following corollaries:

Corollary 10.7. *Let $\mathbf{u}$ and p_r be as in Proposition 10.4 and let j be an integer with $0 \leqslant j \leqslant N$. Fixing $t_1, t_2 \in [0,T]$, normalize p_r so that $M_q(\partial_t^j p_r(t_1) - \partial_t^j p_r(t_2)) = 0$. Then*

$$
\|\partial_t^j p_r(t_1) - \partial_t^j p_r(t_2)\|_{L^q(\Omega)} \leqslant C_2|t_1 - t_2|.
$$

Proof. The proof is the same as that of (10.5), as we note that the constant C_2 already accounts for the presence of ∂_t^j. □

Corollary 10.8. *Let $\mathbf{u}$ and p_r be as in Proposition 10.4. Then $\nabla_\Gamma p_r \in C^{N,\alpha}([0,T] \times \Gamma_+)$ with, recalling Definition 6.1,*

$$
\|\nabla_\Gamma p_r\|_{C^{N,\alpha}([0,T]\times\Gamma_+)} \leqslant c_0 + c_X T^b, \tag{10.13}
$$

where $0 < b \leqslant \alpha$.

Proof. The observations above give that for any $j, k \geqslant 0$ integers with $0 \leqslant j + k \leqslant N$,

$$\begin{aligned} \|\partial_t^j \nabla p_r(t_1) - \partial_t^j \nabla p_r(t_2)\|_{W^{k+1,q}(\Omega')} &\leqslant C_3(N)^2 |t_1 - t_2|^\alpha, \\ \|\partial_t^j \nabla p_r(t_1) - \partial_t^j \nabla p_r(t_2)\|_{C^{k,\alpha}(\Omega')} &\leqslant C_3(N)^2 |t_1 - t_2|^\alpha \end{aligned} \tag{10.14}$$

for all $t_1, t_2 \in [0, T]$, where C_3 is as in (10.8), and we note that

$$C_3(N) \leqslant C \left[\|\boldsymbol{\omega}\|_{C^{N,\alpha}(Q)} + \|\mathbf{U}\|_{C^{N,\alpha}(Q)} + \|\mathbf{u}\|_{S^{N+1,\alpha}} \right] \leqslant c_X,$$

using that $\|\partial_t^N \mathbf{u}\|_{L^\infty(0,T;H)} \leqslant C\|\mathbf{u}\|_{S^{N+1,\alpha}}$.

Letting β be any time-space multi-index with $|\beta| \leqslant N$, it follows from $(10.14)_2$ that

$$\|D^\beta \nabla p_r(t_1) - D^\beta \nabla p_r(t_2)\|_{L^\infty(\Omega')} \leqslant \|D^\beta \nabla p_r(t_1) - D^\beta \nabla p_r(t_2)\|_{C^{j,\alpha}(\Omega')} \leqslant c_X |t_1 - t_2|^\alpha,$$

where $j = N - |\beta|$. Then, letting $Q' = [0, T] \times \Omega'$,

$$\begin{aligned} \|D^\beta \nabla p_r\|_{L^\infty(Q')} &= \sup_{t \in [0,T]} \|D^\beta \nabla p_r(t)\|_{L^\infty(\Omega')} \\ &\leqslant \|D^\beta \nabla p_r(0)\|_{L^\infty(\Omega')} + \sup_{t \in [0,T]} \|D^\beta \nabla p_r(t) - D^\beta \nabla p_r(0)\|_{L^\infty(\Omega')} \\ &\leqslant \|D^\beta \nabla p_r(0)\|_{L^\infty(\Omega')} + c_X T^\alpha \leqslant c_0 + c_X T^\alpha. \end{aligned}$$

Since ∇p_r is continuous and this bound for all $|\beta| \leqslant N$, we see that

$$\|\nabla p_r\|_{C^N(Q')} \leqslant c_0 + c_X T^\alpha. \tag{10.15}$$

Now suppose that $|\beta| = N$, so $j = N - |\beta| = 0$. Then

$$\begin{aligned} \|D^\beta \nabla p_r\|_{\dot{C}^\alpha_x(Q')} &= \sup_{t \in [0,T]} \|D^\beta \nabla p_r(t)\|_{\dot{C}^\alpha(\Omega')} \\ &\leqslant \|D^\beta \nabla p_r(0)\|_{\dot{C}^\alpha(\Omega')} + \sup_{t \in [0,T]} \|D^\beta \nabla p_r(t) - D^\beta \nabla p_r(0)\|_{\dot{C}^\alpha(\Omega')} \\ &\leqslant \|D^\beta \nabla p_r(0)\|_{\dot{C}^\alpha(\Omega')} + c_X T^\alpha \leqslant c_0 + c_X T^\alpha. \end{aligned} \tag{10.16}$$

This gives the spatial C^α-regularity of the highest derivatives of ∇p_r.

For the time regularity, let $f = D^\beta p_r$ and write $(10.14)_2$, noting that $j = 0$, as

$$\|\nabla(f(t_1) - f(t_2))\|_{C^\alpha(\Omega')} \leqslant \|\nabla(f(t_1) - f(t_2))\|_{C^\alpha(\Omega)} \leqslant c_X |t_1 - t_2|^\alpha.$$

Fix $t_1, t_2 \in [0, T]$ and normalize f as in Corollary 10.7, so that $M_q(f(t_1) - f(t_2)) = 0$. Then by Remark 10.5,

$$\|f(t_1) - f(t_2)\|_{C^{1,\alpha}(\Omega')} \leqslant c_X |t_1 - t_2|^\alpha,$$

and Corollary 10.7 gives

$$\|f(t_1) - f(t_2)\|_{L^2(\Omega')} \leqslant \|f(t_1) - f(t_2)\|_{L^2(\Omega)} \leqslant C^2 |t_1 - t_2|.$$

Then, applying the interpolation inequality in Lemma A.4 using Corollary 10.7,

$$\begin{aligned} &\|\nabla(f(t_1) - f(t_2))\|_{C^0(\Gamma_+)} \\ &\qquad \leqslant C \|f(t_1) - f(t_2)\|^a_{C^{1,\alpha}(\Omega')} \|f(t_1) - f(t_2)\|^{1-a}_{L^2(\Omega')} \\ &\qquad \leqslant C \left[c_X |t_1 - t_2|^\alpha \right]^a \left[c_X |t_1 - t_2| \right]^{1-a} \leqslant c_X |t_2 - t_1|^{\alpha'}, \end{aligned}$$

where $\alpha < \alpha' := 1 - a(1 - \alpha) < 1$ ($a = 2\alpha/(3 + 2\alpha)$ from Lemma A.4).

Although this bound was derived using a (t_1, t_2)-dependent normalization of the pressure, the bound itself is independent of that normalization and so applies uniformly for all $t_1, t_2 \in [0, T]$. It follows that

$$\|\nabla f\|_{\dot{C}_t^\alpha(Q')} = \sup_{\substack{t_1 \neq t_2 \\ \mathbf{x} \in \Omega'}} \frac{|\nabla f(t_1, \mathbf{x}) - \nabla f(t_2, \mathbf{x})|}{|t_1 - t_2|^\alpha} \leqslant c_X |t_1 - t_2|^{\alpha' - \alpha}.$$

We conclude that $\|D^\beta \nabla p_r\|_{\dot{C}_t^\alpha(Q')} \leqslant c_X T^{\alpha' - \alpha}$. Combined with (10.15) and (10.16), Lemma 7.1 gives (10.13) with $b = \max\{\alpha, \alpha' - \alpha\} > 0$. □

In Proposition 10.9, we obtain estimates on the difference of the pressure gradients for two velocity fields. These estimates will be used in the proof of Proposition 5.4, which only requires bounding the difference of pressures in $L^\infty([0, T] \times \Gamma_+)$. Hence, we produce the bound in the weakest feasible space, $L^\infty([0, T]; C^\alpha(\Omega))$.

Proposition 10.9. *Let $\mathbf{u}_1, \mathbf{u}_2 \in S^{1,\alpha}$, where $p_{r,1}, p_{r,2}$ solve (3.5) for $\mathbf{u}_1$, $\mathbf{u}_2$, respectively. Then*

$$\|\nabla p_{r,1} - \nabla p_{r,2}\|_{L^\infty([0,T];C^\alpha(\Omega))} \leqslant C_4,$$

where

$$C_4 := C \sum_{j=1}^2 \|\nabla \mathbf{u}_j\|_{L^\infty(Q)} \left[\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(\Omega)} + \|\operatorname{curl}(\mathbf{u}_1 - \mathbf{u}_2)\|_{L^q(\Omega)} \right].$$

Proof. We parallel the proof of Proposition 10.4 for the case $k = 0$.

Letting

$$\overline{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2, \quad P := p_{1,r} - p_{2,r},$$

noting that $\overline{\mathbf{u}} \in H$, the elliptic problem for P can be written,

$$\begin{cases} \Delta P = -\operatorname{div}(\mathbf{u}_1 \cdot \nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u}_2) & \text{in } \Omega, \\ \nabla P \cdot \boldsymbol{n} = -(\mathbf{u}_1 \cdot \nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u}_2) \cdot \boldsymbol{n} - \mathbb{1}_{\Gamma_+} \operatorname{div}_\Gamma(U^{\boldsymbol{n}} \overline{\mathbf{u}}^{\boldsymbol{\tau}}) & \text{on } \Gamma. \end{cases} \tag{10.17}$$

Also, on Γ_+, from (9.2),

$$\nabla P \cdot \boldsymbol{n} = -\overline{\mathbf{u}}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma U^{\boldsymbol{n}} + (\mathbf{u}_1^{\boldsymbol{\tau}} \cdot \mathcal{A} \overline{\mathbf{u}}^{\boldsymbol{\tau}} + \overline{\mathbf{u}}^{\boldsymbol{\tau}} \cdot \mathcal{A} \mathbf{u}_2^{\boldsymbol{\tau}}). \tag{10.18}$$

We can also write (10.17) as

$$\begin{cases} \Delta P = -(\nabla \mathbf{u}_1)^T \cdot \nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T \cdot \nabla \mathbf{u}_2) & \text{in } \overline{Q}, \\ \nabla P \cdot \boldsymbol{n} = -(\mathbf{u}_1 \cdot \nabla \overline{\mathbf{u}} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u}_2) & \text{on } [0, T] \times (\Gamma_- \cup \Gamma_0), \\ \nabla P \cdot \boldsymbol{n} = -\overline{\mathbf{u}}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma U^{\boldsymbol{n}} + \mathbf{u}_1^{\boldsymbol{\tau}} \cdot \mathcal{A} \overline{\mathbf{u}}^{\boldsymbol{\tau}} + \overline{\mathbf{u}}^{\boldsymbol{\tau}} \cdot \mathcal{A} \mathbf{u}_2^{\boldsymbol{\tau}} & \text{on } [0, T] \times \Gamma_+, \end{cases} \tag{10.19}$$

where we used (9.2). Then applying Lemma 10.1 with $q > 3/(1 - \alpha)$, we have

$$\|P\|_{W^{2,q}(\Omega')} \leqslant C \left[\|\Delta P\|_{L^q(\Omega)} + \|\nabla P \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{q},q}(\Gamma_+)} + \|P\|_{L^q(\Omega)} \right].$$

Because $\overline{\mathbf{u}} \in H$,

$$\begin{aligned}\|\Delta P\|_{L^q(\Omega)} &\leqslant 2\sum_{j=1}^{2} \|\nabla \mathbf{u}_j\|_{L^\infty(\Omega)} \|\nabla \overline{\mathbf{u}}\|_{L^q(\Omega)} \\ &\leqslant C \sum_{j=1}^{2} \|\nabla \mathbf{u}_j\|_{L^\infty(\Omega)} \left[\|\operatorname{curl} \overline{\mathbf{u}}\|_{L^q(\Omega)} + \|\nabla P_{H_c} \overline{\mathbf{u}}\|_{L^q(\Omega)}\right] \\ &\leqslant C \sum_{j=1}^{2} \|\nabla \mathbf{u}_j\|_{L^\infty(\Omega)} \left[\|\operatorname{curl} \overline{\mathbf{u}}\|_{L^q(\Omega)} + \|\overline{\mathbf{u}}\|_{H}\right],\end{aligned}$$

where we used Lemma 7.2, and

$$\begin{aligned}\|\nabla P \cdot \boldsymbol{n}\|_{W^{1-\frac{1}{q},q}(\Gamma_+)} &\leqslant \|\nabla P \cdot \boldsymbol{n}\|_{L^\infty(\Gamma_+)} \\ &\leqslant \|\nabla_\Gamma U^{\boldsymbol{n}}\|_{L^\infty(\Gamma_+)} \|\overline{\mathbf{u}}^{\boldsymbol{\tau}}\|_{L^\infty(\Gamma_+)} + C \sum_{j=1}^{2} \|\mathbf{u}_j\|_{L^\infty(\Gamma_+)} \|\overline{\mathbf{u}}^{\boldsymbol{\tau}}\|_{L^\infty(\Gamma_+)} \\ &\leqslant \|\mathbf{U}\|_{C^1(Q)} \|\overline{\mathbf{u}}\|_{L^\infty(\Omega)} + C \sum_{j=1}^{2} \|\mathbf{u}_j\|_{L^\infty(\Omega)} \|\overline{\mathbf{u}}\|_{L^\infty(\Omega)}.\end{aligned}$$

It remains to bound $\|P\|_{L^q(\Omega)}$. We follow the proof of (10.4) of Proposition 10.3, letting β solve

$$\begin{cases} \Delta\beta = P|P|^{q-2} & \text{in } \Omega, \\ \nabla\beta \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases}$$

where P is normalized so that $M_q(P) = 0$. We find that

$$\|P\|_{L^q(\Omega)}^q = -(\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2, \nabla\nabla\beta) - \int_\Gamma U^{\boldsymbol{n}} \overline{\mathbf{u}} \cdot \nabla\beta. \tag{10.20}$$

For the first term on the right-hand side of (10.20), we use that

$$\|\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2\|_{L^q(\Omega)} \leqslant \sum_{j=1}^{2} \|\mathbf{u}_j\|_{L^\infty(\Omega)} \|\overline{\mathbf{u}}\|_{L^{q'}(\Omega)},$$

so

$$-(\mathbf{u}_1 \otimes \mathbf{u}_1 - \mathbf{u}_2 \otimes \mathbf{u}_2, \nabla\nabla\beta) \leqslant C \sum_{j=1}^{2} \|\mathbf{u}_j\|_{L^\infty(\Omega)} \|\overline{\mathbf{u}}\|_{L^{q'}(\Omega)} \|\beta\|_{W^{2,q}(\Omega)}.$$

For the boundary integral in (10.20), we have

$$-\int_\Gamma U^{\boldsymbol{n}} \overline{\mathbf{u}} \cdot \nabla\beta \leqslant C\|\mathbf{U}\|_{L^q(Q)} \|\overline{\mathbf{u}}\|_{L^\infty(\Omega)} \|\beta\|_{W^{2,q'}(\Omega)} \leqslant C\|\mathbf{U}\|_{S^{1,\alpha}} \|\overline{\mathbf{u}}\|_{L^\infty(\Omega)} \|\beta\|_{W^{2,q'}(\Omega)}.$$

Combining these bounds, we have

$$\|P\|_{L^q(\Omega)}^q \leqslant C_4 \|\beta\|_{W^{2,q'}(\Omega)} \leqslant C_4 \|P\|_{L^q(\Omega)}^{q-1}.$$

Since this bound holds uniformly over time, we can use (10.2) to conclude $\|P\|_{L^\infty([0,T];C^\alpha(\Omega))} \leqslant C_4$, completing the proof. □

Part III: Estimates on the Operator A

Organization of Part III. In Section 11 we give the proof of Proposition 5.3 by first obtaining sufficient estimates on the operator A using (primarily) the pressure estimates from Section 10 along with the estimates on the flow map from Section 8. In Section 12, we use these estimates on A to prove Proposition 5.4. In Section 13, we give the proof of Proposition 5.5. Finally, in Section 14, we prove Theorem 1.4.

11. An invariant set

We now make a series of estimates leading, in Proposition 5.3, to the existence of an invariant set for the operator A. Recall that $\mathrm{Dom}_N(A)$ is given in Definition 3.4.

Proposition 11.1. *Assume that* $\mathbf{u} \in \mathrm{Dom}_N(A)$. *Then*

$$\|\mathbf{H}\|_{C^{N,\alpha}([0,T]\times\Gamma_+)} \leqslant c_0 + c_X T^b, \tag{11.1}$$

where $0 < b \leqslant \alpha$ *and* $\mathbf{H}$ *is given by (3.7). Suppose that* $\mathbf{u}_1$, $\mathbf{u}_2$ *both lie in* $\mathrm{Dom}_0(A)$ *with* $\|\mathbf{u}_1\|_{S^{1,\alpha}}, \|\mathbf{u}_1\|_{S^{1,\alpha}} \leqslant M$, *and let* $\mathbf{H}_i$, $i = 1, 2$, *be given by (3.7) with* $\mathbf{u} = \mathbf{u}_i$. *Then, letting* $q > 3/(1-\alpha)$,

$$\|\mathbf{H}_1 - \mathbf{H}_2\|_{L^\infty([0,T]\times\Gamma_+)} \leqslant c_0 M \left[\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(\Omega)} + \|\mathrm{curl}(\mathbf{u}_1 - \mathbf{u}_2)\|_{L^q(\Omega)}\right]. \tag{11.2}$$

Proof. Let $X := C^{N,\alpha}([0,T]\times\Gamma_+)$. From (3.7), we can write,

$$\mathbf{H}^{\boldsymbol{\tau}} = \delta_1 + \delta_2 - \nabla_\Gamma p_r, \quad H^{\boldsymbol{n}} = \mathrm{curl}_\Gamma \, \mathbf{U}^{\boldsymbol{\tau}},$$

where

$$\delta_1 := \frac{1}{U^{\boldsymbol{n}}}\left[-\partial_t \mathbf{U}^{\boldsymbol{\tau}} - \nabla_\Gamma\left(\frac{1}{2}|\mathbf{U}|^2\right) + \mathbf{f}\right]^\perp, \quad \delta_2 := \frac{1}{U^{\boldsymbol{n}}}\,\mathrm{curl}_\Gamma \, \mathbf{U}^{\boldsymbol{\tau}}\mathbf{u}^{\boldsymbol{\tau}}.$$

Hence, $\|H^{\boldsymbol{n}}\|_X \leqslant \|\mathbf{U}\|_{C^{N+1,\alpha}(Q)} + \|\mathbf{f}\|_{C^{N,\alpha}(Q)} \leqslant c_0$, $\|\delta_1\| \leqslant c_0$, and $\|\nabla_\Gamma p_r\|_X \leqslant c_0 + c_X T^b$ by Corollary 10.8. Then, letting $\phi = (U^{\boldsymbol{n}})^{-1}\,\mathrm{curl}_\Gamma \, \mathbf{U}^{\boldsymbol{\tau}}$ and applying Proposition 7.5,

$$\|\delta_2\|_X = \|\phi \mathbf{u}^{\boldsymbol{\tau}}\|_X \leqslant \|\phi\|_X \|\mathbf{u}\|_X \leqslant c_0 + (c_0 + M)\max\{T^{1-\alpha}, T^\alpha\}.$$

Together, these bounds yield (11.1).

Now suppose that $\mathbf{u}_1$, $\mathbf{u}_2$ both lie in $\mathrm{Dom}_0(A)$; hence, they then have the same initial data, and the same $\mathbf{U}$ and $\mathbf{f}$. So reviewing the estimates that led to (11.1), we see that $H_1^n - H_2^n = 0$ and many terms in $\mathbf{H}^{\boldsymbol{\tau}}$ cancel, leaving

$$\mathbf{H}_1^{\boldsymbol{\tau}} - \mathbf{H}_2^{\boldsymbol{\tau}} := \frac{1}{U^{\boldsymbol{n}}}\left[\nabla_\Gamma^\perp (p_{r,1} - p_{r,2}) + \mathrm{curl}_\Gamma \, \mathbf{U}^{\boldsymbol{\tau}}(\mathbf{u}_1^{\boldsymbol{\tau}} - \mathbf{u}_2^{\boldsymbol{\tau}})\right],$$

where $p_{r,j}$ is the pressure corresponding to u_j. The bound in (11.2) follows from this observation and Proposition 10.9. ∎

Proposition 11.2. *Assume that* $\mathbf{u} \in \mathrm{Dom}_N(A)$. *With* Λ *as in (3.10),*

$$\|\Lambda \mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant c_{\mathcal{N}}(c_0, M, T),$$

where $c_{\mathcal{N}}$ *is as in Definition 6.1 and* M *bounds* $\|\mathbf{u}\|_{S^{N+1,\alpha}}$, *as in (6.1).*

Proof. First assume no forcing. Let $\boldsymbol{\omega}_0 = \boldsymbol{\omega}(0)$ and recall the definition of $\boldsymbol{\gamma}_0$ in (8.6). From (8.7), we can write, $\overline{\boldsymbol{\omega}} := \Lambda \mathbf{u} = \overline{\boldsymbol{\omega}}_\pm$ on $U_\pm$, where

$$\begin{aligned} \overline{\boldsymbol{\omega}}_-(t,\mathbf{x}) &= \nabla\eta(0,t;\boldsymbol{\gamma}_0)\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0) \text{ on } U_-, \\ \overline{\boldsymbol{\omega}}_+(t,\mathbf{x}) &= \nabla\eta(\tau(t,\mathbf{x}),t;\boldsymbol{\gamma}(t,\mathbf{x}))\mathbf{H}(\tau(t,\mathbf{x}),\boldsymbol{\gamma}(t,\mathbf{x})) \text{ on } U_+. \end{aligned} \tag{11.3}$$

It follows, using Lemma 8.2, Remark 8.3, Proposition 11.1 that

$$\|\overline{\boldsymbol{\omega}}_-(t,\mathbf{x})\|_{L^\infty(U_-)} \leqslant \|\nabla\eta\|_{L^\infty(Q)}\|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)} \leqslant \|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)}e^{MT},$$
$$\|\overline{\boldsymbol{\omega}}_+(t,\mathbf{x})\|_{L^\infty(U_+)} \leqslant \|\nabla\eta\|_{L^\infty(Q)}\|\mathbf{H}\|_{L^\infty([0,T]\times\Gamma_+)} \leqslant [\|\boldsymbol{\omega}_0\|_{L^\infty(\Gamma_+)} + MT^\alpha]e^{MT},$$

which shows that $\|\Lambda\mathbf{u}\|_{L^\infty(Q)} \leqslant c_{\mathcal{N}}$.

Let us now first treat the case $N = 0$, to get a better understanding of the estimates involved. Using Lemma 8.2 and Remark 8.3 along with Lemmas A.1 and A.2, we see that

$$\begin{aligned}\|\overline{\boldsymbol{\omega}}_-\|_{C^\alpha(U_-)} &\leqslant \|\nabla\eta(0,t;\boldsymbol{\gamma}_0)\|_{C^\alpha(U_-)}\|\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0)\|_{C^\alpha(U_-)}\\ &\leqslant \|\nabla\eta(0,t;\cdot)\|_{C^\alpha(Q)}[\|\nabla\boldsymbol{\gamma}_0\|^\alpha_{L^\infty(U_-)}]^2\|\boldsymbol{\omega}_0\|_{C^\alpha(\Omega)}\\ &\leqslant \|\boldsymbol{\omega}_0\|_{C^\alpha(\Omega)}[1 + Me^{(1+2\alpha)MT}T^{1-\alpha}]e^{2MT} \leqslant c_{\mathcal{N}}(c_0, M, T).\end{aligned}$$

Note that $c_{\mathcal{N}}(c_0, M, 0) = \|\boldsymbol{\omega}_0\|_{C^\alpha(\Omega)}$, giving (6.2).

Similarly,

$$\|\overline{\boldsymbol{\omega}}_+(t,\mathbf{x})\|_{C^\alpha(U_+)} \leqslant \|\nabla\eta(\tau(t,\mathbf{x}),t;\boldsymbol{\gamma}(t,\mathbf{x}))\|_{C^\alpha(U_+)}\|\mathbf{H}(\tau(t,\mathbf{x}),\boldsymbol{\gamma}(t,\mathbf{x}))\|_{C^\alpha(U_+)}.$$

Using Lemmas 8.2 and A.2,

$$\begin{aligned}&\|\nabla\eta(\tau(t,\mathbf{x}),t;\boldsymbol{\gamma}(t,\mathbf{x}))\|_{C^\alpha(U_+)} \leqslant \|\nabla\eta(t_1,t_2;\mathbf{x})\|_{C^\alpha([0,T]^2\times\Omega)}[1 + \|D\mu\|_{L^\infty(U_+)}]^\alpha\\ &\qquad \leqslant [e^{MT} + e^{(1+2\alpha)MT}MT^{1-a}][(1+M^2)e^{MT}]^\alpha.\end{aligned}$$

Then, using Lemma 8.2 with Remark 8.3, and Proposition 11.1,

$$\begin{aligned}&\|\mathbf{H}(\tau(t,\mathbf{x}),\boldsymbol{\gamma}(t,\mathbf{x}))\|_{C^\alpha(U_+)} \leqslant \|\mathbf{H}\|_{C^\alpha([0,T]\times\Gamma_+)}[1 + \|D\mu\|_{L^\infty(U_+)}]^\alpha\\ &\qquad \leqslant [(1+M^2)e^{MT}]^\alpha[c_{\mathcal{N}} + CM(M+1)].\end{aligned}$$

These bounds lead to

$$\|\overline{\boldsymbol{\omega}}_+(t,\mathbf{x})\|_{C^\alpha(U_+)} \leqslant c_{\mathcal{N}}e^{(1+4\alpha)MT}(1+M^2)^{1+\alpha}(1+MT^{1-\alpha}) \leqslant c_{\mathcal{N}}(c_0, M, T),$$

and we can see also that (6.2) holds.

But we know from Theorem 2.2 that $\overline{\boldsymbol{\omega}} \in C^\alpha(Q)$, because we assumed cond$_0$: hence, taking the maximum of the bounds for $\overline{\boldsymbol{\omega}}_\pm$ on $U_\pm$, and using that $\|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)} \leqslant M$ yields the bound on $\|\Lambda\mathbf{u}\|_{C^\alpha(Q)}$.

Now consider $N \geqslant 1$. The expressions for $\overline{\boldsymbol{\omega}}_\pm$ in (11.3) each consist of two factors. We first apply Leibniz's product rule to these expressions then apply the chain rule to each term. For $\overline{\boldsymbol{\omega}}_+$, if β is a time-space multi-index with $|\beta| = N$, then $D^\beta\overline{\boldsymbol{\omega}}_+$ consists of a finite sum of terms of the form,

$$D^{\beta_1}\nabla\eta(\tau(t,\mathbf{x}),t;\boldsymbol{\gamma}(t,\mathbf{x}))D^{\beta_2}\mathbf{H}(\tau(t,\mathbf{x}),\boldsymbol{\gamma}(t,\mathbf{x}))\prod_{\ell=1}^n D^{\beta_3^\ell}\mu(t,\mathbf{x}) \text{ on } U_+,$$

where $\beta_1 + \beta_2 = \beta$ and $\sum_{\ell=1}^n|\beta_3^\ell| = |\beta|$. The factors can be controlled by Proposition 11.1 and Lemma 8.2 with Remark 8.3. Following the similar process for $D^\beta\overline{\boldsymbol{\omega}}_-$ leads to an estimate for $\|\Lambda\mathbf{u}\|_{C^{N,\alpha}(Q)}$ of the same general form as for $\|\Lambda\mathbf{u}\|_{C^\alpha(Q)}$.

This gives the bound $\|\Lambda\mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant c_{\mathcal{N}}(c_0, M, T)$ in the absence of forcing. For forcing, we must bound $\mathbf{G}_\pm$ of (8.7). Now, as noted in Remark 8.5, $\overline{\boldsymbol{\omega}}_\pm$ need not be continuous across $\mathcal{S}$ and $\mathbf{G}_\pm$ need not be continuous across $\mathcal{S}$; rather, $\overline{\boldsymbol{\omega}}_\pm + \mathbf{G}_\pm$ is $C^{N,\alpha}$-continuous across $\mathcal{S}$. Nonetheless, adding separate bounds on $\overline{\boldsymbol{\omega}}_+$, $\overline{\boldsymbol{\omega}}_-$, $\mathbf{G}_+$, and $\mathbf{G}_-$ in $C^{N,\alpha}(U_\pm)$ give the bound on $\Lambda\mathbf{u} = \overline{\boldsymbol{\omega}}$ in $C^{N,\alpha}(Q)$.

It follows from Remark 8.3 and repeated applications of the chain rule that, in fact,

$$\|\mathbf{G}_\pm\|_{C^{N,\alpha}(Q))} \leqslant c_\mathcal{N},$$

which completes the proof. □

Proposition 11.3. *For any* $\mathbf{u} \in \mathrm{Dom}_N(A)$,

$$\|A\mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant c_\mathcal{N}(c_0, M, T).$$

Proof. Let $\mathbf{u} \in \mathrm{Dom}_N(A)$ and $\mathbf{v} = A\mathbf{u}$, which we note satisfies (2.5). Our goal is to bound $\mathbf{v}$ in the space $R^{N+1,\alpha}$ of Definition 7.6 and then apply Lemma 7.8.

From (2.5) we have $\partial_t \mathbf{v} = \mathbf{f} - P_H[\mathbf{u} \cdot \overline{\mathbf{\Omega}}]$. But,

$$P_H\mathbf{v} = P_H(\mathbf{v} - \mathbf{U}) + P_H\mathbf{U} = \mathbf{v} - \mathbf{w}, \quad \mathbf{w} := \mathbf{U} - P_H\mathbf{U},$$

so

$$\partial_t \mathbf{v} = \mathbf{f} + \mathbf{w} - P_H[\mathbf{u} \cdot \overline{\mathbf{\Omega}}], \tag{11.4}$$

and, for any $1 \leqslant j \leqslant N+1$,

$$\partial_t^j \mathbf{v} = \partial_t^{j-1}\mathbf{f} + \partial_t^{j-1}\mathbf{w} - P_H[\partial_t^{j-1}(\mathbf{u} \cdot \overline{\mathbf{\Omega}})].$$

Hence, letting $X^j := L^\infty([0,T]; C^{N+1-j,\alpha}(\Omega))$,

$$\begin{aligned}
\|\partial_t^j \mathbf{v}\|_{L^\infty([0,T];C^{N+1-j,\alpha}(\Omega))} &= \|\partial_t^j \mathbf{v}\|_{X^j} \\
&\leqslant \|\partial_t^{j-1}\mathbf{f}\|_{X^j} + \|\partial_t^{j-1}\mathbf{w}\|_{X^j} + C\|\partial_t^{j-1}(\mathbf{u} \cdot \overline{\mathbf{\Omega}})\|_{X^j} \\
&\leqslant \|\mathbf{f}\|_{C^{N,\alpha}(Q)} + \|\mathbf{w}\|_{C^{N,\alpha}(Q)} + C\|(\mathbf{u} \cdot \overline{\mathbf{\Omega}})\|_{C^{N,\alpha}(Q)} \\
&\leqslant \|\mathbf{f}\|_{C^{N,\alpha}(Q)} + \|\mathbf{w}\|_{C^{N,\alpha}(Q)} + C\|\mathbf{u}\|_{C^{N,\alpha}(Q)}\|\overline{\mathbf{\Omega}}\|_{C^{N,\alpha}(Q)} \\
&\leqslant c_0 + c_\mathcal{N} \leqslant c_\mathcal{N}.
\end{aligned}$$

We used the continuity of P_H in the algebra $C^{N,\alpha}(\Omega)$ (though not in $C^{N,\alpha}(Q)$) and in the last inequality we used that $\|\mathbf{u}\|_{C^{N,\alpha}(Q)} \leqslant c_\mathcal{N}$ by Proposition 7.5 and, because $\overline{\boldsymbol{\omega}} = \Lambda\mathbf{u}$, $\|\overline{\boldsymbol{\omega}}\|_{C^{N,\alpha}(Q)} \leqslant c_\mathcal{N}$ by Proposition 11.2. This same bound follows for $j = 0$ by applying Lemma 7.4, which completes the demonstration that $\mathbf{v} \in R^{N+1,\alpha}$.

We conclude by Lemma 7.8 that $\|A\mathbf{u}\|_{C^{N,\alpha}(Q)} = \|\mathbf{v}\|_{C^{N,\alpha}(Q)} \leqslant c_\mathcal{N}$. □

Corollary 11.4. *For any* $\mathbf{u} \in \mathrm{Dom}_N(A)$, $\|A\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant c_\mathcal{N}(c_0, M, T)$.

Proof. In light of Propositions 11.2 and 11.3, it remains only to bound $\partial_t^{N+1} A\mathbf{u}$ in the space $L^\infty([0,T]; C^\alpha(\Omega))$. For this, we apply the Leray projector to (2.5). Writing $A\mathbf{u} = \mathbf{v} = \overline{\mathbf{v}} + \boldsymbol{\mathcal{V}}$, where $\mathbf{v} \in H$, we have $\partial_t \overline{\mathbf{v}} = -P_H(\mathbf{u} \cdot \overline{\mathbf{\Omega}}) + \mathbf{f}$, using that $P_H\boldsymbol{\mathcal{V}} = 0$ because $\boldsymbol{\mathcal{V}}$ is a gradient. Hence,

$$\partial_t^{N+1}\overline{\mathbf{v}} = -\partial_t^N P_H(\mathbf{u} \cdot \overline{\mathbf{\Omega}}) + \partial_t^N \mathbf{f} = -P_H\left(\partial_t^N(\mathbf{u} \cdot \overline{\mathbf{\Omega}})\right) + \partial_t^N \mathbf{f}.$$

Using that P_H is continuous in $C^\alpha(\Omega)$, we see that

$$\begin{aligned}
\|\partial_t^{N+1}\mathbf{v}\|_{L^\infty([0,T];C^\alpha(\Omega))} &\leqslant C\|\mathbf{u} \cdot \overline{\mathbf{\Omega}}\|_{C^{N,\alpha}(Q)} + \|f\|_{C^{N,\alpha}(Q)} \\
&\leqslant \|\mathbf{u}\|_{C^{N,\alpha}(Q_T)}\|\overline{\boldsymbol{\omega}}\|_{C^{N,\alpha}(Q_T)} + \|f\|_{C^{N,\alpha}(Q)} \\
&\leqslant c_\mathcal{N}(c_0, M, T)c_\mathcal{N}(c_0, M, T) + c_\mathcal{N}(c_0, M, T) \leqslant c_\mathcal{N}(c_0, M, T),
\end{aligned}$$

where we applied Propositions 7.5 and 11.2. This gives the required bound on $\partial_t^{N+1} A\mathbf{u}$ in $L^\infty([0,T]; C^\alpha(\Omega))$. □

Having established our many estimates, we can now give the proof of Proposition 5.3.

Proof of Proposition 5.3. For an arbitrary $T > 0$, recall that we set

$$\mathcal{K} = \mathcal{K}_{M,T} := \{\mathbf{u} \in \mathrm{Dom}_N(A) \colon \|\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant M\}.$$

By Corollary 11.4, for *any* $T, M > 0$,

$$\mathbf{u} \in \mathcal{K}_{M,T} \implies \|A\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant c_{\mathcal{N}}(c_0, M, T).$$

Since $c_{\mathcal{N}}(c_0, M, T) = c_0$, which is independent of M, we can now choose a specific $M > c_0$. Then the continuity of $c_{\mathcal{N}}(c_0, M, T)$ allows us to choose $T > 0$ for which $c_{\mathcal{N}}(c_0, M, T) < M$. But this means that

$$\mathbf{u} \in \mathcal{K}_{M,T} \implies \|A\mathbf{u}\|_{S^{N+1,\alpha}} \leqslant M \implies A\mathbf{u} \in \mathcal{K}_{M,T}.$$

That is, $\mathcal{K}_{M,T}$ is invariant under the operator A. □

12. Continuity of the operator A

Throughout this section, we let M, T, and $\mathcal{K} = \mathcal{K}_{M,T}$ be fixed, as given by Proposition 5.3. We also fix $\beta \in (0, \alpha)$ arbitrarily.

Before giving the proof of Proposition 5.4, we establish a series of estimates on the difference between two velocity fields in $\mathcal{K}$ and the difference of their corresponding flow maps. We assume that

$$\mathbf{u}_1, \mathbf{u}_2 \text{ are two vector fields in } \mathcal{K},$$

and define the following:

- $\boldsymbol{\omega}_j := \operatorname{curl} \mathbf{u}_j$ for $j = 1, 2$,
- η_j, τ_j, $\boldsymbol{\gamma}_j$, $U^j_\pm$, and hypersurface $\mathcal{S}_j$ are defined as in Section 8 for $\mathbf{u}_j$, $j = 1, 2$,
- $V_\pm := U^1_\pm \cap U^2_\pm$,
- $W := Q \setminus (V_+ \cup V_-)$.

We define $\mu_j \colon U_+ \to [0, T] \times \Gamma_+$ by

$$\mu_j(t, \mathbf{x}) := (\tau_j(t, \mathbf{x}), \boldsymbol{\gamma}_j(t, \mathbf{x}))$$

and

$$\mathbf{w} := \mathbf{u}_1 - \mathbf{u}_2, \quad \mu := \mu_1 - \mu_2.$$

We set

$$\theta_\beta := \|\mathbf{w}\|_{\mathring{S}^{0,\beta}} = \|\mathbf{w}\|_{C^\beta(Q)} + \|\operatorname{curl} \mathbf{w}\|_{C^\beta(Q)}. \tag{12.1}$$

Remark 12.1. *In this section, we use the convention that $\widetilde{F}$ stands for a continuous function from $[0, \infty)$ to $[0, \infty)$ with $\widetilde{F}(0) = 0$. Its precise values will be unimportant, and may vary from occurrence to occurrence in expressions.*

Lemma 12.2 gives two interpolation inequalities between $L^\infty(Q)$-based spaces and $\mathring{S}^{N+1,\alpha}$. When applied to $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, we have $\|\mathbf{w}\|_{\mathring{S}^{N+1,\alpha}} \leqslant \|\mathbf{w}\|_{S^{N+1,\alpha}} \leqslant 2M$, so Lemma 12.2 will allow us to control the size of higher norms of $\mathbf{w}$ and $\operatorname{curl} \mathbf{w}$ by the size of their $L^\infty(Q)$ norms. This will greatly simplify our arguments, since estimating $\mathbf{w}$ and $\operatorname{curl} \mathbf{w}$ in L^∞ norms is much easier than in higher norms, and because of this, all of the estimates we obtain following Lemma 12.2 will be in L^∞.

Lemma 12.2. *For any* $\mathbf{u} \in \mathring{S}^{N+1,\alpha}$,

$$\|\operatorname{curl} \mathbf{u}\|_{C^{N,\beta}(Q)} \leqslant \widetilde{F}\left(\|\operatorname{curl} \mathbf{u}\|_{L^\infty(Q)}\right),$$
$$\|\mathbf{u}\|_{\mathring{S}^{N+1,\beta}} \leqslant \widetilde{F}\left(\|\mathbf{u}\|_{L^\infty(Q)} + \|\operatorname{curl} \mathbf{u}\|_{L^\infty(Q)}\right),$$

where the function $\widetilde{F}$ *depends on* $\|\mathbf{u}\|_{\mathring{S}^{N+1,\alpha}} \leqslant M$.

Proof. Let $f \in C^{N,\alpha}(Q)$. By Lemma A.5,

$$\|f\|_{C^{N,\beta}(Q)} \leqslant \|f\|_{L^\infty(Q)} + F_c(\|f\|_{C^{N,\alpha}(Q)})\|f\|_{L^2(Q)}^{1-a}, \tag{12.2}$$

where $F_c(x) = x^{a_1} + x^{a_N} + x^{a'}$, where a_1 and a_N are given in Lemma A.4, and a' is given in Lemma A.5, and each of a_1, a_N, a' lies in $(0,1)$. The exponent a, which also lies in $(0,1)$, depends upon whether $\|f\|_{L^2(Q)}$ is greater or less than 1.

Because $0 < a_1, a_N, a' < 1$ and for any $0 < b < 1$, $(x+y)^b < x^b + y^b \leqslant 2^b(x+y)^b$, we see that $F_c(x+y) \leqslant C(F_c(x) + F_c(y))$. Applying this inequality with (12.2) to $f = \mathbf{u}$ and $f = \operatorname{curl} \mathbf{u}$, and using that $F_c(x+y) \leqslant C(F_c(x) + F_c(y))$ gives the result. □

By Lemma 8.2, we have, for $j = 1, 2$,

$$\|\eta_j(0,\cdot;\cdot)\|_{C^{N+1,\alpha}(Q} \leqslant C(T, M). \tag{12.3}$$

We generally do not state the dependence of constants on T and M, which are fixed and hence have no impact on the proof of Proposition 5.4. We do state such dependence explicitly, however, when it makes the nature of the bound being derived clearer.

Lemma 12.3. *We have,*

$$\|\mu\|_{L^\infty(V_+)} \leqslant C(T, M)T\theta_\beta.$$

Proof. We know from Lemma 3.5 of [11] that μ_j is transported by the flow map for $\mathbf{u}_j$; that is,

$$\partial_t \mu_1 + \mathbf{u}_1 \cdot \nabla \mu_1 = 0,$$
$$\partial_t \mu_2 + \mathbf{u}_2 \cdot \nabla \mu_2 = 0.$$

Hence,

$$\partial_t \mu + \mathbf{u}_1 \cdot \nabla \mu = -\mathbf{w} \cdot \nabla \mu_2,$$

or,

$$\frac{d}{dt}\mu(t, \eta_1(0,t;\mathbf{x})) = -(\mathbf{w} \cdot \nabla \mu_2)(t, \eta_1(0,t;\mathbf{x})).$$

Integrating in time, using that $\mu(t, \eta_1(0,t;\mathbf{x}))|_{t=0} = 0$, and employing Lemma 8.2 gives

$$\begin{aligned}\mu(t, \eta_1(0,t;\mathbf{x})) &= -\int_0^t (\mathbf{w} \cdot \nabla \mu_2)(s, \eta_1(0,s;\mathbf{x})) \leqslant \|\mathbf{w}\|_{L^\infty(Q)} \|\nabla \mu_2\|_{L^\infty(Q)} \\ &\leqslant C(T, M)\theta_\beta.\end{aligned}$$

□

Lemma 12.4. *We have*

$$\|\eta_1 - \eta_2\|_{L^\infty([0,T]^2 \times \Omega)} \leqslant C(T, M)T\theta_\beta,$$
$$\|\nabla \eta_1 - \nabla \eta_2\|_{L^\infty([0,T]^2 \times \Omega)} \leqslant C(T, M)T[\theta_\beta + \theta_\beta^\alpha].$$

Proof. We have,

$$\eta_1(t_1,t_2;\mathbf{x})-\eta_2(t_1,t_2;\mathbf{x})=\int_{t_1}^{t_2}\left[\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x}))-\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x}))\right]\,ds.$$

Fixing t_1, using (12.3), Lemma A.2, Lemma A.3, and applying Minkowski's integral inequality gives

$$\begin{aligned}
&|\eta_1(t_1,t;\mathbf{x})-\eta_2(t_1,t;\mathbf{x})|\\
&\qquad\leqslant \int_{t_1}^{t}|\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))-\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x}))|\,ds\\
&\qquad\qquad+\int_{t_1}^{t}|\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x}))-\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))|\,ds\\
&\qquad\leqslant \int_{t_1}^{t}\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_{L^\infty(\Omega)}\,ds+\int_{t_1}^{t}\|\mathbf{u}_1(s)\|_{\dot{C}^1(\Omega)}|\eta_1(t_1,s;\mathbf{x})-\eta_2(t_1,s;\mathbf{x})|\,ds\\
&\qquad\leqslant T\theta_\beta+C(T,M)\int_{t_1}^{t}\|\eta_1(t_1,s;\cdot)-\eta_2(t_1,s;\cdot)\|_{L^\infty(\Omega)}\,ds.
\end{aligned}$$

Taking the supremum over $\mathbf{x}$ and applying Grönwall's Lemma gives

$$\|\eta_1(t_1,t;\mathbf{x})-\eta_2(t_1,t;\mathbf{x})\|_{C_t([0,T];L^\infty_{\mathbf{x}}(\Omega))}\leqslant Te^{C(M,T)T}\theta_\beta.$$

Since this holds uniformly for all $t_1\in[0,T]$, we obtain the first bound.

Similarly, starting from

$$\begin{aligned}
\nabla\eta_1(t_1,t;\mathbf{x})-\nabla\eta_2(t_1,t;\mathbf{x})&=\int_{t_1}^{t}\left[\nabla_{\mathbf{x}}(\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x})))-\nabla_{\mathbf{x}}(\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x})))\right]\,ds\\
&=\int_{t_1}^{t}\left[\nabla\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x}))\nabla\eta_1(t_1,s;\mathbf{x})-\nabla\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x}))\nabla\eta_2(t_1,s;\mathbf{x})\right]\,ds,
\end{aligned}$$

we find

$$\begin{aligned}
&|\nabla\eta_1(t_1,t;\mathbf{x})-\nabla\eta_2(t_1,t;\mathbf{x})|\\
&\qquad\leqslant\int_{t_1}^{t}|\nabla\mathbf{u}_1(s,\eta_1(t_1,s;\mathbf{x}))\nabla\eta_1(t_1,s;\mathbf{x})-\nabla\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))\nabla\eta_1(t_1,s;\mathbf{x})|\,ds\\
&\qquad\qquad+\int_{t_1}^{t}|(\nabla\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))-\nabla\mathbf{u}_2(s,\eta_2(t_1,s;\mathbf{x})))\nabla\eta_2(t_1,s;\mathbf{x})|\,ds\\
&\qquad\qquad+\int_{t_1}^{t}|\nabla\mathbf{u}_1(s,\eta_2(t_1,s;\mathbf{x}))(\nabla\eta_1(t_1,s;\mathbf{x})-\nabla\eta_2(t_1,s;\mathbf{x}))|\,ds\\
&\qquad\leqslant\int_{t_1}^{t}\|\mathbf{u}_1(s)\|_{\dot{C}^\alpha}\|\eta_1(t_1,s;\cdot)-\eta_2(t_1,s;\cdot)\|^\alpha_{L^\infty(\Omega)}\|\nabla\eta_1(t_1,s;\cdot)\|_{L^\infty(\Omega)}\,ds\\
&\qquad\qquad+\int_{t_1}^{t}\|\nabla\mathbf{u}_1(s)-\nabla\mathbf{u}_2(s)\|_{L^\infty(\Omega)}\|\nabla\eta_2(t_1,s;\cdot)\|_{L^\infty(\Omega)}\,ds\\
&\qquad\qquad+\int_{t_1}^{t}\|\mathbf{u}_1(s)\|_{\dot{C}^1}\|\nabla\eta_1(t_1,s;\cdot)-\nabla\eta_2(t_1,s;\cdot)\|_{L^\infty(\Omega)}\,ds\\
&\qquad\leqslant C(T,M)[Te^{(C(T,M)T}\theta_\beta]^\alpha T+C(M,T)T\theta_\beta
\end{aligned}$$

$$+ C(M,T)\int_{t_1}^{t} \|\nabla\eta_1(t_1,s;\cdot) - \nabla\eta_2(t_1,s;\cdot)\|_{L^\infty(\Omega)}\,ds.$$

In the last inequality, we used Lemma 7.4 to conclude that $\|\nabla\mathbf{u}_1(s) - \nabla\mathbf{u}_2(s)\|_{L^\infty(\Omega)} \leqslant \|\mathbf{w}(s)\|_{C^{1,\beta}(\Omega)} \leqslant C\|\operatorname{curl}\mathbf{w}(s)\|_{C^\beta(\Omega)} + C\|\mathbf{w}(s)\|_H \leqslant C\theta_\beta$. Taking the supremum over $\mathbf{x}$ and applying Grönwall's Lemma as before gives the second bound. □

Lemma 12.5. *Letting $|W|$ be the Lebesgue measure of $W := Q \setminus (V_+ \cup V_-)$, we have*

$$|W| \leqslant C(T,M)T^2\theta_\beta.$$

Proof. The set $W(t) := \{\mathbf{x}\in\Omega\colon (t,\mathbf{x})\in W\}$ consists of all points lying between the surfaces $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$. Any $\mathbf{x}_1 \in \mathcal{S}_1(t)$ is of the form $\mathbf{x}_1 = \eta_1(0,t;\mathbf{y})$ for some $\mathbf{y}\in\Gamma_+$, and by Lemma 12.4, the point $\mathbf{x}_2 = \eta_2(0,t;\mathbf{y})$ is within a distance $\delta = C(T,M)T\theta_\beta$ of $\mathbf{x}_1$. That is, any point in $\mathcal{S}_1(t)$ is within a distance δ of $\mathcal{S}_2(t)$ and the relation is symmetric. So

$$W(t) \subseteq W_\delta(t) := \{x\in\Omega\colon\ \operatorname{dist}(x,\mathcal{S}_1(t)) \leqslant \delta\}.$$

As we observed in Section 8, $\mathcal{S}_1(t)$ is at least $C^{1,\alpha}$ regular as a surface in Ω, and so has finite Hausdorff measure; hence, we can see that $|W_\delta(t)| \leqslant C\delta$. Moreover, this constant can depend upon T and M, but is bounded over $[0,T]$, for as also observed in Section 8, $\mathcal{S}_1$ is at least $C^{1,\alpha}$ regular as a hypersurface in Q. Thus, $|W| \leqslant T|W_\delta(t)| \leqslant C(T,M)T^2\theta_\beta$. □

Proof of Proposition 5.4. We will show that

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|_{L^\infty(Q)} \leqslant \widetilde{F}(\theta_\beta). \tag{12.4}$$

Once we obtain (12.4), we will have

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|_{L^\infty(Q)} \leqslant \widetilde{F}(\theta_\beta) = \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathring{S}^{0,\beta}} \leqslant \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathring{S}^{N+1,\beta}},$$

and continuity of $A\colon \mathcal{K}\to\mathcal{K}$ in $\mathring{S}^{N+1,\beta}$ will follow from Lemma 12.2.

We will obtain the bound on $\|A\mathbf{u}_1 - A\mathbf{u}_2\|_{L^\infty(Q)}$ in (12.4) by the following three steps:

(A) Bound the difference in vorticities, $\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2$, in $L^\infty(Q)$ assuming zero forcing.
(B) Account for forcing in the bound on $\Lambda\mathbf{u}_1 - \Lambda\mathbf{u}_2$ in $L^\infty(Q)$.
(C) Bound $A\mathbf{u}_1 - A\mathbf{u}_2$ in $L^\infty(Q)$ with the help of (B).

(A) Vorticity: Letting $(t,\mathbf{x})\in Q$, we must estimate $|\Lambda\mathbf{u}_1(t,\mathbf{x}) - \Lambda\mathbf{u}_2(t,\mathbf{x})|$. This involves three cases: **(1)** $(t,\mathbf{x})\in V_-$, **(2)** $(t,\mathbf{x})\in V_+$, **(3)** $(t,\mathbf{x})\in W$, which we consider separately. We argue first without forcing.

(1) Define, for $(t,\mathbf{x})\in V_-$, $j=1,2$,

$$\boldsymbol{\gamma}_0^j = \boldsymbol{\gamma}_0^j(t,\mathbf{x}) := \eta_j(t,0;\mathbf{x}). \tag{12.5}$$

From (8.7), we can write,

$$\Lambda\mathbf{u}_1(t,\mathbf{x}) - \Lambda\mathbf{u}_2(t,\mathbf{x}) = \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^1)\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1) - \nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2)\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^2) = I_1 + I_2,$$

where

$$I_1 := \boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1)\cdot(\nabla\eta_1(0,t;\boldsymbol{\gamma}_0^1) - \nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2)),$$
$$I_2 := (\boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1) - \boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^2))\cdot\nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2).$$

We also make the decomposition, $I_1 = \boldsymbol{\omega}_0(\boldsymbol{\gamma}_0^1)\cdot(I_1^1 + I_1^2)$, where

$$I_1^1 := \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^1) - \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^2),$$
$$I_1^2 := \nabla\eta_1(0,t;\boldsymbol{\gamma}_0^2) - \nabla\eta_2(0,t;\boldsymbol{\gamma}_0^2).$$

Then,

$$\|I_1\|_{L^\infty(V_-)} \leqslant \|\boldsymbol{\omega}_0\|_{L^\infty(\Omega)} \left(\|I_1^1\|_{L^\infty(V_-)} + \|I_1^2\|_{L^\infty(V_-)}\right),$$

with

$$\begin{aligned}\|I_1^1\|_{L^\infty(V_-)} &\leqslant \|\nabla\eta_1(0,t;\cdot)\|_{\dot{C}^\alpha(\Omega)}\|\eta_1(t,0;\cdot)-\eta_2(t,0;\cdot)\|^\alpha_{L^\infty(\Omega)}\\ &\leqslant C(T,M)T[T\theta_\beta]^\alpha \leqslant C(T,M)T^{1+\alpha}\theta_\beta^\alpha,\\ \|I_1^2\|_{L^\infty(V_-)} &\leqslant \|\nabla\eta_1(0,t;\cdot)-\nabla\eta_2(0,t;\cdot)\|_{L^\infty(\Omega)} \leqslant C(T,M)T[\theta_\beta+\theta_\beta^\alpha],\end{aligned}$$

where we applied Lemma 12.4. Similarly, applying Lemmas 12.4 and A.3,

$$\begin{aligned}\|I_2\|_{L^\infty(V_-)} &\leqslant \|\boldsymbol{\omega}_0\|_{\dot{C}^\alpha(\Omega)}\|\eta_1(t,0;\cdot)-\eta_2(t,0;\cdot)\|^\alpha_{L^\infty(V_-)}\|\nabla\eta_2(0,t,\cdot)\|_{L^\infty(V_-)}\\ &\leqslant C(T,M)M[C(T,M)T\theta_\beta]^\alpha.\end{aligned}$$

Dropping the dependence upon M or the initial data, which play no role here, we conclude

$$\|\Lambda\mathbf{u}_1(t,\mathbf{x})-\Lambda\mathbf{u}_2(t,\mathbf{x})\|_{L^\infty(V_-)} \leqslant C(T)[\theta_\beta+\theta_\beta^\alpha].$$

(2) For $(t,\mathbf{x})\in V_+$, we have

$$\begin{aligned}\Lambda\mathbf{u}_1(t,\mathbf{x})-\Lambda\mathbf{u}_2(t,\mathbf{x}) &= \mathbf{H}_1(\mu_1(t,\mathbf{x}))\cdot\nabla\eta_1(\tau_1(t,\mathbf{x}),t;\boldsymbol{\gamma}_1(t,\mathbf{x}))\\ &\qquad - \mathbf{H}_2(\mu_2(t,\mathbf{x}))\cdot\nabla\eta_2(\tau_2(t,\mathbf{x}),t;\boldsymbol{\gamma}_2(t,\mathbf{x}))\\ &= J_1+J_2+J_3,\end{aligned}$$

where $\mathbf{H}_j(t,\mathbf{x})$ is defined in (3.7) for $\mathbf{u}_j$, and

$$\begin{aligned}J_1 &:= \mathbf{H}_1(\mu_1(t,\mathbf{x}))\cdot(\nabla\eta_1(\tau_1(t,\mathbf{x}),t;\boldsymbol{\gamma}_1(t,\mathbf{x}))-\nabla\eta_2(\tau_1(t,\mathbf{x}),t;\boldsymbol{\gamma}_1(t,\mathbf{x}))),\\ J_2 &:= \mathbf{H}_1(\mu_1(t,\mathbf{x}))\cdot(\nabla\eta_2(\tau_1(t,\mathbf{x}),t;\boldsymbol{\gamma}_1(t,\mathbf{x}))-\nabla\eta_2(\tau_2(t,\mathbf{x}),t;\boldsymbol{\gamma}_2(t,\mathbf{x}))),\\ J_3 &:= (\mathbf{H}_1(\mu_1(t,\mathbf{x}))-\mathbf{H}_2(\mu_2(t,\mathbf{x}))\cdot\nabla\eta_2(\tau_2(t,\mathbf{x}),t;\boldsymbol{\gamma}_2(t,\mathbf{x})).\end{aligned}$$

Now, since $\mathbf{H}_j(s,\mathbf{y})=\boldsymbol{\omega}_j(s,\mathbf{y})$ for $(s,\mathbf{y})\in[0,T]\times\Gamma_+$, we have, using Lemma 12.4,

$$\|J_1\|_{L^\infty(V_+)} \leqslant \|\boldsymbol{\omega}_1\|_{L^\infty(Q)}\|\nabla\eta_1(\cdot,t;\cdot)-\nabla\eta_2(\cdot,t;\cdot))\|_{L^\infty(Q)} \leqslant C(T,M)[\theta_\beta+\theta_\beta^\alpha],$$

where we also used cond_0. For J_2, we have, using Lemmas 12.3 and A.3,

$$\begin{aligned}\|J_2\|_{L^\infty(V_+)} &\leqslant \|\boldsymbol{\omega}_1\|_{L^\infty(Q)}\|\nabla\eta_2\|_{\dot{C}^\alpha(Q)}\|(\tau_1(t,\mathbf{x}),\boldsymbol{\gamma}_1(t,\mathbf{x}))-(\tau_2(t,\mathbf{x}),\boldsymbol{\gamma}_2(t,\mathbf{x}))\|^\alpha_{L^\infty(Q)}\\ &\leqslant C(T,M)\|\mu\|^\alpha_{L^\infty(U_+)} \leqslant C(T,M)\theta_\beta^\alpha.\end{aligned}$$

For J_3, we have

$$J_3 \leqslant \|\mathbf{H}_1(\mu_1(t,\mathbf{x}))-\mathbf{H}_2(\mu_2(t,\mathbf{x})\|_{L^\infty(U_+)}\|\nabla\eta_2\|_{L^\infty(Q)}.$$

But, $\|\nabla\eta_2\|_{L^\infty(Q)} \leqslant C(T,M)$ by Lemma 8.2, and, using Lemma A.3,

$$\begin{aligned}&\|\mathbf{H}_1(\mu_1(t,\mathbf{x}))-\mathbf{H}_2(\mu_2(t,\mathbf{x})\|_{L^\infty(U_+)}\\ &\quad\leqslant \|\mathbf{H}_1(\mu_1(t,\mathbf{x}))-\mathbf{H}_2(\mu_1(t,\mathbf{x})\|_{L^\infty(U_+)} + \|\mathbf{H}_2(\mu_1(t,\mathbf{x}))-\mathbf{H}_2(\mu_2(t,\mathbf{x})\|_{L^\infty(U_+)}\\ &\quad\leqslant \|\mathbf{H}_1-\mathbf{H}_2\|_{L^\infty([0,T]\times\Gamma_+)} + \|\mathbf{H}_2\|_{\dot{C}^\alpha([0,T]\times\Gamma_+)}\|\mu\|^\alpha_{L^\infty}\\ &\quad\leqslant \|\boldsymbol{\omega}_1-\boldsymbol{\omega}_2\|_{L^\infty([0,T]\times\Gamma_+)} + C(T,M)\theta_\beta^\alpha \leqslant C(T,M)[\theta_\beta+\theta_\beta^\alpha],\end{aligned}$$

where in the second-to-last inequality we used the bounds on $\mathbf{H}_1$ and $\mathbf{H}_2$ from Proposition 11.1 and use that $\mathbf{H}_j=\boldsymbol{\omega}_j$ on $[0,T]\times\Gamma_+$, since $\Lambda\mathbf{u}$ solves (2.1).

Combined, we see that

$$\|\Lambda\mathbf{u}_1(t,\mathbf{x})-\Lambda\mathbf{u}_2(t,\mathbf{x})\|_{L^\infty(V_+)} \leqslant C(T,M)[\theta_\beta+\theta_\beta^\alpha].$$

(3) Now assume $(t, \mathbf{x}) \in W$. Applying Lemma A.9 with the Lipschitz modulus of continuity, $r \mapsto \|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{\dot{C}^\alpha} r \leqslant Mr$,

$$\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(W)} \leqslant \widetilde{F}\left(\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^2(W)}\right).$$

From Lemma 12.5,

$$\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^2(W)} \leqslant \|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(W)} |W|^{\frac{1}{2}} \leqslant CM|W|^{\frac{1}{2}} \leqslant C(T, M)\theta_\beta,$$

which then gives $\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(W)} \leqslant \widetilde{F}(\theta_\beta)$. We conclude that $\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(Q)} \leqslant \widetilde{F}(\theta_\beta)$.

(B) Accounting for forcing: To treat forcing, let $\mathbf{G}^j_\pm$ be given by (8.7) for η_j. Then

$$\begin{aligned}
&\|\mathbf{G}^1_\pm - \mathbf{G}^2_\pm\|_{L^\infty(V_\pm)} \\
&\qquad \leqslant \int_0^T \|\nabla \eta_1(s, t; \eta_1(t, s; \mathbf{x}))\mathbf{g}(s, \eta_1(t, s; \mathbf{x})) - \nabla \eta_2(s, t; \eta_2(t, s; \mathbf{x}))\mathbf{g}(s, \eta_2(t, s; \mathbf{x}))\|_{L^\infty(\Omega)}\, ds.
\end{aligned}$$

But,

$$\begin{aligned}
&\|\nabla \eta_1(s, t; \eta_1(t, s; \mathbf{x}))\mathbf{g}(s, \eta_1(t, s; \mathbf{x})) - \nabla \eta_2(s, t; \eta_2(t, s; \mathbf{x}))\mathbf{g}(s, \eta_2(t, s; \mathbf{x}))\|_{L^\infty(\Omega)} \\
&\quad \leqslant \|\nabla \eta_1(s, t; \eta_1(t, s; \mathbf{x}))\mathbf{g}(s, \eta_1(t, s; \mathbf{x})) - \nabla \eta_2(s, t; \eta_1(t, s; \mathbf{x}))\mathbf{g}(s, \eta_1(t, s; \mathbf{x}))\|_{L^\infty(\Omega)} \\
&\qquad + \|\nabla \eta_2(s, t; \eta_1(t, s; \mathbf{x}))\mathbf{g}(s, \eta_1(t, s; \mathbf{x})) - \nabla \eta_2(s, t; \eta_2(t, s; \mathbf{x}))\mathbf{g}(s, \eta_1(t, s; \mathbf{x}))\|_{L^\infty(\Omega)} \\
&\qquad + \|\nabla \eta_2(s, t; \eta_2(t, s; \mathbf{x}))\mathbf{g}(s, \eta_1(t, s; \mathbf{x})) - \nabla \eta_2(s, t; \eta_2(t, s; \mathbf{x}))\mathbf{g}(s, \eta_2(t, s; \mathbf{x}))\|_{L^\infty(\Omega)} \\
&\quad \leqslant \|\nabla \eta_1 - \nabla \eta_2\|_{L^\infty([0,T]^2 \times \Omega)} \|\mathbf{g}\|_{L^\infty(Q)} \\
&\qquad + \|\nabla \eta_2\|_{\dot{C}^\alpha([0,T]^2 \times \Omega)} \|\nabla \eta_1 - \nabla \eta_2\|^\alpha_{L^\infty([0,T]^2 \times \Omega)} \|\mathbf{g}\|_{L^\infty(Q)} \\
&\qquad + \|\nabla \eta_2\|_{L^\infty([0,T]^2 \times \Omega))} \|\mathbf{g}\|_{\dot{C}^\alpha} \|\eta_1 - \eta_2\|^\alpha_{L^\infty(Q)},
\end{aligned}$$

where we used Lemmas A.2 and A.3.

Since $\mathbf{g} \in L^\infty(Q)$, while $\nabla \eta_1$ and $\nabla \eta_2$ are bounded in $\dot{C}^\alpha([0, T]^2 \times \Omega)$, by Lemma 12.4 we see that

$$\|\mathbf{G}^1_\pm - \mathbf{G}^2_\pm\|_{L^\infty(V_\pm)} \leqslant CT[\theta_\beta + \theta^\alpha_\beta].$$

Hence, the inclusion of forcing does not change our bounds on $\|\Lambda \mathbf{u}_1(t, \mathbf{x}) - \Lambda \mathbf{u}_2(t, \mathbf{x})\|_{L^\infty(V_\pm)}$ in **(1)**, **(2)**. And $\mathbf{G}^1_\pm$, $\mathbf{G}^2_\pm$ are bounded on Q, so the estimate on $\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^2(W)}$ in **(3)** is also unchanged.

In summary, what we have done so far is to show that

$$\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(Q)} \leqslant \widetilde{F}(\theta_\beta). \tag{12.6}$$

(C) Velocity: From (2.5), we have,

$$P_H A\mathbf{u}_j(t, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \int_0^t \mathbf{f}(s, \mathbf{x})\, ds - \int_0^t P_H[\mathbf{u}_j(s) \cdot \overline{\mathbf{\Omega}}_j(s)](\mathbf{x})\, ds.$$

Then because $A\mathbf{u}_1 - A\mathbf{u}_2 = P_H(A\mathbf{u}_1 - A\mathbf{u}_2)$ and P_H is continuous in $C^\beta(\Omega)$, which is an algebra,

$$\begin{aligned}
\|A\mathbf{u}_1(t) - A\mathbf{u}_2(t)\|_{L^\infty(\Omega)} &\leqslant \|A\mathbf{u}_1(t) - A\mathbf{u}_2(t)\|_{C^\beta(\Omega)} \\
&\leqslant C \int_0^t \|\mathbf{w}\|_{C^\beta(\Omega)} \|\overline{\mathbf{\Omega}}_1\|_{C^\beta(\Omega)} + C \int_0^t \|\mathbf{u}_1\|_{C^\beta(\Omega)} \|\overline{\mathbf{\Omega}}_1 - \overline{\mathbf{\Omega}}_2\|_{C^\beta(\Omega)} \\
&= C \int_0^t \|\mathbf{w}\|_{C^\beta(\Omega)} \|\overline{\boldsymbol{\omega}}_1\|_{C^\alpha(\Omega)} + C \int_0^t \|\mathbf{u}_1\|_{C^\alpha(\Omega)} \|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{C^\beta(\Omega)}
\end{aligned}$$

$$\leqslant CT\theta_\beta + CT\widetilde{F}\left(\|\Lambda \mathbf{u}_1 - \Lambda \mathbf{u}_2\|_{L^\infty(Q)}\right) \leqslant \widetilde{F}(\theta_\beta),$$

where we used Proposition 11.3, Proposition 11.2, (12.6), and Lemma 12.2.

This gives (12.4), which completes the proof. □

13. Full inflow boundary condition satisfied

We now prove Proposition 5.5, which shows that a solution satisfying $(1.6)_{1\text{-}4}$ also satisfies $(1.6)_5$, and hence satisfies the full inflow boundary conditions. This can be done by defining $\mathbf{H}$ by (3.7) and recovering the pressure using $N[\mathbf{u}]$ of (3.6), as already observed in [2].

Proof of Proposition 5.5. Our proof is inspired by the proof of Lemma 4.2.1 pages 156-159 of [2]. Let

$$\mathbf{w} = \mathbf{u}^{\boldsymbol{\tau}} - \mathbf{U}^{\boldsymbol{\tau}}, \quad P := p - p_r,$$

where p_r is the regularized pressure given by (3.5). By Proposition 3.1, $\boldsymbol{\omega} = \mathbf{W}[\mathbf{u}, p]$ on $[0, T] \times \Gamma_+$, where we recall that $\mathbf{W}[\mathbf{u}, p]$ is defined in (3.4). From (1.7), (3.5), and (3.6), we see that on Γ_+, $\nabla P \cdot \boldsymbol{n} = \operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{w})$. Hence, P satisfies

$$\begin{cases} \Delta P = 0 & \text{in } \Omega, \\ \nabla P \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_- \cup \Gamma_0, \\ \nabla P \cdot \boldsymbol{n} = \operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{w}) & \text{on } \Gamma_+. \end{cases}$$

Multiplying by P, integrating over Ω, and integrating by parts over Γ_+ gives

$$\|\nabla P\|^2_{L^2(\Omega)} = -(\Delta P, P) + \int_{\Gamma_+} (\nabla P \cdot \boldsymbol{n})P = \int_{\Gamma_+} \operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{w})P = -\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \nabla_\Gamma P. \tag{13.1}$$

By (3.3) and the assumption that $\mathbf{H} = \boldsymbol{\omega} := \operatorname{curl} \mathbf{u}$ on Γ_+, we know that $U^{\boldsymbol{n}}[\mathbf{H}^{\boldsymbol{\tau}}]^\perp = U^{\boldsymbol{n}}[\mathbf{W}^{\boldsymbol{\tau}}[\mathbf{u}, p]]^\perp$. Using also that $(\mathbf{v}^\perp)^\perp = -\mathbf{v}$, we have, from (3.4) and (3.7), that on Γ_+,

$$\begin{aligned} \partial_t \mathbf{U}^{\boldsymbol{\tau}} + \nabla_\Gamma\left(p_r + \frac{1}{2}|\mathbf{U}|^2\right) - \mathbf{f}^{\boldsymbol{\tau}} + \operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}}[\mathbf{u}^{\boldsymbol{\tau}}]^\perp = \mathbf{H} \\ = \boldsymbol{\omega} = \partial_t \mathbf{u}^{\boldsymbol{\tau}} + \nabla_\Gamma\left(p + \frac{1}{2}|\mathbf{u}|^2\right) - \mathbf{f}^{\boldsymbol{\tau}} + \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}[\mathbf{u}^{\boldsymbol{\tau}}]^\perp. \end{aligned}$$

Subtracting the left hand side from the right hand side, we have

$$0 = \nabla_\Gamma P + \frac{1}{2}\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) + \partial_t \mathbf{w} + \operatorname{curl}_\Gamma \mathbf{w}[\mathbf{u}^{\boldsymbol{\tau}}]^\perp.$$

But, $\omega^{\boldsymbol{n}} = H^{\boldsymbol{n}}$ on Γ_+, which gives $\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}}$. Hence, $\operatorname{curl}_\Gamma \mathbf{w} = 0$, so

$$\nabla_\Gamma P = -\partial_t \mathbf{w} - \frac{1}{2}\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2).$$

Returning to (13.1), we thus have

$$\|\nabla P\|^2_{L^2(\Omega)} = \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \partial_t \mathbf{w} + \frac{1}{2}\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2).$$

Now,

$$\begin{aligned} \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w} \cdot \partial_t \mathbf{w} &= \frac{1}{2}\int_{\Gamma_+} U^{\boldsymbol{n}}\partial_t|\mathbf{w}|^2 = \frac{1}{2}\int_{\Gamma_+} \partial_t[U^{\boldsymbol{n}}|\mathbf{w}|^2] - \frac{1}{2}\int_{\Gamma_+} \partial_t U^{\boldsymbol{n}}\,|\mathbf{w}|^2 \\ &= \frac{1}{2}\frac{d}{dt}\int_{\Gamma_+} U^{\boldsymbol{n}}|\mathbf{w}|^2 - \frac{1}{2}\int_{\Gamma_+} \partial_t U^{\boldsymbol{n}}\,|\mathbf{w}|^2, \end{aligned}$$

so

$$\frac{d}{dt}\int_{\Gamma_+} U^{\boldsymbol{n}}|\mathbf{w}|^2 = \int_{\Gamma_+} \partial_t U^{\boldsymbol{n}}|\mathbf{w}|^2 - \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) + 2\|\nabla P\|^2_{L^2(\Omega)}. \tag{13.2}$$

Writing $|\mathbf{U}|^2 - |\mathbf{u}|^2 = |\mathbf{u}^{\boldsymbol{\tau}}|^2 - |\mathbf{U}^{\boldsymbol{\tau}}|^2 = \mathbf{w}\cdot\mathbf{v}$ on Γ_+, since $U^{\boldsymbol{n}} = u^{\boldsymbol{n}}$, where $\mathbf{v} := \mathbf{U}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}}$, we have

$$\begin{aligned}\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) &= \int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(\mathbf{w}\cdot\mathbf{v})\\ &= \int_{\Gamma_+} U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{v})\cdot\mathbf{w} + \int_{\Gamma_+} U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{w})\cdot\mathbf{v}\\ &= \int_{\Gamma_+} U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{v})\cdot\mathbf{w} - \frac{1}{2}\int_{\Gamma_+}|\mathbf{w}|^2\operatorname{div}_\Gamma(U^{\boldsymbol{n}}\mathbf{v}).\end{aligned}$$

For the last term above, we used that $U^{\boldsymbol{n}}(\mathbf{w}\cdot\nabla_\Gamma\mathbf{w})\cdot\mathbf{v} = (1/2)U^{\boldsymbol{n}}\mathbf{v}\cdot\nabla_\Gamma|\mathbf{w}|^2$ and integrated by parts via Lemma B.1. Then because $\mathbf{v}$ and $U^{\boldsymbol{n}}$ are sufficiently regular, we have

$$\left|\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2)\right| \leqslant C\int_{\Gamma_+}|\mathbf{w}|^2.$$

Changing sign in (13.2) and integrating in time, we see that

$$\begin{aligned}\int_{\Gamma_+}|U^{\boldsymbol{n}}(t)||\mathbf{w}(t)|^2 &= -\int_{\Gamma_+} U^{\boldsymbol{n}}(t)|\mathbf{w}(t)|^2\\ &\leqslant -\int_0^t\int_{\Gamma_+}\partial_t U^{\boldsymbol{n}}|\mathbf{w}|^2 + \int_0^t\int_{\Gamma_+} U^{\boldsymbol{n}}\mathbf{w}\cdot\nabla_\Gamma(|\mathbf{u}|^2 - |\mathbf{U}|^2) - 2\int_0^t\|\nabla P\|^2_{L^2(\Omega)}\\ &\leqslant C\int_0^t\int_{\Gamma_+}|\mathbf{w}(s)|^2\,ds - 2\int_0^t\|\nabla P\|^2_{L^2(\Omega)} \leqslant C\int_0^t\int_{\Gamma_+}|\mathbf{w}(s)|^2\,ds.\end{aligned}$$

In the first equality we used that $U^{\boldsymbol{n}} < 0$ on Γ_+, in the second equality we used that $\mathbf{w}(0) = 0$, and in the third equality we used that $\partial_t U^{\boldsymbol{n}}$ is bounded.

Now since $|U^{\boldsymbol{n}}|$ is bounded away from zero, we have

$$\int_{\Gamma_+}|\mathbf{w}(t)|^2 \leqslant C\int_0^t\int_{\Gamma_+}|\mathbf{w}(s)|^2\,ds,$$

and we conclude from Grönwall's Lemma that $\mathbf{w}\equiv 0$. This means that $\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{U}^{\boldsymbol{\tau}}$, so $(1.6)_5$ holds. □

Remark 13.1. *If $\Gamma_0 = \Gamma$, the classical setting of impermeable boundary conditions on the whole boundary, our proof of existence and uniqueness still applies, though a number of things trivialize. First, no vorticity is transported off of the boundary, so there is no need for the pressure estimates in Section 10, and U_- is all of Q, so many of the flow map constructs, such as S, τ, and $\boldsymbol{\gamma}$ are unnecessary. And, of course, none of the estimates involving U_+ are needed.*

14. Vorticity boundary conditions

Proof of Theorem 1.4. The proof of existence is the same as that for Theorem 1.2, though with substantial simplifications. Because $\mathbf{H}$ is given with sufficient regularity, it satisfies

$$\|\mathbf{H}\|_{L^\infty([0,T]\times\Gamma_+)} \leqslant c_0, \quad \|\mathbf{H}\|_{C^{N,\alpha}([0,T]\times\Gamma_+)} \leqslant c_0.$$

Hence, there are no pressure estimates involved, so the condition in (1.17) immediately gives (2.4), and there is no need to appeal to Proposition 3.3. Since we only require $\mathbf{u} \cdot \boldsymbol{n} = U^{\boldsymbol{n}}$ on Γ_+, we simplify the definition of $\mathrm{Dom}_N(A)$ in (3.8) to

$$\mathrm{Dom}_N(A) := \{\mathbf{u} \in S^{N+1,\alpha} \colon \mathbf{u}(0) = \mathbf{u}_0\},$$

and there is no need to invoke Proposition 5.5 or Lemma 7.9. Otherwise, the remainder of the proof of existence proceeds unchanged.

For uniqueness when $N \geqslant 1$, let $\boldsymbol{\omega}_j = \operatorname{curl} \mathbf{u}_j$, $j = 1, 2$, and let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Then $\mathbf{w} \in H_0$, since $\mathbf{u}_1$, $\mathbf{u}_2$ have the same prescribed harmonic component, $\mathbf{u}_c$. Let

$$\boldsymbol{\mu} := \operatorname{curl} \mathbf{w} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2.$$

Since $N \geqslant 1$, we have enough regularity to write $\partial_t \boldsymbol{\omega}_j + \mathbf{u}_j \cdot \nabla \boldsymbol{\omega}_j = \boldsymbol{\omega}_j \cdot \nabla \mathbf{u}_j + \operatorname{curl} \mathbf{f}$, and subtracting this relation for $j = 2$ from that for $j = 1$ gives

$$\partial_t \boldsymbol{\mu} + \mathbf{u}_1 \cdot \nabla \boldsymbol{\mu} + \mathbf{w} \cdot \nabla \boldsymbol{\omega}_2 = \boldsymbol{\omega}_1 \cdot \nabla \mathbf{w} + \boldsymbol{\mu} \cdot \nabla \mathbf{u}_2. \tag{14.1}$$

Multiplying by $\boldsymbol{\mu}$, integrating over Ω, and using that $(\mathbf{u}_1 \cdot \nabla \boldsymbol{\mu}, \boldsymbol{\mu}) = (1/2)(\mathbf{u}_1, \nabla |\boldsymbol{\mu}|^2)$, gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\boldsymbol{\mu}\|^2 + \frac{1}{2} \int_\Omega \mathbf{u}_1 \cdot \nabla |\boldsymbol{\mu}|^2 &= -(\mathbf{w} \cdot \nabla \boldsymbol{\omega}_2, \boldsymbol{\mu}) + (\boldsymbol{\omega}_1 \cdot \nabla \mathbf{w}, \boldsymbol{\mu}) + (\boldsymbol{\mu} \cdot \nabla \mathbf{u}_2, \boldsymbol{\mu}) \\ &\leqslant \frac{1}{2} \|\nabla \boldsymbol{\omega}_2\|_{L^\infty} \|\mathbf{w}\|^2 + \frac{1}{2} \|\boldsymbol{\mu}\|^2 + \frac{1}{2} \|\boldsymbol{\omega}_1\|_{L^\infty} \|\nabla \mathbf{w}\|^2 + \frac{1}{2} \|\boldsymbol{\mu}\|^2 + \|\nabla \mathbf{u}_2\|_{L^\infty} \|\boldsymbol{\mu}\|^2, \end{aligned} \tag{14.2}$$

where $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$ here. Elements of H have mean zero, so by Poincaré's inequality, $\|\mathbf{w}\| \leqslant C \|\nabla \mathbf{w}\|$. Moreover, since $\mathbf{w} \in H_0$, we have $\|\nabla \mathbf{w}\| \leqslant C \|\boldsymbol{\mu}\|$ and so obtain

$$\frac{d}{dt} \|\boldsymbol{\mu}\|^2 \leqslant - \int_\Omega \mathbf{u}_1 \cdot \nabla |\boldsymbol{\mu}|^2 + C \|\boldsymbol{\mu}\|^2.$$

We note that $\nabla \boldsymbol{\omega}_2 \in L^\infty([0, T] \times \Omega)$ by the $N = 1$ existence result. But,

$$- \int_\Omega \mathbf{u}_1 \cdot \nabla |\boldsymbol{\mu}|^2 = \int_\Omega \operatorname{div} \mathbf{u}_1 \, |\boldsymbol{\mu}|^2 - \int_\Gamma U^{\boldsymbol{n}} |\boldsymbol{\mu}|^2 = - \int_{\Gamma_-} U^{\boldsymbol{n}} |\boldsymbol{\mu}|^2 \leqslant 0,$$

so we conclude from Gronwall's lemma, since $\boldsymbol{\mu}(0) = 0$, that $\boldsymbol{\mu} \equiv 0$. That is, $\mathbf{u}_1 = \mathbf{u}_2$.

Finally, from $(1.16)_1$, we have

$$\partial_t \mathbf{u}^{\boldsymbol{\tau}} + (\mathbf{u} \cdot \nabla \mathbf{u})^{\boldsymbol{\tau}} = (\mathbf{f} - \nabla p)^{\boldsymbol{\tau}} + \mathbf{z}^{\boldsymbol{\tau}}.$$

From cond_0, then, we see that $\mathbf{z}^{\boldsymbol{\tau}}(0) = 0$. Since also $z^{\boldsymbol{n}}(0) = 0$, we know that $\mathbf{z}(0) = 0$. □

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Appendix A. Hölder space lemmas

We collect here a number of estimates in Hölder spaces, defined in Section 7, which we use throughout much of this paper. We include proofs only of the less standard ones.

Lemma A.1. *Let $f, g \in C^\alpha(U)$. Then*

$$\begin{aligned}
\|fg\|_{C^\alpha} &\leqslant \|f\|_{C^\alpha}\|g\|_{C^\alpha}, \\
\|fg\|_{\dot{C}^\alpha} &\leqslant \|f\|_{L^\infty}\|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty}\|f\|_{\dot{C}^\alpha}, \\
\|fg\|_{C^\alpha} &\leqslant \|f\|_{L^\infty}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty}\|f\|_{\dot{C}^\alpha}, \\
&\leqslant \|f\|_{L^\infty}\|g\|_{C^\alpha} + \|g\|_{L^\infty}\|f\|_{C^\alpha}, \\
\|fg\|_{C^\alpha} &\leqslant \|f\|_{L^\infty}\|g\|_{\dot{C}^\alpha} + \|g\|_{L^\infty}\|f\|_{C^\alpha},
\end{aligned}$$

where $\dot{C}^\alpha$ is the Hölder semi-norm, as in (7.1). Also, for any $\beta \in (0, \alpha)$, allowing $\alpha = 1$, we have the interpolation inequality,

$$\|f\|_{\dot{C}^\beta} \leqslant 2\|f\|_{\dot{C}^\alpha}^{\frac{\beta}{\alpha}}\|f\|_{L^\infty}^{1-\frac{\beta}{\alpha}}.$$

Lemma A.2. *Let U, V be open subsets of $\mathbb{R}^n$, $\alpha \in (0,1]$, and $k \geqslant 1$ an integer. If $f \in C^{k,\alpha}(U)$ and $g \in C^{k+1}(V)$ with $g(V) \subseteq U$ then*

$$\begin{aligned}
\|f \circ g\|_{\dot{C}^\alpha(V)} &\leqslant \|f\|_{\dot{C}^\alpha(U)}\|g\|_{Lip(V)}^\alpha, \\
\|f \circ g\|_{C^\alpha(V)} &\leqslant \|f\|_{L^\infty(U)} + \|f\|_{\dot{C}^\alpha(U)}\|g\|_{Lip(V)}^\alpha \leqslant \|f\|_{C^\alpha(U)}\left[1 + \|g\|_{Lip(V)}^\alpha\right], \qquad \text{(A.1)} \\
\|f \circ g\|_{C^{k,\alpha}(V)} &\leqslant C(k)\|f\|_{C^{k,\alpha}(U)}\left[1 + \|g\|_{C^{k+1}(V)}\right]^{k+1},
\end{aligned}$$

where Lip is the homogeneous Lipschitz semi-norm and $\dot{C}^\alpha$ is the homogeneous Hölder norm.

Lemma A.3. *Let U, V be open subsets of $\mathbb{R}^d$, $d \geqslant 1$, and let $\alpha \in (0,1]$. Assume that the domain of f is U and the domains of g and h are V, with $g(V), h(V) \subseteq U$. Then*

$$\|f \circ g - f \circ h\|_{L^\infty(V)} \leqslant \|f\|_{\dot{C}^\alpha(U)}\|g - h\|_{L^\infty(V)}^\alpha.$$

We also have the following interpolation-like inequality:

Lemma A.4. *Let U be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$, let $n \geqslant 1$, and $\nabla^n f \in C^\alpha(U)$. Then*

$$\|\nabla^n f\|_{L^\infty(U)} \leqslant C\|f\|_{C^{n,\alpha}(U)}^a\|f\|_{L^2(U)}^{1-a},$$

where

$$a = a_n = \frac{2n + d}{2n + d + 2\alpha} < 1.$$

Proof. First extend f continuously to all of $\mathbb{R}^d$ in all Hölder spaces, as can be done using the extension operator in Theorem $5'$, chapter VI of [29]. Applying a cutoff function, we can insure that the extension, which we continue to call f, has support with a diameter no more than twice diam(U).

Then

$$\|\nabla^n f\|_{L^\infty(U)} = \sup_{\mathbf{x} \in \operatorname{supp} f}|\nabla^n f(\mathbf{x})| = \sup_{\mathbf{x} \in \operatorname{supp} f}|\nabla^n f(\mathbf{x}) - \nabla^n f(\mathbf{x}_0)| \leqslant R,$$

where $\mathbf{x}_0$ is a fixed point in $(\operatorname{supp} f)^C$ (so $\nabla \mathbf{f}(x_0) = 0$) and

$$R = \sup_{\mathbf{x} \in \operatorname{supp} f} |\mathbf{x} - \mathbf{x}_0|^\alpha \sup_{\mathbf{x} \in \operatorname{supp} f} \frac{|\nabla^n f(\mathbf{x}) - \nabla^n f(\mathbf{x}_0)|}{|\mathbf{x} - \mathbf{x}_0|^\alpha} = \sup_{\mathbf{x} \in \operatorname{supp} f} |\mathbf{x} - \mathbf{x}_0|^\alpha \|\nabla^n (f(s\cdot))\|_{\dot{C}^\alpha(\mathbb{R}^d)}.$$

In particular,

$$\|\nabla^n f\|_{L^\infty(\mathbb{R}^d)} \leqslant R + \|f\|_{L^2(\mathbb{R}^d)} \tag{A.2}$$

for all $f \in C_0^\infty(\mathbb{R}^d)$.

Following the scaling argument in the proof of Proposition 13.3.4 of [31], we write (A.2) schematically in the form $Q \leqslant R + P$. Replacing $f(\cdot)$ with $f(s\cdot)$, we have $\nabla^n(f(s\mathbf{x})) = s^n \nabla f(s\mathbf{x})$. This gives $\|\nabla^n(f(s\cdot))\|_{L^\infty(\mathbb{R}^d)} = s^n \|\nabla f\|_{L^\infty(\mathbb{R}^d)}$ and $\|f(s\cdot)\|_{L^2(\mathbb{R}^d)} = s^{-\frac{d}{2}} \|f\|_{L^2(\mathbb{R}^d)}$. Also, R becomes

$$\sup_{\mathbf{x} \in \operatorname{supp} f} |s\mathbf{x} - s\mathbf{x}_0|^\alpha \sup_{\mathbf{x} \in \operatorname{supp} f} s^n \frac{|\nabla^n f(s\mathbf{x}) - \nabla^n f(s\mathbf{x}_0)|}{|s\mathbf{x} - s\mathbf{x}_0|^\alpha} = s^{n+\alpha} R.$$

Thus, $Q \leqslant R + P$ becomes

$$s^n Q \leqslant s^{n+\alpha} R + s^{-\frac{d}{2}} P \implies Q \leqslant s^\alpha R + s^{-(n+\frac{d}{2})} P.$$

As in [31], we conclude that

$$\|\nabla^n f\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\nabla^n f\|^a_{\dot{C}^\alpha(\mathbb{R}^d)} \|f\|^{1-a}_{L^2(\mathbb{R}^d)} \leqslant C \|\nabla^n f\|^a_{\dot{C}^\alpha(U)} \|f\|^{1-a}_{L^2(U)}$$

as long as $\alpha a = (n + \frac{d}{2})(1 - a)$, which gives the stated value of a and the stated estimate, using the continuity of the extension operator. □

The inequality in Lemma A.4 is similar to that in the lemma on page 126 of [25], used by the authors of [2] (for $N = 0$).

Lemma A.5. *Let U be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$, let $n \geqslant 1$, and suppose that $f \in C^{n,\alpha}(U)$. Let a_n be as in Lemma A.4. For any $\beta \in (0, \alpha)$,*

$$\|f\|_{C^{n,\beta}(U)} \leqslant \|f\|_{L^\infty(U)} + C \left[\|f\|^{a_1}_{C^{n,\alpha}(U)} + \|f\|^{a_n}_{C^{n,\alpha}(U)} \right] \left[\|f\|^{1-a_1}_{L^2(U)} + \|f\|^{1-a_n}_{L^2(U)} \right] + C \|f\|^{a'}_{C^{n,\alpha}(U)} \|f\|^{1-a'}_{L^2(U)},$$

where

$$a' = (\beta/\alpha) + a_n(1 - \beta/\alpha) < 1.$$

Lemma A.6. *Recalling the definition of $\dot{C}^{0,\beta}_t(Q)$ in (7.2), for any $\beta \in (0, 1]$ and $f \in \dot{C}^{0,\beta}(Q)$,*

$$\|f(t_1) - f(t_2)\|_{L^\infty(\Omega)} \leqslant \|f\|_{\dot{C}^{0,\beta}_t(Q)} |t_1 - t_2|^\beta.$$

Proof. For any $t_1, t_2 \in [0, T]$ with $t_1 \neq t_2$,

$$\|f(t_1) - f(t_2)\|_{L^\infty(\Omega)} = \sup_{\mathbf{x} \in \Omega} \frac{|f(t_1, \mathbf{x}) - f(t_2, \mathbf{x})|}{|t_1 - t_2|^\beta} |t_1 - t_2|^\beta \leqslant \|f\|_{\dot{C}_t(Q)} |t_1 - t_2|^\beta.$$ □

Lemma A.7. *Let $f \in C^{N,\alpha}(Q)$ for some $N \geqslant 0$ and $\alpha \in (0, 1]$. Then for any $t_1, t_2 \in [0, T]$,*

$$\|f(t_1) - f(t_2)\|_{C^N(\Omega)} \leqslant \|f\|_{C^{N,\alpha}(Q)} |t_1 - t_2|^\alpha.$$

Proof. We have, applying Lemma A.6,

$$\begin{aligned}\|f(t_1)-f(t_2)\|_{C^N(\Omega)} &= \sum_{k=0}^{N}\|\nabla^k(f(t_1)-f(t_2))\|_{L^\infty(\Omega)}\\ &\leqslant \sum_{k=0}^{N-1}\|\nabla^k f\|_{\dot{C}_t^{0,1}(Q)}|t_1-t_2| + \|\nabla^N f\|_{\dot{C}_t^{0,\alpha}(Q)}|t_1-t_2|^\alpha\\ &\leqslant \sum_{k=0}^{N}\|\nabla^k f\|_{\dot{C}_t^{\alpha}(Q)}|t_1-t_2|^\alpha \leqslant \|f\|_{C^{N,\alpha}(Q)}|t_1-t_2|^\alpha.\end{aligned}$$

□

Corollary A.8. *If $f \in C^{N,\alpha}(Q)$ for some $N \geqslant 0$ and $\alpha \in (0,1]$ then*

$$\|f(t)-f(0)\|_{C^N(Q)} \leqslant C\|f\|_{C^{N,\alpha}(Q)}T^\alpha.$$

Lemma A.9 is adapted from Lemma 8.3 of [15].

Lemma A.9. *Suppose that $f_j\colon \mathbb{R}^d \to \mathbb{R}$, $j = 1, 2$, each have the modulus of continuity Θ, with $\Theta\colon [0,\infty) \to [0,\infty)$ continuous and increasing with $\Theta(0) = 0$. There exists a continuous increasing function $\widetilde{F}\colon [0,\infty) \to \infty$, depending on Θ, with $\widetilde{F}(0) = 0$ for which*

$$\|f_1-f_2\|_{L^\infty(\mathbb{R}^d)} \leqslant \widetilde{F}(\|f_1-f_2\|_{L^2(\mathbb{R}^d)}).$$

Proof. Fix $x \in \mathbb{R}^d$ arbitrarily and suppose that $\delta = |f_1(x)-f_2(x)| > 0$. Let y be in the ball B of radius $a = \Theta^{-1}(\delta/4)$ about x, so that $|f_1(x)-f_1(y)|, |f_2(x)-f_2(y)| \leqslant \delta/4$. Then

$$|f_1(y)-f_2(y)| \geqslant \delta - |f_1(x)-f_1(y)| - |f_2(x)-f_2(y)| = \frac{\delta}{2}.$$

Hence,

$$\|f_1-f_2\|_{L^2(\mathbb{R}^d)} \geqslant \|f_1-f_2\|_{L^2(B)} \geqslant \left(\int_B \left(\frac{\delta}{2}\right)^2\right)^{\frac{1}{2}} = \frac{\delta}{2}\sqrt{\pi}a,$$

or,

$$h(\delta) := \frac{\sqrt{\pi}}{2}\delta\Theta^{-1}(\delta/4) \leqslant \|f_1-f_2\|_{L^2(\mathbb{R}^d)}.$$

Since Θ^{-1} must be increasing, so must h, so setting $\widetilde{F} = h^{-1}$ (noting that $\widetilde{F}(0) = 0$) we have

$$|f_1(x)-f_2(x)| = \delta \leqslant \widetilde{F}(\|f_1-f_2\|_{L^2(\mathbb{R}^d)}).$$

This inequality applies for all x even when $\delta = |f_1(x)-f_2(x)| = 0$, giving the result. □

Appendix B. Boundary differential operators

We can define differential operators up to order two on $\partial\Omega$ by treating it as a manifold having at least C^2 regularity. In this appendix, we describe the properties that we need of the first-order differential operators, ∇_Γ, $\operatorname{div}_\Gamma$, and $\operatorname{curl}_\Gamma$. We refer the reader to standard references for such operators (for instance, Section 2.2 of [30]).

We will also have the need to calculate ∇, div, and curl in 3-space, but restricted to the boundary. This can be done by introducing a convenient coordinate system in a tubular neighborhood of the boundary in such a way that on the boundary itself, the coordinates reduce to a convenient coordinate system on the boundary. This is as done, for instance, in [10], drawing upon [16], and we refer the reader to those references for details.

We can define ∇_Γ—and then from it, $\operatorname{div}_\Gamma$ and $\operatorname{curl}_\Gamma$—in a coordinate-free manner by requiring that for any $f \in C^\infty(\Gamma)$ and any smooth curve $\mathbf{x}(s)$ on Γ parameterized by arc length,

$$\nabla_\Gamma f \cdot \mathbf{x}'(0) = \lim_{s\to 0} \frac{f(\mathbf{x}(s)) - f(\mathbf{x}(0))}{s}.$$

We then define $\operatorname{div}_\Gamma$ as the adjoint of ∇_Γ, in the sense of Lemma B.1:

Lemma B.1. *Let $f \in C^1(\Gamma)$, $\mathbf{v} \in (C^1(\Gamma))^d$. Then*

$$\int_\Gamma \mathbf{v} \cdot \nabla_\Gamma f = -\int_\Gamma \operatorname{div}_\Gamma \mathbf{v}\, f.$$

Moreover,

$$\operatorname{div}_\Gamma(f\mathbf{v}) = f \operatorname{div}_\Gamma \mathbf{v} + \mathbf{v} \cdot \nabla_\Gamma f. \tag{B.1}$$

Proof. This is classical for smooth functions (see, for instance, Proposition 2.2.2 of [30]), and follows in the same way for C^1 functions, integrating by parts on the boundary in charts. □

Finally, we define (with the $\perp$ operator as in Definition 9.1)

$$\operatorname{curl}_\Gamma \mathbf{v} := -\operatorname{div}_\Gamma \mathbf{v}^\perp.$$

We collect now a few useful facts.

For $\mathbf{u}$, $\mathbf{v}$ tangent vectors,

$$(\mathbf{u} \cdot \nabla_\Gamma \mathbf{v}) \cdot \mathbf{v} = \frac{1}{a_j} u^j \partial_j v^i\, v^i = \frac{1}{2a_j} u^i \partial_j |\mathbf{v}|^2 = \frac{1}{2} \mathbf{u} \cdot \nabla |\mathbf{v}|^2,$$

so for any component Γ_n of the boundary,

$$\int_{\Gamma_n} (\mathbf{u} \cdot \nabla_\Gamma \mathbf{v}) \cdot \mathbf{v} = \frac{1}{2} \int_{\Gamma_n} \mathbf{u} \cdot \nabla_\Gamma |\mathbf{v}|^2.$$

For a vector field $\mathbf{v}$ on $\overline{\Omega}$,

$$\operatorname{curl}_\Gamma \mathbf{v}^{\boldsymbol{\tau}} = (\operatorname{curl} \mathbf{v}) \cdot \boldsymbol{n} \tag{B.2}$$

and

$$\operatorname{div} \mathbf{v} = \operatorname{div}_\Gamma \mathbf{v}^{\boldsymbol{\tau}} + \partial_{\boldsymbol{n}} v^{\boldsymbol{n}} + (\kappa_1 + \kappa_2) v^{\boldsymbol{n}} \text{ on } \Gamma, \tag{B.3}$$

where κ_1, κ_2 are the principal curvatures on Γ.

Lemma B.2. *Let $\mathbf{u}, \mathbf{v}$ be vector fields on $\overline{\Omega} \subseteq \mathbb{R}^3$. Then*

$$[\mathbf{u} \times \mathbf{v}]^{\boldsymbol{\tau}} = u^{\boldsymbol{n}} [\mathbf{v}^{\boldsymbol{\tau}}]^\perp - v^{\boldsymbol{n}} [\mathbf{u}^{\boldsymbol{\tau}}]^\perp, \quad u^{\boldsymbol{n}} \mathbf{v}^{\boldsymbol{\tau}} - v^{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{\tau}} = [(\mathbf{v} \times \mathbf{u})^{\boldsymbol{\tau}}]^\perp.$$

Proof. We have,

$$\mathbf{u} \times \mathbf{v} = (\mathbf{u}^{\boldsymbol{n}} + \mathbf{u}^{\boldsymbol{\tau}}) \times (\mathbf{v}^{\boldsymbol{n}} + \mathbf{v}^{\boldsymbol{\tau}}) = \mathbf{u}^{\boldsymbol{n}} \times \mathbf{v}^{\boldsymbol{\tau}} - \mathbf{v}^{\boldsymbol{n}} \times \mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}} \times \mathbf{v}^{\boldsymbol{\tau}},$$

since $\mathbf{u}^{\boldsymbol{n}} \times \mathbf{v}^{\boldsymbol{n}} = 0$. Now, $\mathbf{u}^{\boldsymbol{\tau}} \times \mathbf{v}^{\boldsymbol{\tau}}$ is parallel to $\boldsymbol{n}$, so we see that

$$[\mathbf{u} \times \mathbf{v}]^{\boldsymbol{\tau}} = \mathbf{u}^{\boldsymbol{n}} \times \mathbf{v}^{\boldsymbol{\tau}} - \mathbf{v}^{\boldsymbol{n}} \times \mathbf{u}^{\boldsymbol{\tau}}.$$

But, $\mathbf{u}^{\boldsymbol{n}}$ is perpendicular to $\mathbf{v}^{\boldsymbol{\tau}}$, so we see that $\mathbf{u}^{\boldsymbol{n}} \times \mathbf{v}^{\boldsymbol{\tau}} = u^{\boldsymbol{n}}[\mathbf{v}^{\boldsymbol{\tau}}]^\perp$, and similarly, $\mathbf{v}^{\boldsymbol{n}} \times \mathbf{u}^{\boldsymbol{\tau}} = v^{\boldsymbol{n}}[\mathbf{u}^{\boldsymbol{\tau}}]^\perp$. Hence, $[\mathbf{u} \times \mathbf{v}]^{\boldsymbol{\tau}} = u^{\boldsymbol{n}}[\mathbf{v}^{\boldsymbol{\tau}}]^\perp - v^{\boldsymbol{n}}[\mathbf{u}^{\boldsymbol{\tau}}]^\perp$, giving also $u^{\boldsymbol{n}}\mathbf{v}^{\boldsymbol{\tau}} - v^{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{\tau}} = [(\mathbf{v} \times \mathbf{u})^{\boldsymbol{\tau}}]^\perp$. □

Proof of Proposition 9.2. All the following calculations are on Γ. We start with a short calculation *in rectangular coordinates*, using that $\operatorname{div} \mathbf{u} = \partial_i u^i = 0$:

$$\begin{aligned}(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} &= u^i \partial_i u^j n^j = \partial_i (u^i u^j n^j) - u^j u^i \partial_i n^j = \operatorname{div}(u^{\boldsymbol{n}} \mathbf{u}) - \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \boldsymbol{n}) \\ &= \operatorname{div}(u^{\boldsymbol{n}} \mathbf{u}) - \mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A} \mathbf{u}^{\boldsymbol{\tau}}.\end{aligned}$$

In the last equality, we used that because $\boldsymbol{n}$ does not change in the direction of $\boldsymbol{n}$,

$$\mathbf{u} \cdot \nabla \boldsymbol{n} = (\mathbf{u}^{\boldsymbol{n}} \cdot \nabla) \boldsymbol{n} + \mathbf{u}^{\boldsymbol{\tau}} \cdot \nabla \boldsymbol{n} = \mathcal{A} \mathbf{u}^{\boldsymbol{\tau}},$$

which is a tangent vector.

From (B.3) followed by (B.1), then,

$$\begin{aligned}\operatorname{div}(u^{\boldsymbol{n}} \mathbf{u}) &= \operatorname{div}_\Gamma (u^n \mathbf{u}^{\boldsymbol{\tau}}) + \partial_{\boldsymbol{n}} (u^{\boldsymbol{n}})^2 + (\kappa_1 + \kappa_2)(u^{\boldsymbol{n}})^2 \\ &= u^{\boldsymbol{n}} \operatorname{div}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma u^{\boldsymbol{n}} + \partial_{\boldsymbol{n}} (u^{\boldsymbol{n}})^2 + (\kappa_1 + \kappa_2)(u^{\boldsymbol{n}})^2.\end{aligned}$$

Using (B.3) again,

$$0 = (\operatorname{div} \mathbf{u}) u^{\boldsymbol{n}} = (\operatorname{div}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} + \partial_{\boldsymbol{n}} u^{\boldsymbol{n}} + (\kappa_1 + \kappa_2) u^{\boldsymbol{n}}) u^{\boldsymbol{n}},$$

so

$$\partial_{\boldsymbol{n}} (u^{\boldsymbol{n}})^2 = 2 u^{\boldsymbol{n}} \partial_{\boldsymbol{n}} u^{\boldsymbol{n}} = -2 u^{\boldsymbol{n}} \operatorname{div}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} - 2(\kappa_1 + \kappa_2)(u^{\boldsymbol{n}})^2.$$

Hence,

$$(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{n} = -u^{\boldsymbol{n}} \operatorname{div}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} + \mathbf{u}^{\boldsymbol{\tau}} \cdot \nabla_\Gamma u^{\boldsymbol{n}} - (\kappa_1 + \kappa_2)(u^{\boldsymbol{n}})^2 - \mathbf{u}^{\boldsymbol{\tau}} \cdot \mathcal{A} \mathbf{u}^{\boldsymbol{\tau}}. \qquad \square$$

Appendix C. Compatibility conditions: special case

In [32], Temam and Wang consider a periodic domain with $\mathbf{U} = (0, 0, -1)$, so $\mathbf{U}^{\boldsymbol{\tau}} = 0$ for all time. More generally, the authors of [8] consider $\mathbf{U} = -U^I \boldsymbol{n}$, where $U^I > 0$ is constant, so $\mathbf{U}^{\boldsymbol{\tau}} = 0$ on Γ_+ for all time. The compatibility conditions simplify in these settings.

Proposition C.1. *Assume that $\mathbf{U}^{\boldsymbol{\tau}} \equiv 0$ and $U^{\boldsymbol{n}}$ is spatially constant along Γ_+ ($U^{\boldsymbol{n}}$ need not be constant in time). Then the compatibility condition cond_N for $N \geqslant 0$ is*

$$\partial_t^j \mathbf{f}^{\boldsymbol{\tau}}|_{t=0} = \partial_t^j \nabla_\Gamma p|_{t=0} - U_0^{\boldsymbol{n}} (\partial_t^j \boldsymbol{\omega}^{\boldsymbol{\tau}})^\perp|_{t=0} \text{ for all } 0 \leqslant j \leqslant N, \qquad \text{(C.1)}$$

where $\partial_t^j \nabla_\Gamma p|_{t=0}$ and $\partial_t^j \boldsymbol{\omega}|_{t=0}$ must be treated as explained following (1.14).

Proof. Since $\mathbf{u}^{\boldsymbol{\tau}} = \mathbf{U}^{\boldsymbol{\tau}} = 0$, (B.2) gives that on Γ_+,

$$\omega^{\boldsymbol{n}} = \boldsymbol{\omega} \cdot \boldsymbol{n} = \operatorname{curl}_\Gamma \mathbf{u}^{\boldsymbol{\tau}} = 0.$$

In particular, this holds at time zero. Both $\partial_t \mathbf{U}^{\boldsymbol{\tau}} = 0$ and $\operatorname{curl}_\Gamma \mathbf{U}^{\boldsymbol{\tau}} = 0$, while $|\mathbf{U}|^2 = (U^{\boldsymbol{n}})^2$ is constant on Γ_+, so also $\nabla_\Gamma |\mathbf{U}|^2 = 0$. We see, then, that $\mathbf{H}^{\boldsymbol{\tau}}$ simplifies to $\mathbf{H}^{\boldsymbol{\tau}} = (U^{\boldsymbol{n}})^{-1} \left[\mathbf{f}^{\boldsymbol{\tau}} - \nabla_\Gamma p\right]^\perp$, so $\operatorname{lincond}_0$ (which follows from cond_0 by Proposition 4.4) becomes

$$\left[\mathbf{f}^{\boldsymbol{\tau}} - \nabla_\Gamma p\right]_{t=0}^\perp = U_0^{\boldsymbol{n}} \boldsymbol{\omega}_0^{\boldsymbol{\tau}},$$

which is (C.1) for $N = 0$. The inductive extension of this to higher N follows readily, leading to (C.1) for $N \geqslant 0$. $\square$

The condition in (C.1) for $N = 0$ also follows from cond_0 with slightly more work, though the inductive extension to higher N is not so transparent as it is starting from cond_0'.

Because $\operatorname{div} \mathbf{f} = 0$ with $\mathbf{f} \cdot \boldsymbol{n} = 0$ on Γ, $\mathbf{f}$ plays no role in the calculation of $\nabla_\Gamma p$ for $N = 0$. By writing the condition in (C.1) as we do, we are stressing that, given initial data one can always choose a forcing at time zero so that cond_0 is satisfied.

For all $N \geqslant 1$, though, forcing enters into the calculation of $\partial_t \nabla_\Gamma p$, when $\partial_t \mathbf{u}_0$ is replaced by $\mathbf{f}(0) - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \nabla p^0$: even though $\mathbf{f} \cdot \boldsymbol{n} = 0$, the forcing still does not, in general, vanish from even the $N = 1$ condition. Because of this fact, the forcing is intimately entwined in cond_N for $N \geqslant 1$, appearing on both sides of the condition, even for the simplest nontrivial case considered in [32]. These same comments hold in the general setting, but are more transparent in this simplified setting.

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