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Analytic continuation of multiple Hurwitz zeta functions

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Abstract

We use a variant of a method of Goncharov, Kontsevich and Zhao [**5**, **16**] to meromorphically continue the multiple Hurwitz zeta function

$$\zeta_d(s;\theta) = \sum_{0<n_1<\cdots<n_d} (n_1+\theta_1)^{-s_1}\cdots(n_d+\theta_d)^{-s_d}, \quad \theta_k \in [0,1),$$

to $\mathbb{C}^d$, to locate the hyperplanes containing its possible poles and to compute the residues at the poles. We explain how to use the residues to locate trivial zeros of $\zeta_d(s;\theta)$.

1. *Introduction and statements of results*

Let θ_k, $k = 1, \ldots, d$, be real numbers in $[0, 1)$. The *multiple Hurwitz zeta function* is defined by

$$\zeta_d(s;\theta) = \sum_{0<n_1<\cdots<n_d} (n_1+\theta_1)^{-s_1}\cdots(n_d+\theta_d)^{-s_d}, \tag{1·1}$$

which is absolutely convergent and analytic in the region

$$\mathrm{Re}(s_k+\cdots+s_d) > d-k+1 \quad \text{for} \quad k = 1,\ldots,d.$$

When $\theta_k = 0$, $k = 1, \ldots, d$, the multiple Hurwitz zeta function equals the multiple zeta function $\zeta_d(s)$, which was defined by D. Zagier in [**14**] and has been the focus of intense study in recent years, appearing in connection with arithmetic and hyperbolic geometry, moduli spaces, number theory, and quantum physics (see for example [**2**, **4**, **8**, **9**, **13**, **15**]). The many interesting algebraic and combinatorial aspects of the multiple Hurwitz zeta function have been studied in [**3**] and [**11**].

In this paper we will use a variant of a method of Goncharov, Kontsevich and Zhao [**5**, **16**] to give a proof of the meromorphic continuation of $\zeta_d(s;\theta)$, and to locate the hyperplanes containing its possible poles.

THEOREM 1·1. *The multiple Hurwitz zeta function $\zeta_d(s;\theta)$ meromorphically continues to $\mathbb{C}^d$ with the following possible poles*:

(i) *a simple pole along the hyperplane $s_d = 1$;*

(ii) *a simple pole along the hyperplane $s_k + \cdots + s_d - d + k - 1 = n$ for all integers $n \leqslant 0$ for $k = 1, \ldots, d-1$.*

Remark 1·2. In Theorem 1·1 we are using the definition given in [**7**, p. 168] of a pole as a holomorphic subvariety of dimension $d-1$.

To prove Theorem 1·1 we use a d-dimensional Mellin transform to show that $\zeta_d(s;\theta)$ equals, in its region of absolute convergence, a meromorphic distribution applied to a test function. This gives an explicit continuation of $\zeta_d(s;\theta)$ to $\mathbb{C}^d$, the poles being the same as those of the distribution.

The meromorphic distribution is

$$\Psi(x_1,\ldots,x_d;s_1,\ldots,s_d) = \frac{\Gamma(u_1)\cdots\Gamma(u_{d-1})}{s_d-1}\psi(x_1,\ldots,x_d;s_1,\ldots,s_d), \tag{1·2}$$

where

$$u_k = s_k + \cdots + s_d - d + k - 1 \tag{1·3}$$

and

$$\psi(x_1,\ldots,x_d;s_1,\ldots,s_d) = \frac{(x_1)_+^{u_1-1}}{\Gamma(u_1)}\prod_{k=2}^{d}\frac{(1-x_k)_+^{s_{k-1}-1}(x_k)_+^{u_k-1}}{\Gamma(s_{k-1})\Gamma(u_k)}, \tag{1·4}$$

where $t_+ := t\mathbf{1}_{(0,\infty)}$. This is a regular distribution in the domain of $\mathbb{C}^d$ defined by $\mathrm{Re}(s_1) > 0, \ldots, \mathrm{Re}(s_{d-1}) > 0$ and $\mathrm{Re}(u_1) > 0, \ldots, \mathrm{Re}(u_d) > 0$. In Lemma 3·1 we show that ψ continues to an entire distribution. Thus, the poles of Ψ arise from the other factors in (1.2).

The test function is defined on

$$R = (0,\infty)\times(0,1)^{d-1} \tag{1·5}$$

by

$$h(x) = \nu(x)g(x),$$

where

$$\nu(x) = \nu(x_1,\ldots,x_d) = e^{-\theta_1 x_1(1-x_2)}\cdots e^{-\theta_{d-1}x_1\cdots x_{d-1}(1-x_d)}e^{-\theta_d x_1\cdots x_d}$$

and

$$g(x) = g(x_1,\ldots,x_d) = \frac{x_1^d x_2^{d-1}\cdots x_d}{(e^{x_1}-1)\cdots(e^{x_1\cdots x_d}-1)}.$$

Let $S(\Omega)$ be the space of Schwartz class functions on an open subset Ω of $\mathbb{R}^d$ (see Section 2 for the definition). Since Ψ is zero for x outside of R, the value of the pairing (Ψ, h) does not depend on the value of h outside of R. Nonetheless, to complete the continuation argument it is *essential* that h extend to a test function on all of $\mathbb{R}^d$. To prove that h extends we show in Lemma 3·3 that g (and hence h) is in $S(R)$, and then construct in Theorem 3·4 a continuous linear extension operator $\mathcal{E}$ from $S(R)$ to $S(\mathbb{R}^d)$.

The meromorphic continuation of $\zeta_d(s;\theta)$ to $\mathbb{C}^d$ has also been accomplished by Akiyama and Ishikawa [**1**] using the Euler–Maclaurin summation formula, and by Murty and Sinha [**10**] using the binomial theorem and Hartog's theorem. The main advantage of our proof is that we are able to use the pairing (Ψ, h) and some combinatorial analysis to compute the residues at the poles of $\zeta_d(s;\theta)$ (see also [**5**, section 2] and [**16**, section 4]). This in turn provides a way to locate trivial zeros of $\zeta_d(s;\theta)$ (see the discussion below).

THEOREM 1·3. *The residue of the multiple Hurwitz zeta function $\zeta_d(s;\theta)$ on the hyperplane $s_d = 1$ is*

$$\mathrm{Res}_{s_d=1}\zeta_d(s_1,\ldots,s_d;\theta_1,\ldots,\theta_d) = \begin{cases} 1, & \text{if } d = 1,\\ \zeta_{d-1}(s_1,\ldots,s_{d-1};\theta_1,\ldots,\theta_{d-1}), & \text{if } d > 1.\end{cases}$$

THEOREM 1·4. *For $d \geqslant 2$ and any integers $1 \leqslant k \leqslant d-1$ and $n \geqslant 1$, the residue of the multiple Hurwitz zeta function $\zeta_d(s;\theta)$ on the hyperplane*

$$s_d(k) = d - k + 2 - n$$

is equal to (using the convention $\zeta_0(s_0;\theta_0) = 1$)

$$\zeta_{k-1}(s_1,\ldots,s_{k-1};\theta_1,\ldots,\theta_{k-1}) \sum_{\substack{a_d(k+1)=n-1\\ a_{k+1},\ldots,a_d\geqslant 0}} \left\{ \prod_{j=k+1}^{d} \frac{B_{a_j}(\theta_{j-1}-\theta_j)\Gamma(a_d(j)+u_j)}{a_j!\Gamma(a_d(j+1)+u_j+1)} \right\},$$

where

$$B_{a_j}(x) = \sum_{k=0}^{a_j} \binom{a_j}{k} B_k x^{a_j-k}$$

is the a_jth Bernoulli polynomial (here B_k is the kth Bernoulli number), and we have set $s_d(k) = s_k + \cdots + s_d$, $a_d(j) = a_j + \cdots + a_d$, $a_d(d+1) = 0$, and

$$u_j = s_d(j) - d + j - 1.$$

Example 1·5. The residue of the double Hurwitz zeta function $\zeta_2(s;\theta)$ on the hyperplane

$$s_1 + s_2 = 3 - n, \quad n \geqslant 1,$$

equals

$$\frac{B_{n-1}(\theta_1-\theta_2)\Gamma(s_2+n-2)}{(n-1)!\Gamma(s_2)}.$$

Example 1·6. The residue of the triple Hurwitz zeta function $\zeta_3(s;\theta)$ on the hyperplane

$$s_1 + s_2 + s_3 = 4 - n, \quad n \geqslant 1,$$

equals

$$\sum_{\substack{a_2+a_3=n-1\\ a_2,a_3\geqslant 0}} \frac{B_{a_2}(\theta_1-\theta_2)B_{a_3}(\theta_2-\theta_3)\Gamma(a_2+a_3+s_2+s_3-2)\Gamma(a_3+s_3-1)}{a_2!a_3!\Gamma(a_3+s_2+s_3-1)\Gamma(s_3)},$$

and the residue on the hyperplane

$$s_2 + s_3 = 3 - n, \quad n \geqslant 1,$$

equals

$$\zeta_1(s_1;\theta_1)\sum_{\substack{a_3=n-1\\a_3\geqslant 0}}\frac{B_{a_3}(\theta_2-\theta_3)\Gamma(a_3+s_3-1)}{a_3!\Gamma(s_3)}.$$

As mentioned, the formula for the residues in Theorem 1·4 can be used to locate trivial zeros of $\zeta_d(s;\theta)$. For example, the following sets of points (s_1, s_2) in $\mathbb{C}^2$ are trivial zeros of the double Hurwitz zeta function $\zeta_2(s;\theta)$:

$$\begin{aligned}
&(s_1,s_2)=(0,-k), \quad k\in\mathbb{Z}_{\geqslant 0};\\
&(s_1,s_2)=(-1,1-k), \quad k\in\mathbb{Z}_{\geqslant 1};\\
&(s_1,s_2)=(-2j,1-k), \quad (j,k)\in\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}_{\geqslant 1};\\
&(s_1,s_2)=(-2j-1,2-k), \quad (j,k)\in\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}_{\geqslant 2}.
\end{aligned}$$

To prove these are zeros, choose $n\geqslant 1$ so that (s_1, s_2) lies on the hyperplane $s_1+s_2=3-n$, and then verify that the residue

$$\frac{B_{n-1}(\theta_1-\theta_2)\Gamma(s_2+n-2)}{(n-1)!\Gamma(s_2)}$$

is zero. This method can be used, along with properties of the Bernoulli polynomials, to locate other trivial zeros of $\zeta_d(s;\theta)$ in dimensions $d\geqslant 2$ (see also [**16**, section 5]).

2. *Analytic continuation of tempered distributions*

In this section we give a brief overview of the analytic continuation of tempered distributions. Let Ω be an open subset of $\mathbb{R}^d$. Then $\mathcal{S}(\Omega)$, the *Schwartz-class functions* on Ω, are defined to be the set of all complex-valued C^∞–functions f on Ω such that

$$\rho_{\alpha,\beta}(f):=\sup_{x\in\Omega}|x^\alpha D^\beta f(x)|<\infty$$

for all (d-dimensional) multi-indices α and β. A multi-index α is an ordered pair of d non-negative integers $(\alpha_1,\dots,\alpha_d)$, $x^\alpha := x_1^{\alpha_1}\cdots x_d^{\alpha_d}$, and

$$D^\alpha f := \frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}},$$

where $|\alpha|=\alpha_1+\cdots+\alpha_d$.

Endowed with the sufficient family of semi-norms, $\{\rho_{\alpha,\beta}\}$, $S(\mathbb{R}^d)$ is a Fréchet space.

A (*tempered*) *distribution* is an element of $\mathcal{S}'(\mathbb{R}^d)$, the dual space of $\mathcal{S}(\mathbb{R}^d)$; that is, the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$, continuity being with respect to all the semi-norms $\rho_{\alpha,\beta}$ separately. A distribution ψ applied to a test function φ in $\mathcal{S}(\mathbb{R}^d)$ is written as (ψ,φ), the operation $(\cdot,\cdot)$ or, more explicitly, $(\cdot,\cdot)_{\mathcal{S}'(\mathbb{R}^d),\mathcal{S}(\mathbb{R}^d)}$, defining a pairing of $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. If, for some locally integrable function $\overline{\psi}$, $(\psi,\varphi)=\int_{\mathbb{R}^d}\overline{\psi}\varphi$ for all φ in $\mathcal{S}(\mathbb{R}^d)$, then the distribution is called a *regular* distribution, and ψ and $\overline{\psi}$ are normally identified.

A distribution ψ is analytic (meromorphic) if for any test function φ in $\mathcal{S}(\mathbb{R}^d)$, (ψ,φ) is analytic (meromorphic) in some domain in $\mathbb{C}^d$. If ψ is regular and analytic on some domain of $\mathbb{C}^d$ and, for any test function φ in $\mathcal{S}(\mathbb{R}^d)$, (ψ,φ) analytically continues to an analytic or meromorphic function, then ψ is said to analytically continue to an analytic or meromorphic distribution. A region on which ψ is regular and analytic is called a region of absolute convergence of ψ.

We will also have a need for the tensor product of distributions, which we define as follows. Let ψ_1 and ψ_2 be distributions in $\mathcal{S}'(\mathbb{R}^{d_1})$ and $\mathcal{S}'(\mathbb{R}^{d_2})$, and let φ be in $\mathcal{S}(\mathbb{R}^{d_1+d_2})$. Then the functions

$$\varphi_1(x_1) = (\psi_2(\cdot), \varphi(x_1, \cdot))_{\mathcal{S}'(\mathbb{R}^{d_2}),\mathcal{S}(\mathbb{R}^{d_2})}, \quad \varphi_2(x_2) = (\psi_1(\cdot), \varphi(\cdot, x_2))_{\mathcal{S}'(\mathbb{R}^{d_1}),\mathcal{S}(\mathbb{R}^{d_1})}$$

are in $\mathcal{S}(\mathbb{R}^{d_1})$, $\mathcal{S}(\mathbb{R}^{d_2})$, respectively: this follows from the general fact that for any fixed f in $\mathcal{S}'(\mathbb{R}^{d_2})$ and φ in $\mathcal{S}(\mathbb{R}^{d_1+d_2})$, $x \mapsto (f(\cdot), \varphi(x, \cdot))$ lies in $\mathcal{S}(\mathbb{R}^{d_1})$, along with the symmetric relation with the order of the variables transposed. Then we define $\psi_1 \otimes \psi_2$ by

$$(\psi_1 \otimes \psi_2, \varphi)_{\mathcal{S}'(\mathbb{R}^{d_1+d_2}),\mathcal{S}(\mathbb{R}^{d_1+d_2})} := (\psi_1, \varphi_1)_{\mathcal{S}'(\mathbb{R}^{d_1}),\mathcal{S}(\mathbb{R}^{d_1})} = (\psi_2, \varphi_2)_{\mathcal{S}'(\mathbb{R}^{d_2}),\mathcal{S}(\mathbb{R}^{d_2})},$$

which we can write more concisely as

$$(\psi_1 \otimes \psi_2, \varphi) = (\psi_1, (\psi_2, \varphi)) = (\psi_2, (\psi_1, \varphi)). \qquad (2\cdot1)$$

To show that this definition is consistent, we must show that equality holds in the last two expressions in (2.1). So suppose first that $\varphi = \varphi_1 \otimes \varphi_2$. Then

$$(\psi_1, (\psi_2, \varphi)) = (\psi_1, (\psi_2, \varphi_1 \otimes \varphi_2)) = (\psi_1, \varphi_1(\psi_2, \varphi_2)) = (\psi_2, \varphi_2)(\psi_1, \varphi_1) = (\psi_2, (\psi_1, \varphi)),$$

where in the second and third equalities we used the linearity of the pairings involved, and in the final equality we used the symmetric equality with the order of ψ_1 and ψ_2 transposed. This shows that (2·1) is well-defined for test functions that are product-form and hence by linearity for all test functions in $\mathcal{S}(\mathbb{R}^{d_1}) \otimes \mathcal{S}(\mathbb{R}^{d_2})$. But $\mathcal{S}(\mathbb{R}^{d_1}) \otimes \mathcal{S}(\mathbb{R}^{d_2})$ is dense[1] in $\mathcal{S}(\mathbb{R}^{d_1+d_2})$ so the definition is, in fact, well-defined for all distributions in $\mathcal{S}(\mathbb{R}^{d_1+d_2})$.

Equation (2·1) can also be seen as the analog of Fubini's theorem for tempered distributions. In fact, it follows for regular distributions by an application of Fubini's theorem, and hence is a natural definition of the tensor product of two distributions.

3. *Analytic preliminaries*

In this section we establish some analytic results to be used in the proof of Theorem 1·1.

LEMMA 3·1. *The distribution* ψ *of* (1.4) *is absolutely convergent on* $\mathrm{Re}(u_k) > 0$, $k = 1, \ldots, d$, *and* $\mathrm{Re}(s_k) > 0$, $k = 1, \ldots, d-1$, *and continues to an entire distribution.*

Proof. We leave the proof of absolute convergence to the reader. To prove that ψ continues to an entire distribution, we may assume that $d = 2$, the proof being entirely analogous for $d > 2$. The distribution ψ_1 is analytic on $\mathrm{Re}u_1 > 0$ and continues to an entire distribution on u_1 by [**16**, lemma 3], and ψ_2 is analytic on $\mathrm{Re}s_1 > 0$, $\mathrm{Re}u_2 > 0$ and continues to an entire distribution on (s_1, u_2) by [**16**, lemma 4]. Then $\psi = \psi_1 \otimes \psi_2$, and we can write, for any φ in $\mathcal{S}(\mathbb{R}^2)$,

$$(\psi, \varphi) = (\psi_1, (\psi_2, \varphi)) = (\psi_2, (\psi_1, \varphi)).$$

Since $(\psi, \varphi) = (\psi_1, (\psi_2, \varphi))$, it is entire in u_1; since $(\psi, \varphi) = (\psi_2, (\psi_1, \varphi))$ it is entire in s_1 and u_2 as well. But a complex-valued function that is entire in each variable separately is entire: this follows from Hartog's theorem (for instance, see [**6**, theorem B·6, p. 15]). Hence, (ψ, φ) is entire in (u_1, s_1, u_2) and so is entire on the subvariety defined by $s_1 = u_1 - u_2 + 1$, which, with the change of variables $s_1 = s_1$, $s_2 = u_2 + 1$, means that (ψ, φ) is entire when

[1] This fact, which we do not prove, is nonetheless key, because it is where the real machinery of analysis is being used.

viewed as a function of (s_1, s_2). (These relations come from solving for s_1 and s_2 in (1·3).) Since this is true for all φ in $\mathcal{S}(\mathbb{R}^2)$, the distribution ψ is entire.

LEMMA 3·2. *Let $y = y(x)$ be the transformation*

$$y_k = y_k(x) = x_1 \cdots x_k, \quad k = 1, \ldots, d,$$

and let R be defined as in (1.5). *If f is in $\mathcal{S}(y(R))$, then $f \circ y$ is in $\mathcal{S}(R)$.*

Proof. Let f be in $\mathcal{S}(y(R))$ and let $\overline{f} = f \circ y$. Applying the chain rule, we can see that for any multi-index β,

$$D^\beta \overline{f}(x) = \sum_{j=1}^N C_j x^{\gamma^j} \big(D^{\gamma^j} f\big)(y(x))$$

for some positive integers N and (C_j) and multi-indices (γ^j) with each $\gamma^j \leqslant \beta$. It follows that for any multi-indices α and β,

$$\begin{aligned}\sup_{x\in R} |x^\alpha D^\beta \overline{f}(x)| &\leqslant \sum_{j=1}^N C_j \sup_{x\in R} \left|x^{\alpha+\gamma^j} \big(D^{\gamma^j} f\big)(y(x))\right| \\ &\leqslant \sum_{j=1}^N C_j \sup_{x\in R} \left|x_1^{\alpha_1+\gamma_1^j} \big(D^{\gamma^j} f\big)(y(x))\right| \\ &= \sum_{j=1}^N C_j \sup_{y\in y(R)} \left|y_1^{\alpha_1+\gamma_1^j} D^{\gamma^j} f(y)\right|,\end{aligned}$$

where we used the fact that $|x_k| < 1$ for all $k = 2, \ldots, d$. But this is finite because f is in $\mathcal{S}(y(R))$, and we conclude that $\overline{f}$ is in $\mathcal{S}(R)$.

LEMMA 3·3. *The function*

$$g(x) = g(x_1, \ldots, x_d) = \frac{y_1(x) \cdots y_d(x)}{(e^{y_1(x)} - 1) \cdots (e^{y_d(x)} - 1)}$$

is in $S(R)$.

Proof. To prove this we view g as a function of $y = y(x)$. The function $g(y)$ then factors into a product, each factor of which is in $S((0, \infty))$. Hence $g(y)$ is in $S((0, \infty)^d) \subset S(y(R))$, which by Lemma 3·2 implies that g is in $S(R)$.

THEOREM 3·4. *There exists a continuous linear extension operator $\mathcal{E}$ that maps $\mathcal{S}(R)$ to $\mathcal{S}(\mathbb{R}^d)$.*

Proof. For simplicity of notation, we give the proof for $d = 2$; nothing significant changes for $d > 2$. Also, the proof for $d = 1$ is an obvious simplification of the argument for $d = 2$.

Let f be in $\mathcal{S}(R)$. We prove the existence of the extension $\widetilde{f} := \mathcal{E}f$ in three steps, as follows:

Step 1: we extend f to a function u in $C^\infty(\mathbb{R}^2)$ much as in the proof of [**12**, theorem 5', chapter VI], though we do so explicitly so we can more easily make the calculations required to establish Schwartz decay.

Because f is in $C^\infty(R)$, we can extend f continuously to the boundary of R. We then define u on $(-\infty, 0] \times (0, 1)$ as in [**12**, equation (24), p. 182] by

$$u(x, y) = \int_1^\infty f(x - \lambda x, y)\psi(\lambda)\, d\lambda \tag{3·1}$$

and on R by $u(x, y) = f(x, y)$. Here, ψ is as in [**12**, lemma 1, p. 182].

Because f is in $\mathcal{S}(R)$, u and all its derivatives are continuous, as we can verify directly from (3.1) hence, u is in $C^\infty((-\infty, \infty) \times (0, 1))$.

Next we extend u to $\mathbb{R}^2$ as follows. Let $\{\phi_-, \phi_+\}$ be a partition of unity of R defined so that ϕ_+ equals 1 on the set $\{(x, y) \in R : 3/4 \leqslant y < 1\}$, ϕ^- equals 1 on the set $\{(x, y) \in R : 0 < y \leqslant 1/4\}$, and both are constant along horizontal lines. Then define u_- and u_+ in $C^\infty(\mathbb{R}^2)$ by

$$u_-(x, y) = \begin{cases} \displaystyle\int_1^\infty (u\phi_-)(x, y - \lambda y)\psi(\lambda)\, d\lambda, & y \leqslant 0, \\ u(x, y)\phi_-(x, y), & 0 < y < 1, \\ 0, & y \geqslant 1, \end{cases}$$

$$u_+(x, y) = \begin{cases} \displaystyle\int_1^\infty (u\phi_+)(x, y + \lambda(y - 1))\psi(\lambda)\, d\lambda, & y \geqslant 1, \\ u(x, y)\phi_+(x, y), & 0 < y < 1, \\ 0, & y \leqslant 0. \end{cases}$$

In both integrals above we treat u as being zero whenever ϕ_- or ϕ_+ is zero (the value we choose for u does not matter).

Finally, define u in $C^\infty(\mathbb{R}^2)$ by

$$u(x, y) = u_-(x, y) + u_+(x, y),$$

and observe that u is an extension of f to all of $\mathbb{R}^2$, and u is in $C^\infty(\mathbb{R}^2)$ by the same reasoning as before.

Step 2: let φ_h and φ_v in $C^\infty(\mathbb{R}^2)$ assume values in $[0, 1]$ and be such that $\varphi_h \equiv 1$ on $[0, \infty)$, $\varphi_h \equiv 0$ on $(-\infty, -1]$, $\varphi_v \equiv 1$ on $[0, 1]$, and $\varphi_v \equiv 0$ on $[2, \infty)$ and on $(-\infty, -1]$. Then $\varphi := \varphi_h\varphi_v$ is in $C^\infty(\mathbb{R}^2)$ and assumes values in $[0, 1]$, is identically 1 on R, and is identically 0 on the complement in $\mathbb{R}^2$ of $(-1, \infty) \times (-1, 2)$.

Define $\widetilde{f}$ in $C^\infty(\mathbb{R}^2)$ by

$$\widetilde{f} = \varphi u.$$

Step 3: the function $\widetilde{f}$ has Schwartz decay in all directions except possibly along the positive x-axis when y is in $[1, 2)$ or in $(-1, 0]$, because in all other directions, $\widetilde{f}$ either equals f, which has Schwartz decay, or becomes zero after a finite distance. So we need only show that $|x^m y^n \partial_x^j \partial_y^k \widetilde{f}(x, y)|$ is bounded for all nonnegative integers m, n, j, and k on two subsets of $\mathbb{R}^d$: $R_1 = (0, \infty) \times (-1, 0)$ and $R_2 = (0, \infty) \times (1, 2)$.

First we consider only partial derivatives of x. Assume that (x, y) is in R_1, and that m, n, and j are nonnegative integers. Then, since φ is constant along horizontal rays in R_1,

$$\begin{aligned}
\left|x^m y^n \partial_x^j \widetilde{f}(x, y)\right| &= \left|\varphi(x, y) x^m y^n \partial_x^j u(x, y)\right| \\
&\leqslant \left|x^m y^n \partial_x^j u(x, y)\right| \\
&= \left|x^m y^n \partial_x^j u_-(x, y)\right| \\
&= \left|x^m y^n \partial_x^j \int_1^\infty (f\phi_-)(x, y - \lambda y)\psi(\lambda)\, d\lambda\right| \\
&= \left|\int_1^\infty \big(x^m y^n \partial_x^j f(x, y - \lambda y)\big)\phi_-(x, y - \lambda y)\psi(\lambda)\, d\lambda\right| \\
&\leqslant \sup|\psi| \sup_{y' \in (0,1)} \left|x^m y^n \partial_x^j f(x, y')\right| \left|\int_1^\infty \phi_-(x, y - \lambda y)\right| \\
&\leqslant \sup|\psi| \sup_{y' \in (0,1)} \left|x^m y^n \partial_x^j f(x, y')\right|.
\end{aligned}$$

The second and third equalities follow from the definitions of u and u_- (and u becomes f in the integral because $x > 0$). The fourth equality uses the constancy of ϕ_- along horizontal lines. The last inequality follows by a change of variables and the observation that ϕ_- is supported in a strip of vertical width less than 1.

Thus,

$$\begin{aligned}
\sup_{(x,y) \in R_1} \left|x^m y^n \partial_x^j \widetilde{f}(x, y)\right| &\leqslant \sup|\psi| \sup_{x>0} \sup_{y' \in (0,1)} \left|x^m y^n \partial_x^j f(x, y')\right| \\
&= \sup|\psi| \sup_{(x,y') \in R} \left|x^m y^n \partial_x^j f(x, y')\right|,
\end{aligned}$$

which is finite by the assumption that f is in $\mathcal{S}(R)$. The bound on R_2 is obtained similarly.

Bounding $|x^m y^n \partial_x^j \partial_y^k \widetilde{f}(x, y)|$ is more tedious, because both φ and ϕ_- have nonzero partial derivatives in the y-direction. If we write this as $|x^m y^n \partial_y^k \partial_x^j \widetilde{f}(x, y)|$, we can start with the calculation above then perform the partial derivatives in y. This will result in a sum of terms including partial derivatives of φ, ϕ_-, and f. Each term, however, will be just as above, with φ and ϕ_- replaced by partial derivatives of these functions, and with partial derivatives in both x and y. Since all the partial derivatives of φ and ϕ_- are bounded, this does not change the argument for each term, and we see that $|x^m y^n \partial_x^j \partial_y^k \widetilde{f}(x, y)|$ is bounded as well.

The linearity of the extension operator $\mathcal{E}f = \widetilde{f}$ is clear from the definition of $\widetilde{f}$, and its continuity follows from the bounds we established above.

4. *Proof of Theorem* 1·1

From (1·1) and the identity

$$\Gamma(s) = \int_0^\infty w^{s-1} e^{-w} dw$$

we have

$$\zeta_d(s;\theta)\prod_{j=1}^{d}\Gamma(s_j)$$
$$=\int_0^\infty\cdots\int_0^\infty\sum_{0<n_1\cdots<n_d}(n_1+\theta_1)^{-s_1}\cdots(n_d+\theta_d)^{-s_d}w_1^{s_1-1}e^{-w_1}\cdots w_d^{s_d-1}e^{-w_d}dw_1\cdots dw_d.$$

Make the change of variables $w_k=(n_k+\theta_k)t_k$,

$$\begin{aligned}(n_k+\theta_k)^{-s_k}w_k^{s_k-1}e^{-w_k}dw_k &= (n_k+\theta_k)^{-s_k}(n_k+\theta_k)^{s_k-1}t_k^{s_k-1}e^{-(n_k+\theta_k)t_k}(n_k+\theta_k)dt_k\\ &= t_k^{s_k-1}e^{-(n_k+\theta_k)t_k}dt_k,\end{aligned}$$

to obtain

$$\begin{aligned}&\zeta_d(s;\theta)\prod_{j=1}^{d}\Gamma(s_j)\\ &=\int_0^\infty\cdots\int_0^\infty\sum_{0<n_1<\cdots<n_d}t_1^{s_1-1}e^{-(n_1+\theta_1)t_1}\cdots t_d^{s_d-1}e^{-(n_d+\theta_d)t_d}dt_1\cdots dt_d\\ &=\int_0^\infty\cdots\int_0^\infty\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}t_1^{s_1-1}e^{-(n_1+\theta_1)t_1}\cdots t_d^{s_d-1}e^{-(n_1+\cdots+n_d+\theta_d)t_d}dt_1\cdots dt_d\\ &=\int_0^\infty\cdots\int_0^\infty t_1^{s_1-1}\cdots t_d^{s_d-1}e^{-\theta_1t_1}\cdots e^{-\theta_dt_d}\phi(t_1,\ldots,t_d)dt_1\cdots dt_d, \qquad (4{\cdot}1)\end{aligned}$$

where

$$\phi(t_1,\ldots,t_d)=\sum_{n_1=1}^{\infty}\cdots\sum_{n_d=1}^{\infty}e^{-n_1(t_1+\cdots+t_d)}\cdots e^{-n_dt_d}.$$

Set $x_{d+1}=0$ and define the change of variables

$$x_1\cdots x_k=t_k+\cdots+t_d \quad \text{if and only if} \quad t_k=x_1\cdots x_k(1-x_{k+1}) \qquad (4{\cdot}2)$$

for $1\leqslant k\leqslant d$. Notice that $t_k\geqslant 0$ for $1\leqslant k\leqslant d$ if and only if $0\leqslant x_1<\infty$ and $0\leqslant x_k\leqslant 1$ for $2\leqslant k\leqslant d$. The k,ℓth entry of the Jacobian of the transformation defined by (4·2) is

$$\frac{\partial t_k}{\partial x_\ell}(x_1,\ldots,x_d)=\begin{cases}x_1\cdots x_{\ell-1}x_{\ell+1}\cdots x_k(1-x_{k+1}), & \text{if } \ell<k;\\ x_1\cdots x_{k-1}(1-x_{k+1}), & \text{if } \ell=k;\\ -x_1\cdots x_k, & \text{if } \ell=k+1;\\ 0, & \text{if } \ell>k+1.\end{cases}$$

It can be shown that the determinant of the Jacobian is

$$\left|\left(\frac{\partial t_k}{\partial x_\ell}(x_1,\ldots,x_d)\right)\right|=x_1^{d-1}\cdots x_{d-1}.$$

Make the change of variables defined by (4·2) in (4·1) and use the functional equation $\Gamma(s) = (s-1)\Gamma(s-1)$ to obtain

$$\begin{aligned}
&\zeta_d(s;\theta)\frac{s_d - 1}{\Gamma(u_1)\cdots\Gamma(u_{d-1})} \\
&= \int_0^1 \cdots \int_0^1 \int_0^\infty \frac{x_1^{u_1-1}}{\Gamma(u_1)} \prod_{k=2}^{d} \frac{(1-x_k)^{s_{k-1}-1} x_k^{u_k-1}}{\Gamma(s_{k-1})\Gamma(u_k)} h(x_1,\ldots,x_d)dx_1\cdots dx_d \\
&= \int_0^1 \cdots \int_0^1 \int_0^\infty \psi(x_1,\ldots,x_d; s_1,\ldots,s_d) h(x_1,\ldots,x_d)dx_1\cdots dx_d, \qquad (4\cdot3)
\end{aligned}$$

where

$$h(x_1,\ldots,x_d) = \nu(x_1,\ldots,x_d)g(x_1,\ldots,x_d),$$

$$\nu(x_1,\ldots,x_d) = e^{-\theta_1 x_1(1-x_2)} \cdots e^{-\theta_{d-1}x_1\cdots x_{d-1}(1-x_d)} e^{-\theta_d x_1 \cdots x_d}$$

and

$$\begin{aligned}
g(x_1,\ldots,x_d) &= x_1^d x_2^{d-1} \cdots x_d \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} e^{-n_1 x_1} \cdots e^{-n_d x_1 \cdots x_d} \\
&= \frac{x_1^d x_2^{d-1} \cdots x_d}{(e^{x_1}-1)\cdots(e^{x_1\cdots x_d}-1)}.
\end{aligned}$$

By Lemma 3·3 we know that g is in $S(R)$. Since ν is a bounded, C^∞ function on R, all of whose derivatives are bounded on R, and $S(R)$ is closed under multiplication by such functions, it follows that h is in $S(R)$. Therefore, by Theorem 3·4 there exists an extension $f = \mathcal{E}(h)$ of h to $S(\mathbb{R}^d)$. Solve for $\zeta_d(s;\theta)$ in (4·3) to obtain

$$\zeta_d(s;\theta) = \frac{\Gamma(u_1)\cdots\Gamma(u_{d-1})}{s_d - 1}(\psi(\cdot, s_1,\ldots,s_d), f(\cdot)). \qquad (4\cdot4)$$

Finally, by Lemma 3·1 we see that (4·4) gives an explicit expression for the continuation of $\zeta_d(s;\theta)$ with the possible poles along the stated hyperplanes.

5. *Proof of Theorem* 1·3

The case $d = 1$ is a classical result. Assume that $d > 1$. By (4·4),

$$\begin{aligned}
\mathrm{Res}_{s_d=1}\zeta_d(s;\theta) &= \Gamma(s_1+\cdots+s_{d-1}-(d-1))\cdots\Gamma(s_{d-2}+s_{d-1}-2)\Gamma(s_{d-1}-1) \\
&\quad\times \lim_{s_d\to 1}(\psi(\cdot, s_1,\ldots,s_d), h(\cdot)).
\end{aligned}$$

Now, a straightforward calculation yields

$$\begin{aligned}
&\lim_{s_d\to 1}(\psi(\cdot, s_1,\ldots,s_d), h(\cdot)) \\
&= \lim_{s_d\to 1}\int_0^1 \cdots \int_0^1 \int_0^\infty \frac{x_1^{s_1+\cdots+s_{d-1}-d}}{\Gamma(s_1+\cdots+s_{d-1}-(d-1))}
\end{aligned}$$

$$\times \prod_{k=2}^{d-1} \frac{(1-x_k)^{s_{k-1}-1} x_k^{s_k+\cdots+s_{d-1}-(d-1)+k-2}}{\Gamma(s_{k-1})\Gamma(s_k+\cdots+s_{d-1}-(d-1)+k-1)}$$

$$\times e^{-\theta_1 x_1(1-x_2)} \cdots e^{-\theta_{d-1}x_1\cdots x_{d-1}} \frac{x_1^{d-1}x_2^{d-1}\cdots x_{d-1}}{(e^{x_1}-1)\cdots(e^{x_1\cdots x_{d-1}}-1)}$$

$$\times \frac{1}{(s_{d-1}-1)\Gamma(s_{d-1}-1)} \frac{(1-x_d)^{s_{d-1}-1}x_d^{s_d-2}}{\Gamma(s_d-1)} e^{-(\theta_d-\theta_{d-1})x_1\cdots x_d} \frac{x_1\cdots x_d}{e^{x_1\cdots x_d}-1} dx_1\cdots dx_d,$$

where we have used the functional equation $\Gamma(s) = (s-1)\Gamma(s-1)$. From (4·4) we see that it suffices to show that

$$\lim_{s_d\to 1} \int_0^1 \frac{(1-x_d)^{s_{d-1}-1}x_d^{s_d-2}}{\Gamma(s_d-1)} e^{-(\theta_d-\theta_{d-1})x_1\cdots x_d} \frac{x_1\cdots x_d}{e^{x_1\cdots x_d}-1} dx_d = 1.$$

This follows from a calculation similar to that in the proof of [**16**, lemma 4].

6. *Proof of Theorem* 1·4

First, assume that $k \geqslant 2$. Using (4·3) we may express the multiple Hurwitz zeta function $\zeta_d(s;\theta)$ as

$$\zeta_d(s;\theta) = \int_0^1 \cdots \int_0^1 \int_0^\infty \prod_{j=1}^{k-1} \frac{x_j^{s_{k-1}(j)+u_k-1}}{e^{\theta_j x_1\cdots x_j(1-x_{j+1})}(e^{x_1\cdots x_j}-1)}$$

$$\times \prod_{j=2}^{k-1}(1-x_j)^{s_{j-1}-1} \frac{f(x_1,\ldots,x_{k-1};s)}{\prod_{j=1}^d \Gamma(s_j)} dx_1\cdots dx_{k-1}, \tag{6·1}$$

where

$$f(x_1,\ldots,x_{k-1};s) = \int_0^1 \cdots \int_0^1 \prod_{j=k}^d x_j^{u_j-1}(1-x_j)^{s_{j-1}-1} \frac{x_1\cdots x_j e^{(\theta_{j-1}-\theta_j)x_1\cdots x_j}}{e^{x_1\cdots x_j}-1} dx_k\cdots dx_d$$

(here we have set $x_{j+1}=0$ for $j+1 \geqslant i$). We want to compute the residue of $f(x_1,\ldots, x_{k-1};s)$ on the hyperplane

$$s_d(k) = d-k+2-n.$$

Recall that the Bernoulli polynomial generating function is

$$\frac{xe^{tx}}{e^x-1} = \sum_{a=0}^\infty B_a(t)\frac{x^a}{a!}.$$

Therefore

$$\prod_{j=k}^d \frac{x_1\cdots x_j e^{(\theta_{j-1}-\theta_j)x_1\cdots x_j}}{e^{x_1\cdots x_j}-1} = \sum_{a_k,\ldots,a_d\geqslant 0} \left\{(x_1\cdots x_{k-1})^{a_d(k)} \prod_{j=k}^d \frac{B_{a_j}(\theta_{j-1}-\theta_j)}{a_j!} x_k^{a_d(j)}\right\}. \tag{6·2}$$

Substitute (6·2) into $f(x_1, \dots, x_{k-1}; s)$ and use the identity

$$u_j + s_{j-1} = u_{j-1} + 1$$

and the beta integral

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

to obtain

$$\begin{aligned} &f(x_1, \dots, x_{k-1}; s) \\ &= \sum_{a_k,\dots,a_d \geqslant 0} \left\{ (x_1 \cdots x_{k-1})^{a_d(k)} \prod_{j=k}^{d} \frac{B_{a_j}(\theta_{j-1}-\theta_j)}{a_j!} \frac{\Gamma(a_d(j)+u_j)\Gamma(s_{j-1})}{\Gamma(a_d(j)+u_j+s_{j-1})} \right\} \\ &= \sum_{a_k,\dots,a_d \geqslant 0} \left\{ (x_1 \cdots x_{k-1})^{a_d(k)} \frac{B_{a_k}(\theta_{k-1}-\theta_k)}{a_k!} \frac{\Gamma(a_d(k)+u_k)}{\Gamma(a_d(k+1)+u_k+1)} \right. \\ &\left. \times \frac{1}{\Gamma(a_d(k)+u_k+s_{k-1})} \prod_{j=k+1}^{d} \frac{B_{a_j}(\theta_{j-1}-\theta_j)}{a_j!} \frac{\Gamma(a_d(j)+u_j)}{\Gamma(a_d(j+1)+u_j+1)} \prod_{j=k-1}^{d} \Gamma(s_j) \right\}. \end{aligned} \tag{6·3}$$

Observe that in (6·3) only the terms with $a_d(k) = n-1$ contribute to the residue on the hyperplane $s_d(k) = d-k+2-n$. Furthermore, for every such term with $a_k > 0$ the function

$$\frac{\Gamma(a_d(k)+u_k)}{\Gamma(a_d(k+1)+u_k+1)} = \prod_{\ell=1}^{a_k-1}(a_d(k)+u_k-\ell)$$

has *no* poles. Using these facts we obtain the residue in the statement of the theorem.

Next, assume that $k=1$. The proof in this case is essentially the same as for $k \geqslant 2$. One simply needs to set $k=2$ in (6·1) and use

$$\lim_{u\to 1}(u-1)\int_0^\infty \frac{x^{u-1}}{e^{\theta_1 x}(e^x-1)}dx = 1.$$

To calculate this limit we use the Mellin transform of the classical one-dimensional Hurwitz zeta function

$$\zeta(s;q) = \sum_{n=0}^{\infty}(n+q)^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^{qx}(1-e^{-x})}dx$$

and the fact that $\zeta(s;q)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$ with residue 1. In particular,

$$\int_0^\infty \frac{x^{u-1}}{e^{\theta_1 x}(e^x-1)} = \int_0^\infty \frac{x^{u-1}}{e^{(\theta_1+1)x}(1-e^{-x})},$$

so that

$$\lim_{u\to 1}(u-1)\int_0^\infty \frac{x^{u-1}}{e^{\theta_1 x}(e^x-1)}dx = \lim_{u\to 1}(u-1)\Gamma(u)\zeta(u;\theta_1+1) = 1.$$

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