GAME THEORY — THE GRAND THEOREM
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Definition 3. Given a zero-sum 2-player normal form game and a mixed strategy p for player A, we define **A's security level** to be

$$\min_{q'} \ p \cdot Aq'$$

Definition 4. Given a zero-sum 2-player normal form game, we say a mixed strategy p for player A is a **maximin strategy** if

$$\min_{q'} \ p \cdot Aq' = \max_{p'} \min_{q'} \ p' \cdot Aq'$$

Definition 5. Given a zero-sum 2-player normal form game and a mixed strategy q for player B, we define **B's security level** to be

$$\min_{p'} \ -p' \cdot Aq$$

(since $B = -A$).

Definition 6. Given a zero-sum 2-player normal form game, we say a mixed strategy q' for B is a **maximin strategy for B** if

$$\min_{p'} \ -p' \cdot Aq = \max_{q'} \min_{p'} \ -p' \cdot Aq'$$

Theorem 0. Given a zero-sum 2-player normal form game, q is a maximin strategy for B if and only if:

$$\max_{p'} \ p' \cdot Aq = \min_{q'} \max_{p'} \ p' \cdot Aq'$$

Now for the big theorem we're going to prove:

Grand Theorem. For every zero-sum 2-player normal-form game, a Nash equilibrium exists. Moreover, a pair of mixed strategies (p, q) for the two players is a Nash equilibrium if and only if each strategy is a maximin strategy.

We prove this in 6 steps:

Theorem 1. For any zero-sum 2-player normal form game,

$$\min_{q'} \max_{p'} p' \cdot Aq' \geq \max_{p'} \min_{q'} \ p' \cdot Aq'$$

Theorem 2. Given a zero-sum 2-player normal form game for which a Nash equilibrium exists,

$$\min_{q'} \max_{p'} \ p' \cdot Aq' = \max_{p'} \min_{q'} \ p' \cdot Aq'$$

Theorem 3. If (p, q) is a Nash equilibrium for a zero-sum 2-player normal-form game, then p is a maximin strategy for player A and q is a maximin strategy for player B.

Theorem 4. Suppose we have a zero-sum 2-player normal form game for which

$$\min_{q'} \max_{p'} \ p' \cdot Aq' = \max_{p'} \min_{q'} \ p' \cdot Aq' \qquad \star$$

holds. If p is a maximin strategy for player A and q is a maximin strategy for player B, then (p, q) is a Nash equilibrium.

Theorem 5. For every zero-sum 2-player normal-form game, a maximin strategy exists for each player.

Theorem 6. For every zero-sum 2-player normal-form game, $\star$ holds.