Higher-Dimensional Algebra II:
2-Hilbert Spaces

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Abstract

A 2-Hilbert space is a category with structures and properties analogous to those of a Hilbert space. More precisely, we define a 2-Hilbert space to be an abelian category enriched over Hilb with a $*$-structure, conjugate-linear on the hom-sets, satisfying $\langle fg, h\rangle = \langle g, f^*h\rangle = \langle f, hg^*\rangle$. We also define monoidal, braided monoidal, and symmetric monoidal versions of 2-Hilbert spaces, which we call 2-H*-algebras, braided 2-H*-algebras, and symmetric 2-H*-algebras, and we describe the relation between these and tangles in 2, 3, and 4 dimensions, respectively. We prove a generalized Doplicher-Roberts theorem stating that every symmetric 2-H*-algebra is equivalent to the category $\mathrm{Rep}(G)$ of continuous unitary finite-dimensional representations of some compact supergroupoid G. The equivalence is given by a categorified version of the Gelfand transform; we also construct a categorified version of the Fourier transform when G is a compact abelian group. Finally, we characterize $\mathrm{Rep}(G)$ by its universal properties when G is a compact classical group. For example, $\mathrm{Rep}(\mathrm{U}(n))$ is the free connected symmetric 2-H*-algebra on one even object of dimension n.

1 Introduction

A common theme in higher-dimensional algebra is 'categorification': the formation of $(n+1)$-categorical analogs of n-categorical algebraic structures. This amounts to replacing equations between n-morphisms by specified $(n+1)$-isomorphisms, in accord with the philosophy that any *interesting* equation — as opposed to one of the form $x = x$ — is better understood as an isomorphism, or more generally an equivalence.

In their work on categorification in topological quantum field theory, Freed [10] and Crane [5] have, in an informal way, used the concept of a '2-Hilbert space': a category with structures and properties analogous to those of a Hilbert space. Our goal here is to define 2-Hilbert spaces precisely and begin to study them. We concentrate on the finite-dimensional case, as the infinite-dimensional case introduces extra issues that we are not yet ready to handle. We must start by categorifying the various ingredients in the definition of Hilbert space. These are: 1) the zero element, 2) addition, 3) subtraction, 4) scalar multiplication, and 5) the inner product. The first four have well-known categorical analogs.

1) The analog of the zero vector is a 'zero object'. A zero object in a category is an object that is both initial and terminal. That is, there is exactly one morphism from it to any object, and exactly one morphism to it from any object. Consider for example the category Hilb having finite-dimensional Hilbert spaces as objects, and linear maps between them as morphisms. In Hilb, any zero-dimensional Hilbert space is a zero object.

2) The analog of adding two vectors is forming the direct sum, or more precisely the 'coproduct', of two objects. A coproduct of the objects x and y is an object $x \oplus y$, equipped with morphisms from x and y to it, that is universal with respect to this property. In Hilb, for example, any Hilbert space equipped with an isomorphism to the direct sum of x and y is a coproduct of x and y.

3) The analog of subtracting vectors is forming the 'cokernel' of a morphism $f: x \to y$. This makes sense only in a category with a zero object. A cokernel of $f: x \to y$ is an object $\mathrm{cok} f$ equipped with an epimorphism $g: y \to \mathrm{cok} f$ for which the composite of f and g factors through the zero object, that is universal with respect to this property. Note that while we can simply subtract a number x from a number y, to form a cokernel we need to say how the object x is mapped to the object y. In Hilb, for example, any space equipped with an isomorphism to the orthogonal complement of $\mathrm{im} f$ in y is a cokernel of $f: x \to y$. If f is an inclusion, so that x is a subspace of y, its cokernel is sometimes written as the 'direct difference' $y \ominus x$ to emphasize the analogy with subtraction.

An important difference between zero, addition and subtraction and their categorical analogs is that these operations represent extra *structure* on a set, while having a zero object, binary coproducts or cokernels is merely a *property* of a category. Thus these concepts are in some sense more intrinsic to categories than to sets. On the other hand, one pays a price for this: while the zero element, sums, and differences are unique in a Hilbert space, the zero object, coproducts, and cokernels are typically unique only up to canonical isomorphism.

4) The analog of multiplying a vector by a complex number is tensoring an object by a Hilbert space. Besides its additive properties (zero object, binary coproducts, and cokernels), Hilb also has a compatible multiplicative structure, that is, tensor products and a unit object for the tensor product. In other words, Hilb is a 'ring category', as defined by Laplaza and Kelly [19, 20]. We expect it to play a role in 2-Hilbert space theory analogous to the role played by the ring $\mathbb{C}$ of complex numbers in Hilbert space theory. Thus we expect 2-Hilbert spaces to be 'module categories' over Hilb, as defined by Kapranov and Voevodsky [17].

An important part of our philosophy here is that $\mathbb{C}$ is the primordial Hilbert space: the simplest one, upon which the rest are modelled. By analogy, we expect Hilb to be the primordial 2-Hilbert space. This is part of a general pattern pervading higher-dimensional algebra; for example, there is a sense in which $n\mathrm{Cat}$ is the primordial

$(n+1)$-category. The real significance of this pattern remains somewhat mysterious.

5) Finally, what is the categorification of the inner product in a Hilbert space? It appears to be the 'hom functor'. The inner product in a Hilbert space x is a bilinear map

$$\langle \cdot \, , \cdot \rangle \colon \overline{x} \times x \to \mathbb{C}$$

taking each pair of elements $v, w \in x$ to the inner product $\langle v, w \rangle$. Here $\overline{x}$ denotes the conjugate of the Hilbert space x. Similarly, the hom functor in a category C is a bifunctor

$$\mathrm{hom}(\cdot \, , \cdot) \colon C^{\mathrm{op}} \times C \to \mathrm{Set}$$

taking each pair of objects $c, d \in C$ to the set $\mathrm{hom}(c, d)$ of morphisms from c to d. This analogy clarifies the relation between category theory and quantum theory that is so important in topological quantum field theory. In quantum theory the inner product $\langle v, w \rangle$ is a *number* representing the amplitude to pass from v to w, while in category theory $\mathrm{hom}(c, d)$ is a *set* of morphisms passing from c to d.

To understand this analogy better, note that any morphism $f \colon x \to y$ in Hilb can be turned around or 'dualized' to obtain a morphism $f^* \colon y \to x$. The morphism f^* is called the adjoint of f, and satisfies

$$\langle fv, w \rangle = \langle v, f^* w \rangle$$

for all $v \in x$, $w \in y$. The ability to dualize morphisms in this way is crucial to quantum theory. For example, observables are represented by self-adjoint morphisms, while symmetries are represented by unitary morphisms, whose adjoint equals their inverse.

The ability to dualize morphisms in Hilb makes this category very different from the category Set, in which the only morphisms $f \colon x \to y$ admitting any natural sort of 'dual' are the invertible ones. There is, however, duals for certain noninvertible morphisms in Cat — namely, adjoint functors. The functor $F^* \colon D \to C$ is said to be a right adjoint of the functor $F \colon C \to D$ if there is a natural isomorphism

$$\mathrm{hom}(Fc, d) \cong \mathrm{hom}(c, F^* d)$$

for all $c \in C$, $d \in D$. The analogy to adjoints of operators between Hilbert spaces is clear. Our main point here is that that this analogy relies on the more fundamental analogy between the inner product and the hom functor.

One twist in the analogy between the inner product and the hom functor is that the inner product for a Hilbert space takes values in $\mathbb{C}$. Since we are treating Hilb as the categorification of $\mathbb{C}$, the hom-functor for a 2-Hilbert space should take values in Hilb rather than Set. In technical terms [18], this suggests that a 2-Hilbert space should be enriched over Hilb.

To summarize, we expect that a 2-Hilbert space should be some sort of category with 1) a zero object, 2) binary coproducts, and 3) cokernels, which is 4) a Hilb-module and 5) enriched over Hilb. However, we also need a categorical analog for the equation

$$\langle v, w\rangle = \overline{\langle w, v\rangle}$$

satisfied by the inner product in a Hilbert space. That is, for any two objects x, y in a 2-Hilbert space there should be a natural isomorphism

$$\hom(x, y) \cong \overline{\hom(y, x)}$$

where $\overline{\hom(y, x)}$ is the complex conjugate of the Hilbert space $\hom(y, x)$. (The fact that objects in Hilb have complex conjugates is a categorification of the fact that elements of $\mathbb{C}$ have complex conjugates.) This natural isomorphism should also satisfy some coherence laws, which we describe in Section 2. We put these ingredients together and give a precise definition of 2-Hilbert spaces in Section 3.

Why bother categorifying the notion of Hilbert space? As already noted, one motivation comes from the study of topological quantum field theories, or TQFTs. In the introduction to this series of papers [2], we proposed that n-dimensional unitary extended TQFTs should be treated as n-functors from a certain n-category nCob to a certain n-category nHilb. Roughly speaking, the n-category nCob should have 0-dimensional manifolds as objects, 1-dimensional cobordisms between these as morphisms, 2-dimensional cobordisms between these as 2-morphisms, and so on up to dimension n. The n-category nHilb, on the other hand, should have 'n-Hilbert spaces' as objects, these being $(n-1)$-categories with structures and properties analogous to those of Hilbert spaces. (Note that an ordinary Hilbert space is a '1-Hilbert space', and is a 0-category, or set, with extra structures and properties.)

An eventual goal of this series is to develop the framework needed to make these ideas precise. This will require work both on n-categories in general — especially 'weak' n-categories, which are poorly understood for $n > 3$ — and also on the particular n-categories nCob and nHilb. One of the guiding lights of weak n-category theory is the chart shown in Figure 1. This describes 'k-tuply monoidal n-categories' — that is, $(n+k)$-categories with only one j-morphism for $j < k$. The entries only correspond to theorems for $n + k \leq 3$, but there is evidence that the pattern continues for arbitrarily large values of n, k. Note in particular how as we descend each column, the n-categories first acquire a 'monoidal' or tensor product structure, which then becomes increasingly 'commutative' in character with increasing k, stabilizing at $k = n + 2$.

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	2-categories
$k = 1$	monoids	monoidal categories	monoidal 2-categories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k = 3$	‘’	symmetric monoidal categories	weakly involutory monoidal 2-categories
$k = 4$	‘’	‘’	strongly involutory monoidal 2-categories
$k = 5$	‘’	‘’	‘’

1. The category-theoretic hierarchy: expected results

At least in the low-dimensional cases examined so far, the n-categories of interest in topological quantum field theory have simple algebraic descriptions. For example, knot theorists are familiar with the category of framed oriented 1-dimensional cobordisms embedded in $[0,1]^3$. We would call these ‘1-tangles in 3 dimensions’. They form not merely a category, but a braided monoidal category. In fact, they form the ‘free braided monoidal category with duals on one object’, the object corresponding to the positively oriented point. More generally, we expect that n-tangles in $n+k$ dimensions form the ‘free k-tuply monoidal n-category with duals on one object’, $C_{n,k}$. By its freeness, we should be able to obtain a representation of $C_{n,k}$ in any k-tuply monoidal n-category with duals by specifying a particular object therein.

When the codimension k enters the stable range $k \geq n+2$ we hope to obtain the ‘free stable n-category with duals on one object’, $C_{n,\infty}$. A unitary extended TQFT should be a representation of this in nHilb. If as expected nHilb is a stable n-category with duals, to specify a unitary extended TQFT would then simply be to specify a particular n-Hilbert space. More generally, we expect an entire hierarchy of k-tuply monoidal n-Hilbert spaces in analogy to the category-theoretic hierarchy, as shown in Figure 2. We also hope that an object in a k-tuply monoidal n-Hilbert space H will determine a representation of $C_{n,k}$ in H, and thus an invariant of n-tangles in $(n+k)$ dimensions.

	$n = 1$	$n = 2$	$n = 3$
$k = 0$	Hilbert spaces	2-Hilbert spaces	3-Hilbert spaces
$k = 1$	H*-algebras	2-H*-algebras	3-H*-algebras
$k = 2$	commutative H*-algebras	braided 2-H*-algebras	braided 3-H*-algebras
$k = 3$	‘’	symmetric 2-H*-algebras	weakly involutory 3-H*-algebras
$k = 4$	‘’	‘’	strongly involutory 3-H*-algebras
$k = 5$	‘’	‘’	‘’

2. The quantum-theoretic hierarchy: expected results

We are far from proving general results along these lines! However, in Section 4 we sketch the structure of 2Hilb as a strongly involutory 3-H*-algebra, and in Section 5 we define 2-H*-algebras, braided 2-H*-algebra, and symmetric 2-H*-algebras, and describe their relationships to 1-tangles in 2, 3, and 4 dimensions, respectively.

An exciting fact about the quantum-theoretic hierarchy is that it automatically subsumes various branches of representation theory. 2-H*-algebras arise naturally as categories of unitary representations of certain Hopf algebras, or more generally ‘Hopf algebroids’, which are to groupoids as Hopf algebras are to groups [21]. Braided 2-H*-algebras arise in a similar way from certain quasitriangular Hopf algebroids — for example, quantum groups — while symmetric 2-H*-algebras arise from certain triangular Hopf algebroids — for example, groups.

In Section 6 of this paper we concentrate on the symmetric case. Generalizing the Doplicher-Roberts theorem [8], we prove that all symmetric 2-H*-algebras are equivalent to categories of representations of ‘compact supergroupoids’. If a symmetric 2-H*-algebra is ‘purely bosonic’, it is equivalent to a category of representations of a compact groupoid; if it is ‘connected’, it is equivalent to a category of representations of a compact supergroup. In particular, any connected even symmetric 2-H*-algebra is equivalent to the category $\mathrm{Rep}(G)$ of continuous unitary finite-dimensional representations of a compact group G. This is the original Doplicher-Roberts theorem.

One can view our generalized Doplicher-Roberts theorem as a categorified version of the Gelfand-Naimark theorem. The Gelfand-Naimark theorem applies to commutative C*-algebras, but one can easily deduce a version for commutative H*-algebras. Roughly speaking, this says that every commutative H*-algebra H is isomorphic to a commutative H*-algebra of *functions* from some set $\mathrm{Spec}(H)$ to $\mathbb{C}$. Similarly, our theorem implies that every even symmetric 2-H*-algebra H is equivalent to a symmetric 2-H*-algebra of *functors* from some groupoid $\mathrm{Spec}(H)$ to Hilb. The equivalence is given explicitly by a categorified version of the Gelfand transform. We also construct a categorified version of the Fourier transform, applicable to the representation theory of compact abelian groups.

These links between the quantum-theoretic hierarchy and representation theory give new insight into the representation theory of classical groups. The designation of a group as 'classical' is more a matter of tradition than of some conceptual definition, but in practice what makes a group 'classical' is that it has a nice right universal property. In other words, there is a simple description of homomorphisms into it. Using the fact that group homomorphisms from G to H determine symmetric 2-H*-algebra homomorphisms from $\mathrm{Rep}(H)$ to $\mathrm{Rep}(G)$, one can show that for a classical group H the symmetric 2-H*-algebra $\mathrm{Rep}(H)$ has nice left universal property: there is a simple description of homomorphisms out of it.

For example, the group $\mathrm{U}(n)$ has a distinguished n-dimensional unitary representation ρ, its fundamental representation on $\mathbb{C}^n$. An n-dimensional unitary representation of any group G is essentially the same as a homomorphism from G to $\mathrm{U}(n)$. Using this right universal property of $\mathrm{U}(n)$, we show in Section 6 that the category of unitary representations of $\mathrm{U}(n)$ is the 'free symmetric 2-H*-algebra on one object of dimension n'. This statement tersely encodes the usual description of the representations of $\mathrm{U}(n)$ in terms of Young diagrams. We also give similar characterizations of the categories of representations of other classical groups.

In what follows, we denote the composition of 1-morphisms, the horizontal composition of a 1-morphism and a 2-morphism (in either order) and the horizontal composition of 2-morphisms is denoted by $\circ$ or simply juxtaposition. Vertical composition of 2-morphisms is denoted by $\cdot$. *Nota bene*: in composition we use the ordering in which, for example, the composite of $f\colon x \to y$ and $g\colon y \to z$ is denoted $f \circ g$. We denote the identity morphism of an object x either as 1_x or, if there is no danger of confusion, simply as x. We refer to our earlier papers on higher-dimensional algebra as HDA0 [2] and HDA1 [3].

2 H*-Categories

Let Hilb denote the category whose objects are finite-dimensional Hilbert spaces, and whose morphisms are arbitrary linear maps. (Henceforth, all Hilbert spaces will taken as finite-dimensional unless otherwise specified.) The category Hilb is symmetric monoidal, with $\mathbb{C}$ as the unit object, the usual tensor product of Hilbert spaces as the monoidal structure, and the maps

$$S_{x,y}(v \otimes w) = w \otimes v$$

as the symmetry, where $x, y \in \mathrm{Hilb}$, $v \in x$, and $w \in y$. Using enriched category theory [18] we may thus define the notion of a category enriched over Hilb, or Hilb-category. Concretely, this amounts to the following:

Definition 1. *A* Hilb-category H *is a category such that for any pair of objects* $x, y \in H$ *the set of morphisms* $\hom(x,y)$ *is equipped with the structure of a Hilbert space, and for any objects* $x, y, z \in H$ *the composition map*

$$\circ : \hom(x,y) \times \hom(y,z) \to \hom(x,z)$$

is bilinear.

We may think of the 'hom' in a Hilb-category H as a functor

$$\hom\colon H^{\mathrm{op}} \times H \to \mathrm{Hilb}$$

as follows. An object in $H^{\mathrm{op}} \times H$ is just a pair of objects (x,y) in H, and the hom functor assigns to this the object $\hom(x,y) \in \mathrm{Hilb}$. A morphism $F\colon (x,y) \to (x',y')$ in $H^{\mathrm{op}} \times H$ is just a pair of morphisms $f\colon x' \to x$, $g\colon y \to y'$ in H, and the hom functor assigns to F the morphism $\hom(F)\colon \hom(x,y) \to \hom(x',y')$ given by

$$\hom(F)(h) = fhg.$$

As described in the introduction, we may regard Hilb as the categorification of $\mathbb{C}$. A structure on $\mathbb{C}$ which is crucial for Hilbert space theory is complex conjugation,

$$\overline{}\colon \mathbb{C} \to \mathbb{C}.$$

The categorification of this map is a functor

$$\overline{}\colon \mathrm{Hilb} \to \mathrm{Hilb}$$

called *conjugation*, defined as follows. First, for any Hilbert space x, there is a conjugate Hilbert space $\overline{x}$. This has the same underlying abelian group as x, but to keep things straight let us temporarily write $\overline{v}$ for the element of $\overline{x}$ corresponding to $v \in x$. Scalar multiplication in $\overline{x}$ is then given by

$$c\overline{v} = \overline{(\overline{c}v)}$$

for any $c \in \mathbb{C}$, while the inner product is given by

$$\langle \overline{v}, \overline{w} \rangle = \overline{\langle v, w \rangle}.$$

Second, for any morphism $f\colon x \to y$ in Hilb, there is a conjugate morphism $\overline{f}\colon \overline{x} \to \overline{y}$, given by

$$\overline{f}(\overline{v}) = \overline{f(v)}$$

for all $v \in x$. One can easily check that with these definitions conjugation is a covariant functor. Note that the square of this functor is equal to the identity. Also note that a linear map $f: x \to \overline{y}$ is the same thing as an antilinear (i.e., conjugate-linear) map from x to y, while a unitary map $f: x \to \overline{y}$ is the same thing as an antiunitary map from x to y.

Now, just as in a Hilbert space we have the equation

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

for any pair of elements, in a 2-Hilbert space we would like an isomorphism

$$\hom(x, y) \cong \overline{\hom(y, x)}$$

for every pair of objects. This isomorphism should be be 'natural' in some sense, but $\hom(x, y)$ is contravariant in x and covariant in y, while $\overline{\hom(y, x)}$ is covariant in x and contravariant in y. Luckily Hilb is a $*$-category, which allows us to define 'antinatural isomorphisms' between covariant functors and contravariant functors from any category to Hilb.

This works as follows. In general, a *$*$-structure* for a category C is defined as a contravariant functor $*: C \to C$ which acts as the identity on the objects of C and satisfies $*^2 = 1_C$. A *$*$-category* is a category equipped with a $*$-structure. For example, Hilb is a $*$-category where for any morphism $f: x \to y$ we define $f^*: y \to x$ to be the Hilbert space adjoint of f:

$$\langle fv, w \rangle = \langle v, f^* w \rangle$$

for all $v \in x$, $w \in y$.

Now suppose that D is a $*$-category and $F: C \to D$ is a covariant functor, while $G: C \to D$ is a contravariant functor. We define an *antinatural transformation* $\alpha: F \Rightarrow G$ to be a natural transformation from F to $G \circ *$. Similarly, an antinatural transformation from G to F is defined to be a natural transformation from G to $F \circ *$.

As a step towards defining a 2-Hilbert space we now define an H*-category.

Definition 2. *An* H*-category *is a* Hilb-*category with a $*$-structure that defines an antinatural transformation from* $\hom(x, y)$ *to* $\overline{\hom(y, x)}$.

This may require some clarification. Given a $*$-structure $*: H \to H$, we obtain for any objects $x, y \in H$ a function $*: \hom(x, y) \to \hom(y, x)$. By abuse of notation we may also regard this as a function

$$*: \hom(x, y) \to \overline{\hom(y, x)}.$$

We then demand that this define an antinatural transformation between the covariant functor $\hom: H^{\mathrm{op}} \times H \to \mathrm{Hilb}$ to the contravariant functor sending $(x, y) \in H^{\mathrm{op}} \times H$ to $\overline{\hom(y, x)} \in \mathrm{Hilb}$.

The following proposition gives a more concrete description of H*-categories:

Proposition 3. *An H^*-category H is the same as a* Hilb*-category equipped with antilinear maps* $*\colon \hom(x,y) \to \hom(y,x)$ *for all* $x, y \in H$*, such that*

1. $f^{**} = f$,
2. $(fg)^* = g^* f^*$,
3. $\langle fg, h\rangle = \langle g, f^* h\rangle$,
4. $\langle fg, h\rangle = \langle f, hg^*\rangle$

whenever both sides of the equation are well-defined.

Proof - First suppose that H is an H*-category. By the antinaturality of $*$, for all $x, y \in H$ there is a linear map $*\colon \hom(x,y) \to \overline{\hom(y,x)}$, which is the same as an antilinear map $*\colon \hom(x,y) \to \hom(y,x)$. The fact that $*$ is a $*$-structure implies properties 1 and 2. As for 3 and 4, suppose (x,y) and (x',y') are objects in $H^{\mathrm{op}} \times H$, and let (f,g) be a morphism from (x,y) to (x',y'). The fact that $*$ is an antinatural transformation means that the following diagram commutes:

$$\begin{tikzcd}
\hom(x,y) \arrow[r, "*"] \arrow[d, "V(f{,}g)"'] & \overline{\hom(y,x)} \arrow[d, "W(f{,}g)^*"] \\
\hom(x',y') \arrow[r, "*"] & \overline{\hom(y',x')}
\end{tikzcd}$$

where V is the covariant functor

$$H^{\mathrm{op}} \times H \xrightarrow{\hom} \mathrm{Hilb}$$

and W is the contravariant functor

$$H^{\mathrm{op}} \times H \xrightarrow{S_{H^{\mathrm{op}},H}} H \times H^{\mathrm{op}} \xrightarrow{\hom} \mathrm{Hilb} \xrightarrow{\quad - \quad} \mathrm{Hilb},$$

where in this latter diagram S denotes the symmetry in Cat, hom is regarded as a contravariant functor from $(H^{\mathrm{op}} \times H)^{\mathrm{op}} \cong H \times H^{\mathrm{op}}$ to Hilb, and the overline denotes conjugation. This is true if and only if for all $h \in \hom(x,y)$ and $k \in \overline{\hom(y',x')}$,

$$\langle (V(f,g)h)^*, k\rangle = \langle W(f,g)^* h^*, k\rangle$$

or in other words,

$$\langle (fhg)^*, k\rangle = \langle h^*, gkf\rangle$$

or

$$\langle g^* h^* f^*, k\rangle = \langle h^*, gkf\rangle.$$

Here the inner products are taken in $\overline{\hom(y', x')}$, but the equations also hold with the inner product taken in $\hom(y', x')$. Taking either f or g to be the identity, we obtain 3 and 4 after some relabelling of variables.

Conversely, given antilinear maps $*\colon \hom(x, y) \to \hom(y, x)$ for all $x, y \in H$, properties 1 and 2 say that these define a $*$-structure for H, and using 3 and 4 we obtain

$$\begin{aligned}\langle g^* h^* f^*, k\rangle &= \langle g^* h^*, kf\rangle \\ &= \langle h^*, gkf\rangle,\end{aligned}$$

showing that $*$ is antinatural. $\square$

Corollary 4. *If H is an H^*-category, for all objects $x, y \in H$ the map $*\colon \hom(x, y) \to \hom(y, x)$ is antiunitary.*

Proof - The map $*\colon \hom(x, y) \to \hom(y, x)$ is antilinear, and by 3 and 4 of Proposition 3 we have

$$\langle f, g\rangle = \langle g^*, f^*\rangle = \overline{\langle f^*, g^*\rangle}$$

for all $f, g \in \hom(x, y)$, so $*$ is antiunitary. $\square$

Next we give a structure theorem for H*-categories. This relies heavily on the theory of 'H*-algebras' due to Ambrose [1], so let us first recall this theory. For our convenience, we use a somewhat different definition of H*-algebra than that given by Ambrose. Namely, we restrict our attention to finite-dimensional H*-algebras with multiplicative unit, and we do not require the inequality $\|ab\| \leq \|a\| \, \|b\|$.

Definition 5. *An H^*-algebra A is a Hilbert space that is also an associative algebra with unit, equipped with an antilinear involution $*\colon A \to A$ satisfying*

$$\begin{aligned}\langle ab, c\rangle &= \langle b, a^* c\rangle \\ \langle ab, c\rangle &= \langle a, cb^*\rangle\end{aligned}$$

for all $a, b, c \in A$. An isomorphism *of H^*-algebras is a unitary operator that is also an involution-preserving algebra isomorphism.*

The basic example of an H*-algebra is the space of linear operators on a Hilbert space H. Here the product is the usual product of operators, the involution is the usual adjoint of operators, and the inner product is given by

$$\langle a, b\rangle = k \,\mathrm{tr}(a^* b)$$

where $k > 0$. We denote this H*-algebra by $L^2(H, k)$. It follows from the work of Ambrose that all H*-algebras can be built out of H*-algebras of this form. More precisely, every H*-algebra A is the orthogonal direct sum of finitely many minimal

2-sided ideals I_i, each of which is isomorphic as an H*-algebra to $L^2(H_i, k_i)$ for some Hilbert space H_i and some positive real number k_i.

This result immediately classifies H*-categories with one object. Given an H*-category with one object x, $\mathrm{end}(x)$ is an H*-algebra, and is thus of the above form. Conversely, any H*-algebra is isomorphic to $\mathrm{end}(x)$ for some H*-category with one object x.

We generalize this to arbitrary H*-categories as follows. Suppose first that H is an H*-category with finitely many objects. Let A denote the orthogonal direct sum

$$A = \bigoplus_{x,y} \mathrm{hom}(x, y).$$

Then A becomes an H*-algebra if we define the product in A of morphisms in H to be their composite when the composite exists, and zero otherwise, and define the involution in A using the $*$-structure of H. A is thus the orthogonal direct sum of finitely many minimal 2-sided ideals:

$$A = \bigoplus_{i=1}^{n} L^2(H_i, k_i).$$

For each object $x \in H$, the identity morphism 1_x can be regarded as an element of A. This element is a self-adjoint projection, meaning that

$$1_x^* = 1_x, \qquad 1_x^2 = 1_x.$$

It follows that we may write

$$1_x = \bigoplus_{i=1}^{n} p_i^x$$

where $p_i^x \in L^2(H_i, k_i)$ is the self-adjoint projection onto some subspace $H_i^x \subseteq H_i$. Note that the elements 1_x, $x \in H$, form a complete orthogonal set of projections in A. In other words, $1_x 1_y = 0$ if $x \neq y$, and

$$\sum_{x \in H} 1_x = 1.$$

Thus each Hilbert space H_i is the orthogonal direct sum of the subspaces H_i^x.

This gives the following structure theorem for H*-categories:

Theorem 6. *Let H be an H*-category and S any finite set of objects of H. Then for some n, there exist positive numbers $k_i > 0$ and Hilbert spaces H_i^x for $i = 1, \ldots, n$ and $x \in S$, such that the following hold:*

1. For $i = 1, \ldots, n$, let

$$H_i = \bigoplus_{x \in S} H_i^x$$

denote the orthogonal direct sum, and let p_i^x be the self-adjoint projection from H_i to H_i^x. Then for any objects $x, y \in S$, there is a unitary isomorphism between the Hilbert space $\hom(x, y)$ and the subspace

$$\bigoplus_i p_i^x L^2(H_i, k_i) p_i^y \ \subseteq\ \bigoplus_i L^2(H_i, k_i),$$

Thus we may write any morphism $f\colon x \to y$ as

$$f = \bigoplus_i f_i$$

where $f_i\colon H_i^x \to H_i^y$.

2. Via the above isomorphism, the composition map

$$\circ : \hom(x, y) \times \hom(y, z) \to \hom(x, z)$$

is given by

$$f \circ g = \bigoplus_i f_i g_i.$$

3. Via the same isomorphism, the $*$-structure

$$*\colon \hom(x, y) \to \hom(y, x)$$

is given by

$$f^* = \bigoplus_i f_i^*.$$

Conversely, given a Hilb-*category H with $*$-structure such that the above holds for any finite subset S of its objects, H is an H*-category.*

Proof - If H has finitely many objects and we take S to be the set of all objects of H, properties 1-3 follow from the remarks preceding the theorem. More generally, by Proposition 3 any full subcategory of an H*-category is an H*-category, so 1-3 hold for any finite subset S of the objects of H.

Conversely, given a Hilb-category H with a $*$-structure, if every full subcategory of H with finitely many objects is an H*-category, then H itself is an H*-category. One may check using Proposition 3 that if S is any finite subset of the objects of H, properties 1-3 imply the full subcategory of H with S as its set of objects is an H*-category. Thus H is an H*-category. $\Box$

The notions of unitarity and self-adjointness will be important in all that follows.

Definition 7. *Let x and y be objects of a $*$-category. A morphism $u\colon x \to y$ is* unitary *if $uu^* = 1_x$ and $u^*u = 1_y$. A morphism $a\colon x \to x$ is* self-adjoint *if $a^* = a$.*

Note that every unitary morphism is an isomorphism. Conversely, the following proposition implies that in an H*-category, isomorphic objects are isomorphic by a unitary.

Proposition 8. *Suppose $f: x \to y$ is an isomorphism in the H*-category H. Then $f = au$ where $a: x \to x$ is self-adjoint and $u: x \to y$ is unitary.*

Proof - Suppose that $f: x \to y$ is an isomorphism. Then applying Theorem 6 to the full subcategory of H with x and y as its only objects, we have $f = \bigoplus f_i$ with $f_i: H_i^x \to H_i^y$ an isomorphism for all i. Using the polar decomposition theorem we may write $f_i = a_i u_i$, where $a_i: H_i^x \to H_i^x$ is the positive square root of $f_i {f_i}^*$, and $u_i: H_i^x \to H_i^y$ is a unitary operator given by $u_i = a_i^{-1} f_i$. Then defining $a = \bigoplus a_i$ and $u = \bigoplus u_i$, we have $f = au$ where a is self-adjoint and u is unitary. $\square$

One can prove a more general polar decomposition theorem allowing one to write any morphism $f: x \to y$ in an H*-category as the product of a self-adjoint morphism $a: x \to x$ and a partial isometry $i: x \to y$, that is, a morphism for which ii^* and i^*i are self-adjoint idempotents. However, we will not need this result here.

3 2-Hilbert Spaces

The notion of 2-Hilbert space is intended to be the categorification of the notion of Hilbert space. As such, it should be a category having a zero object, direct sums and 'direct differences' of objects, tensor products of Hilbert spaces with objects, and 'inner products' of objects. So far, with our definition of H*-category, we have formalized the notion of a category in which the 'inner product' $\hom(x, y)$ of any two objects x and y is a Hilbert space. Now we deal with the rest of the properties:

Definition 9. *A* 2-Hilbert space *is an abelian H*-category.*

Recall that an abelian category is an Ab-category (a category enriched over the category Ab of abelian groups) such that

1. There exists an initial and terminal object.
2. Any pair of objects has a biproduct.
3. Every morphism has a kernel and cokernel.
4. Every monomorphism is a kernel, and every epimorphism is a cokernel.

Let us comment a bit on what this amounts to. Since an H*-category is enriched over Hilb it is automatically enriched over Ab. We call an initial and terminal object a *zero object*, and denote it by 0. The zero object in a 2-Hilbert space is the analog of the zero vector in a Hilbert space. We call the biproduct of x and y the *direct sum*, and denote it by $x \oplus y$. Recall that by definition, this is equipped with morphisms $p_x: x \oplus y \to x$, $p_x: x \oplus y \to y$, $i_x: x \to x \oplus y$, $i_y: y \to x \oplus y$ such that

$$i_x p_x = 1_x, \qquad i_y p_y = 1_y, \qquad p_x i_x + p_y i_y = 1_{x \oplus y}.$$

The direct sums in a 2-Hilbert space are the analog of addition in a Hilbert space. Similarly, the cokernels in a 2-Hilbert space are the analogs of differences in a Hilbert space. Finally, the ability to tensor objects in a 2-Hilbert space by Hilbert spaces (the analog of scalar multiplication) will follow from the other properties, so we do not need to include it in the definition of 2-Hilbert space.

Some aspects of our definition of 2-Hilbert space may seem unmotivated by the analogy with Hilbert spaces. Why should a 2-Hilbert space have kernels, and why should it satisfy clause 4 in the definition of abelian category? In fact, these properties follow from the rest.

Proposition 10. *Let H be an H^*-category. Then the following are equivalent:*

1. *There exists an initial object.*
2. *There exists a terminal object.*
3. *There exists a zero object.*

Moreover, the following are equivalent:

1. *Every pair of objects has a product.*
2. *Every pair of objects has a coproduct.*
3. *Every pair of objects has a direct sum.*

Moreover, the following are equivalent:

1. *Every morphism has a kernel.*
2. *Every morphism has a cokernel.*

Finally, if H has a zero object, every pair of objects in H has a direct sum, and every morphism in H has a cokernel, then H is a semisimple abelian category.

Proof - It is well-known [23] that an initial or terminal object in an Ab-category is automatically a zero object. Alternatively, this is true in every $*$-category, using the bijection $*\colon \hom(x, y) \to \hom(y, x)$. It is also well-known that in an Ab-category, a binary product or coproduct is automatically a binary biproduct. Furthermore, it is easy to check that in any $*$-category, the morphism $j\colon k \to x$ is a kernel of $f\colon x \to y$ if and only if $j^*\colon x \to k$ is a cokernel of $f^*\colon y \to x$. Thus a $*$-category has kernels if and only if it has cokernels.

Now suppose that H is an H*-category with a zero object, direct sums, and cokernels. Then H has kernels as well, so to show H is abelian we merely need to prove that every monomorphism is a kernel and every epimorphism is a cokernel. Let us show a monomorphism $f\colon x \to y$ is a kernel; it follows using the $*$-structure that every epimorphism is a cokernel.

It suffices to show this result for any full subcategory of H with finitely many objects, so by Theorem 6 we may write

$$f = \bigoplus_i f_i$$

where $f_i\colon H_i^x \to H_i^y$ is a linear operator. Let $p\colon y \to y$ be given by $\bigoplus p_i$ where p_i is the projection onto the orthogonal complement of the range of f_i. We claim that $f\colon x \to y$ is a kernel of p. Since $f_i p_i = 0$ for all i we have $fp = 0$. We also need to show that if $f'\colon x' \to y$ is any morphism with $f'p = 0$, then there is a unique $g\colon x' \to x$ with $f' = gf$. Writing $f' = \bigoplus f_i'$, the fact that $f'p = 0$ implies that the range of f_i' is contained in the range of f_i. Thus by linear algebra there exists $g_i\colon H_i^{x'} \to H_i^x$ such that $f_i' = g_i f_i$. Letting $g = \bigoplus g_i$, we have $f' = gf$, and g is unique with this property because f is monic.

Finally, note that H is semisimple, i.e., every short exact sequence splits. This follows from Theorem 6 and elementary linear algebra. □

Given a 2-Hilbert space H, the fact that H is semisimple implies that every object is isomorphic to a direct sum of *simple* objects, that is, objects x for which $\mathrm{end}(x)$ is isomorphic as an algebra to $\mathbb{C}$. This fact lets us reason about 2-Hilbert spaces using bases:

Definition 11. *Given a 2-Hilbert space H, a set of nonisomorphic simple objects of H is called a* basis *if every object of H is isomorphic to a finite direct sum of objects in that set.*

Corollary 12. *Every 2-Hilbert space H has a basis, and any two bases of H have the same cardinality.*

Proof - The 2-Hilbert space H has a basis because it is semisimple: given two bases $\{e_\alpha\}$ and $\{f_\beta\}$, each object e_α is isomorphic to a direct sum of copies of the objects f_β, but as the e_α and f_β are simple we must actually have an isomorphism $e_\alpha \cong f_\beta$ for some β. This β is unique since no distinct f_β's are isomorphic. This sets up a function from $\{e_\alpha\}$ to $\{f_\beta\}$, and similar reasoning gives us the inverse function. □

Definition 13. *The* dimension *of a 2-Hilbert space is the cardinality of any basis of it.*

Note that every basis $\{e_\alpha\}$ of a 2-Hilbert space is 'orthogonal' in the sense that

$$\hom(e_\alpha, e_\beta) \cong \begin{cases} L^2(\mathbb{C}, k_\alpha) & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

where the isomorphism is one of H*-algebras, and k_α are certain positive constants. Moreover, up to reordering, the constants k_α are independent of the choice of basis. For suppose x, y are two isomorphic objects in an H*-category. By Proposition 8 there is a unitary isomorphism $f: x \to y$. Then there is an H*-algebra isomorphism $\alpha: \mathrm{end}(x) \to \mathrm{end}(y)$ given by $\alpha(g) = f^{-1}gf$.

One would also like to be able to tensor objects in a 2-Hilbert space with Hilbert spaces, but this is a consequence of the definition we have given, since one may define the tensor product of an object x in a 2-Hilbert space with an n-dimensional Hilbert space to be the direct sum of n copies of x. In fact, Hilb has a structure analogous to that of an algebra, with tensor product and direct sum playing the roles of multiplication and addition. In the terminology we introduce in Section 5, one says that Hilb is a '2-H*-algebra'. One can develop a theory of modules of 2-H*-algebras following the ideas of Kapranov and Voevodsky [17] and Yetter [30]. Every 2-Hilbert space H is then a module over Hilb. We will not pursue this further here.

4 2Hilb as a 2-Category

We now investigate a certain 2-category 2Hilb of 2-Hilbert spaces. To keep things simple we take as its objects only finite-dimensional 2-Hilbert spaces. Nonetheless we prove theorems more generally whenever possible.

Definition 14. *A morphism $F: H \to H'$ between 2-Hilbert spaces H and H' is an exact functor such that $F: \hom(x, y) \to \hom(F(x), F(y))$ is linear and $F(f^*) = F(f)^*$ for all $f \in \hom(x, y)$.*

Recall that an exact functor is one preserving short exact sequences. Exactness is an natural sort of condition for functors between abelian categories. Similarly, the requirement that $F: \hom(x, y) \to \hom(F(x), F(y))$ be linear is a natural condition for functors between Hilb-categories; one calls such a functor a Hilb-*functor*. Finally, $F(f^*) = F(f)^*$ is a natural condition for functors between $*$-categories, and functors satisfying it are called *$*$-functors*.

The following fact is occasionally handy:

Proposition 15. *Let $F: H \to H'$ be a functor between 2-Hilbert spaces such that for all $x, y \in H$, $F: \hom(x, y) \to \hom(F(x), F(y))$ is linear. Then the following are equivalent:*

1. *F is exact.*

2. *F is left exact.*

3. *F is right exact.*

4. *F preserves direct sums.*

Proof - Following Yetter [30], we use the fact that every short exact sequence splits. $\square$

Definition 16. *A* 2-morphism *$\alpha: F \Rightarrow F'$ between morphisms $F, F': H \to H'$ between 2-Hilbert spaces H and H' is a natural transformation.*

Definition 17. *We define the 2-category* 2Hilb *to be that for which objects are finite-dimensional 2-Hilbert spaces, while morphisms and 2-morphisms are defined as above.*

Now, just as in some sense $\mathbb{C}$ is the primordial Hilbert space and Hilb is the primordial 2-Hilbert space, 2Hilb should be the primordial 3-Hilbert space. The study of 2Hilb should thus shed light on the properties of the still poorly understood 3-Hilbert spaces. However, note that $\mathbb{C}$ is not merely a Hilbert space, but also a commutative monoid, in fact a commutative H*-algebra. Similarly, Hilb is not merely a 2-Hilbert space, but also a symmetric monoidal category when equipped with its usual tensor product. Indeed, in Section 5 we show that Hilb is a 'symmetric 2-H*-algebra'. Likewise, we expect 2Hilb to be not only a 3-Hilbert space, but also a strongly involutory monoidal 2-category, in fact a 'strongly involutory 3-H*-algebra'. As sketched in HDA0, commutative monoids, symmetric monoidal categories, and strongly involutory monoidal 2-categories are all examples of 'stable' n-categories. In general we expect nHilb to be a 'stable $(n+1)$-H*-algebra.' The results below offer some support for this expectation.

We begin with a study of duality in 2Hilb, as this is the most distinctive aspect of Hilbert space theory. Note that every element $x \in \mathbb{C}$ has a kind of 'dual' element, namely, its complex conjugate $\overline{x}$. Similarly, the category Hilb has duality both for objects and for morphisms. At the level of morphisms, each linear map $f: x \to y$ between Hilbert spaces has a dual $f^*: y \to x$, the usual Hilbert space adjoint of f. This defines a $*$-structure on H. Duality at the level of objects can be regarded either as a contravariant functor assigning to each each Hilbert space x its dual x^*, or as a covariant functor assigning to each Hilbert space x its conjugate $\overline{x}$. These two viewpoints become equivalent if we take advantage of duality at the morphism level, since x^* and $\overline{x}$ are antinaturally isomorphic.

Similarly, 2Hilb has duality for objects, morphisms, and 2-morphisms. As in Hilb, we can use duality at a given level to reinterpret dualities at lower levels in various ways. This recursive process can become rather confusing unless we choose by convention to take certain dualities as 'basic' and others as derived. Here we follow the philosophy of HDA0: any 2-morphism $\alpha: F \Rightarrow G$ has a dual $\alpha^*: G \Rightarrow F$, any morphism $F: H \to H'$ has a dual $F^*: H' \to H$, and every object H has a dual H^*. (Our notation differs from HDA0 in that we use the same symbol to denote all these different levels of duality.)

4.1 Duality for 2-morphisms

Duals of 2-morphisms are the easiest to define. It pays to do so in the greatest possible generality:

Definition 18. *Given $*$-categories C, D and $*$-functors $F, G\colon C \to D$, the* dual α^* *of a natural transformation $\alpha\colon F \Rightarrow G$ is the natural transformation with $(\alpha^*)_c = (\alpha_c)^*$ for all $c \in C$.*

It is easy to check that α^* is a natural transformation when α is, and that

$$(\alpha^*)^* = \alpha, \qquad 1^* = 1.$$

The vertical composite of natural transformations satisfies

$$(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*$$

when this is defined. When D is a $*$-category, the horizontal composite of a functor $F\colon B \to C$ and a natural transformation $\alpha\colon G \Rightarrow H$ with $G, H\colon C \to D$ satisfies

$$(F\alpha)^* = F\alpha^*.$$

Similarly, when $F\colon C \to D$ is a $*$-functor and $\alpha\colon G \Rightarrow H$ is a natural transformation between $G, H\colon B \to C$, we have

$$(\alpha F)^* = \alpha^* F.$$

In particular, taking C, D to be 2-Hilbert spaces, we obtain the definition of the dual of a 2-morphism in 2Hilb. We also obtain the notion of 'unitary' and 'self-adjoint' natural transformations:

Definition 19. *Given $*$-categories C, D and $*$-functors $F, G\colon C \to D$, a natural transformation $\alpha\colon F \Rightarrow G$ is* unitary *if*

$$\alpha\alpha^* = 1_F, \qquad \alpha^*\alpha = 1_G.$$

A natural transformation $\alpha\colon F \Rightarrow F$ is self-adjoint *if*

$$\alpha^* = \alpha.$$

Equivalently, α is unitary if α_c is a unitary morphism in D for all objects $c \in C$, and self-adjoint if α_c is self-adjoint for all $c \in C$.

Note that every unitary natural transformation is a natural isomorphism. Conversely:

Proposition 20. *Suppose $F, G\colon H \to H'$ are morphisms between 2-Hilbert spaces and $\alpha\colon F \Rightarrow G$ is a natural isomorphism. Then $\alpha = \beta \cdot \gamma$ where $\beta\colon F \Rightarrow F$ is self-adjoint and $\gamma\colon F \Rightarrow G$ is unitary.*

Proof - By Proposition 8, for any $x \in H$ we can write the isomorphism $\alpha_x \colon F(x) \to G(x)$ as the composite $\beta_x \gamma_x$, where $\beta_x \colon F(x) \to F(x)$ is self-adjoint and $\gamma_x \colon F(x) \to G(x)$ is unitary. More importantly, the polar decomposition gives a natural way to construct β_x and γ_x from α_x: we take β_x to be the positive square root of $\alpha_x {\alpha_x}^*$, and take $\gamma_x = \beta_x^{-1} \alpha_x$.

Since $\alpha\alpha^*$ is a natural transformation from F to itself, if we define $P(\alpha\alpha^*)_x = P(\alpha_x {\alpha_x}^*)$ for any polynomial P, we have

$$P(\alpha_x {\alpha_x}^*) F(f) = F(f) P(\alpha_y {\alpha_y}^*)$$

for any morphism $f \colon x \to y$. By the finite-dimensional spectral theorem, we can find a sequence of polynomials P_i such that $P_i(\alpha_x {\alpha_x}^*) \to \beta_x$ and $P_i(\alpha_y {\alpha_y}^*) \to \beta_y$. Thus

$$\beta_x F(f) = F(f) \beta_y,$$

so β is a natural transformation from F to itself. It follows that $\gamma = \beta^{-1} \cdot \alpha$ is a natural transformation from F to G. Clearly β is self-adjoint and γ is unitary. $\square$

4.2 Duality for morphisms

Duals of morphisms in 2Hilb are just adjoint functors. Normally one needs to distinguish between left and right adjoint functors, but duality at the 2-morphism level allows us to turn left adjoints into right adjoints, and vice versa:

Proposition 21. *Suppose $F \colon H \to H'$, $G \colon H' \to H$ are morphisms in* 2Hilb. *Then F is left adjoint to G with unit $\iota \colon 1_H \Rightarrow FG$ and counit $\epsilon \colon GF \Rightarrow 1_{H'}$ if and only if F is right adjoint to G with unit $\epsilon^* \colon 1_{H'} \Rightarrow GF$ and counit $\iota^* \colon FG \Rightarrow 1_H$.*

Proof - The triangle equations for ι and ϵ:

$$(\iota F) \cdot (F\epsilon) = 1_F, \qquad (G\iota) \cdot (\epsilon G) = 1_G,$$

become equivalent to those for ϵ^* and ι^*:

$$(\epsilon^* G) \cdot (G\iota^*) = 1_G, \qquad (F\epsilon^*) \cdot (\iota^* F) = 1_F,$$

by taking duals. $\square$

As noted by Dolan [7], it is probably quite generally true in n-categories that duality for j-morphisms allows us to turn 'left duals' of $(j-1)$-morphisms into 'right duals' and vice versa. This should give the theory of n-Hilbert spaces quite a different flavor from general n-category theory.

Every morphism in 2Hilb has an adjoint. We prove this using bases and the concept of a skeletal 2-Hilbert space.

Definition 22. *A category is* skeletal *if all isomorphic objects are equal.*

Definition 23. *A* unitary equivalence *between 2-Hilbert spaces H and H' consists of morphisms $U\colon H \to H'$, $V\colon H' \to H$ and unitary natural transformations $\iota\colon 1_H \Rightarrow UV$, $\epsilon\colon VU \Rightarrow 1_{H'}$ forming an adjunction. If there exists a unitary equivalence between H and H', we say they are* unitarily equivalent.

Proposition 24. *Any 2-Hilbert space is unitarily equivalent to a skeletal 2-Hilbert space.*

Proof - Let $\{e_\lambda\}$ be a basis for the 2-Hilbert space H. For any nonnegative integers $\{n^\lambda\}$ with only finitely many nonzero, make a choice of direct sum

$$\bigoplus_\lambda n^\lambda e_\lambda,$$

where $n^\lambda e_\lambda$ denotes the direct sum of n^λ copies of e_λ. (Recall that the direct sum is an object equipped with particular morphisms; it is only unique up to isomorphism, but here we fix a particular choice.) Let H_0 denote the full subcategory of H with only these direct sums as objects. Note that H_0 inherits a 2-Hilbert space structure from H, and it is skeletal. Let $V\colon H_0 \to H$ denote the inclusion functor.

For any $x \in H$ there is a unique object $U(x) \in H_0$ for which $V(U(x))$ is isomorphic to x. By Proposition 8, we may choose a unitary isomorphism

$$\iota_x\colon x \to V(U(x)).$$

For $x = V(y)$ we have $U(x) = y$, so we choose ι_x to be the identity in this case. For each morphism $f\colon x \to y$ define $U(f)\colon U(x) \to U(x')$ so that the following diagram commutes:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\iota_x"'] & y \arrow[d, "\iota_y"] \\ V(U(x)) \arrow[r, "V(U(f))"'] & V(U(y)) \end{tikzcd}$$

It follows that $U\colon H \to H_0$ is a functor.

One may check that U and V are actually morphisms of 2-Hilbert spaces. Moreover, one may check that there is a natural isomorphism

$$\hom(Ux, y) \cong \hom(x, Vy)$$

given by

$$f \mapsto \iota_x V(f).$$

It follows that U is left adjoint to V. The unit of this adjunction is ι, while the counit is the identity. These are both unitary natural transformations. □

Just as with Hilbert spaces, phrasing definitions and theorems about 2-Hilbert spaces in terms of a basis is usually a mistake, since they should be manifestly invariant under unitary equivalence. In comparison, the use of bases to prove theorems is at worst a minor lapse of taste, and sometimes convenient. This is facilitated by the use of skeletal 2-Hilbert spaces.

Proposition 25. *Let $F\colon H \to H'$ be a morphism in* 2Hilb. *Then there is a morphism $F^*\colon H' \to H$ that is left and right adjoint to F.*

Proof - Here we opt for a lowbrow proof using bases, to illustrate the analogy between an adjoint functor and the adjoint of a matrix. By Proposition 24 it suffices to consider the case where H and H' are skeletal. Let $\{e_\lambda\}$ be a basis for H and $\{e'_\mu\}$ a basis for H'. Write

$$F(e_\lambda) = \bigoplus_\mu F_{\lambda\mu} e'_\mu$$

where $F_{\lambda\mu}$ are nonnegative integers and $F_{\lambda\mu}e'_\mu$ denotes the direct sum of $F_{\lambda\mu}$ copies of e'_μ. Let

$${F_{\mu\lambda}}^* = F_{\lambda\mu}.$$

Defining

$$F^*(e'_\mu) = \bigoplus_\mu F^*_{\mu\lambda} e_\lambda,$$

one may check that F^* extends uniquely to a morphism from H' to H. Note that both $\hom(Fe_\lambda, e'_\mu)$ and $\hom(e_\lambda, F^*e'_\mu)$ may be naturally identified with a direct sum of $F_{\lambda\mu}$ copies of $\mathbb{C}$, which sets up an isomorphism $\hom(Fe_\lambda, e'_\mu) \cong \hom(e_\lambda, F^*e'_\mu)$. One can check that this extends uniquely to a natural isomorphism

$$\hom(Fx, y) \cong \hom(x, F^*y),$$

so F^* is a right adjoint, and by Proposition 21 also a left adjoint, of F. □

A basic fact in Hilbert space theory is that two objects in Hilb are isomorphic if and only if there is a unitary morphism between them. The same is true of objects in any other 2-Hilbert space, by Proposition 8. Similarly, two morphisms in 2Hilb are isomorphic if and only if there is a unitary natural transformation between them, by Proposition 20. Below we show a similar result for objects in 2Hilb. In general, we expect a recursively defined notion of 'equivalence' of j-morphisms in an n-category: two n-morphisms are equivalent if they are equal, while two $(j-1)$-morphisms x, y are equivalent if there exist $f\colon x \to y$ and $g\colon y \to x$ with gf and fg equivalent to the identity on x and y, respectively. In an n-Hilbert space we also expect a similar notion of 'unitary equivalence': two $(n-1)$-morphisms are unitarily equivalent if they are equal, while two $(j-1)$-morphisms x, y are unitarily equivalent if there exists $u\colon x \to y$ with uu^* and u^*u unitarily equivalent to 1_x and 1_y, respectively. Our results so far lead us to suspect that, quite generally, equivalent j-morphisms in an n-Hilbert space will be unitarily equivalent.

Definition 26. *An* equivalence *between 2-Hilbert spaces H and H' is an pair of morphisms $F\colon H \to H'$, $G\colon H' \to H$ together with natural isomorphisms $\alpha\colon 1_H \Rightarrow FG$, $\beta\colon GF \Rightarrow 1_{H'}$. If there is an equivalence between H and H', we say they are* equivalent.

Note that a unitary equivalence is automatically an equivalence. Conversely:

Proposition 27. *Suppose H and H' are 2-Hilbert spaces and the morphisms $F\colon H \to H'$, $G\colon H' \to H$ can be extended to an equivalence between H and H'. Then F and G can be extended to a unitary equivalence between H and H'.*

Proof - Suppose $\alpha\colon 1_H \Rightarrow FG$, $\beta\colon GF \Rightarrow 1_{H'}$ are natural isomorphisms. By Proposition 20 we can find unitary natural transformations $\gamma\colon 1_H \Rightarrow FG$, $\delta\colon GF \Rightarrow 1_{H'}$. We may then obtain an adjunction by replacing γ with the composite γ' given by

$$1_H \xrightarrow{\quad\gamma\quad} FG = F1_{H'}G \xrightarrow{F\delta^{-1}G} FGFG \xrightarrow{\quad\gamma^{-1}FG\quad} FG$$

Checking that this is an adjunction is a lengthy but straightforward calculation. Noting that γ' is unitary, we conclude that (F, G, ι, ϵ) is a unitary equivalence. ☐

When we are being less pedantic, we call a 2-Hilbert space morphism $F\colon H \to H'$ an *equivalence* if it can be extended to an equivalence in the sense of Definition 26.

Just as Hilbert spaces are classified by their dimension, we have:

Corollary 28. *Two 2-Hilbert spaces are equivalent if and only if they have the same dimension.*

Proof - Since an equivalence between H and H' carries a basis of H to a basis of H', Proposition 12 implies that dimension is preserved by equivalence. By Proposition 24 it thus suffices to show two skeletal 2-Hilbert spaces are equivalent if they have the same dimension. Let $\{e_\lambda\}$ be a basis of H and $\{e'_\lambda\}$ a corresponding basis of H'. Then there is a unique 2-Hilbert space morphism with $F(e_\lambda) = e'_\lambda$, and the adjunction constructed as in the proof of Proposition 25 is a unitary equivalence. ☐

4.3 Duality for objects

Finally, duals of objects in 2Hilb are defined using an 'internal hom'. Given 2-Hilbert spaces H and H', let $\hom(H, H')$ be the category having 2-Hilbert space morphisms $F\colon H \to H'$ as objects and 2-morphisms between these as morphisms.

Proposition 29. *Suppose H is a finite-dimensional 2-Hilbert space and H' is a 2-Hilbert space. Then the category* $\hom(H, H')$ *becomes a* Hilb*-category if for any $F, G \in$* $\hom(H, H')$ *we make* $\hom(F, G)$ *into a Hilbert space with the obvious linear structure and the inner product given by*

$$\langle \alpha, \beta \rangle = \sum_\lambda \langle \alpha_{e_\lambda}, \beta_{e_\lambda} \rangle$$

for any basis $\{e_\lambda\}$ of H. Moreover, $\hom(H,H')$ *becomes a 2-Hilbert space if we define the dual of $\alpha\colon F \Rightarrow G$ by $(\alpha^*)_x = (\alpha_x)^*$.*

Proof - Note first that $\hom(F,G)$ becomes a vector space if we define

$$(\alpha+\beta)_x = \alpha_x + \beta_x, \qquad (c\alpha)_x = c(\alpha_x)$$

for any $\alpha, \beta\colon F \Rightarrow G$ and $c \in \mathbb{C}$. Note also that the inner product described above is nondegenerate, since if $\alpha_{e_\lambda} = 0$ for all objects e_λ in a basis, then $\alpha = 0$. Finally, note that the inner product is independent of the choice of basis: if $\{e'_\lambda\}$ is another basis we may assume after reordering that $e_\lambda \cong e'_\lambda$, and by Proposition 8 we may choose unitary isomorphisms $u_\lambda\colon e_\lambda \to e'_\lambda$, so that

$$\alpha_{e'_\lambda} = F(u_\lambda)^* \alpha_{e_\lambda} G(u_\lambda)$$

and similarly for β. It follows that

$$\begin{aligned} \langle \alpha_{e'_\lambda}, \beta_{e'_\lambda} \rangle &= \langle F(u_\lambda)^* \alpha_{e_\lambda} G(u_\lambda), F(u_\lambda)^* \beta_{e_\lambda} G(u_\lambda) \rangle \\ &= \langle \alpha_{e_\lambda}, \beta_{e_\lambda} \rangle. \end{aligned}$$

Since composition of morphisms in $\hom(H,H')$ is bilinear, it becomes a Hilb-category.

It is easy to check that defining $(\alpha^*)_x = (\alpha_x)^*$ makes $\hom(H,H')$ into a $*$-category, and using Proposition 3 one can also check it is an H*-category. To check that it is a 2-Hilbert space it suffices by Proposition 10 to check that it has a zero object, direct sums and kernels. Any functor $0\colon H \to H'$ mapping all objects in H to zero objects in H' is initial in $\hom(H,H')$. Given $F, F' \in \hom(H,H')$, we may take as the direct sum $F \oplus F'$ any functor with $(F \oplus F')(x) = F(x) \oplus F(x')$ for any object $x \in H$ and $(F \oplus F')(f) = F(f) \oplus F(f')$ for any morphism f. Similarly, given $\alpha\colon F \to F'$, we may construct $\ker\alpha \in \hom(H,H')$ by letting $(\ker\alpha)(x) = \ker\alpha_x$ for any object x and defining $(\ker\alpha)(f)$ for any morphism using the universal property of the kernel. $\square$

Definition 30. *Given a finite-dimensional 2-Hilbert space H, the* dual H^* *is the 2-Hilbert space* $\hom(H, \mathrm{Hilb})$.

The following is an analog of the Riesz representation theorem for finite-dimensional 2-Hilbert spaces. In its finite-dimensional form, the Riesz representation theorem says if x is a Hilbert space, any morphism $f\colon x \to \mathbb{C}$ is equal to one of the form

$$\langle v, \cdot \rangle$$

for some $v \in H$. This determines an isomorphism $\overline{x} \cong x^*$. Similarly, given a 2-Hilbert space H, we say a morphism $F\colon H \to \mathrm{Hilb}$ is *representable* if it is naturally isomorphic to one of the form

$$\hom(x, \cdot)$$

for some $x \in H$. The essence of the Riesz representation theorem for 2-Hilbert spaces is that every morphism $F\colon H \to \mathrm{Hilb}$ is representable. This yields an equivalence between H^{op} and H^*.

Proposition 31. *For any finite-dimensional 2-Hilbert space H, the morphism $U\colon H^{\mathrm{op}} \to H^*$ given by*

$$U(x) = \hom(x, \cdot), \qquad U(f) = \hom(f, \cdot)$$

is an equivalence between H^{op} and H^*.

Proof - It suffices to show that U is fully faithful and essentially surjective. We can check both of these using a basis $\{e_\lambda\}$ of H. We leave the full faithfulness to the reader. Checking that U is essentially surjective amounts to checking that any $F \in H^*$ is representable. Note there is a 'dual basis' of 2-Hilbert space morphisms $f^\lambda \in \hom(H, \mathrm{Hilb})$ with

$$f^\mu(e_\lambda) \cong \begin{cases} \mathbb{C} & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$$

Since any morphism $F\colon H \to \mathrm{Hilb}$ is determined up to natural isomorphism by its value on the basis $\{e_\lambda\}$, any $F \in H^*$ is isomorphic to a direct sum of the $\{f^\lambda\}$. But f^λ is isomorphic to $U(e_\lambda)$, so U is essentially surjective. $\Box$

4.4 The tensor product

Next we develop the tensor product of 2-Hilbert spaces. For this we need the analog of a bilinear map:

Definition 32. *Given 2-Hilbert spaces H, H', K, a functor $F\colon H \times H' \to K$ is a* bimorphism *of 2-Hilbert spaces if for any objects $x \in H$, $x' \in H'$ the functors $F(x \otimes \cdot)\colon H' \to K$ and $F(\cdot \otimes x')\colon H \to K$ are 2-Hilbert space morphisms. We write* $\mathrm{bihom}(H \times H', K)$ *for the category having bimorphisms $F\colon H \times H' \to K$ as objects and natural transformations between these as morphisms.*

Proposition 33. *Suppose H and H' are finite-dimensional 2-Hilbert spaces and K is a 2-Hilbert space. Then* $\mathrm{bihom}(H \times H', K)$ *becomes a* Hilb*-category if for any $F, G \in \mathrm{bihom}(H \times H', K)$ we make $\hom(F, G)$ into a Hilbert space with the obvious linear structure and the inner product given by*

$$\langle \alpha, \beta \rangle = \sum_{\lambda,\mu} \langle \alpha_{(e_\lambda, f_\mu)}, \beta_{(e_\lambda, f_\mu)} \rangle$$

for any bases $\{e_\lambda\}$ of H and $\{f_\mu\}$ of H'. Moreover, $\mathrm{bihom}(H \times H', K)$ *becomes a 2-Hilbert space if we define the dual of $\alpha\colon F \Rightarrow G$ by $(\alpha^*)_x = (\alpha_x)^*$.*

Proof - The proof is analogous to that of Proposition 29. $\Box$

Given 2-Hilbert spaces H, H' and L, note that a bimorphism $T\colon H \times H' \to L$ induces a morphism

$$T^*\colon \hom(L, K) \to \mathrm{bihom}(H \times H', K).$$

Definition 34. *Given 2-Hilbert space H, H', a* tensor product *of H and H' is a bimorphism $T: H \times H' \to L$ together with a choice for each 2-Hilbert space K of an equivalence of 2-Hilbert spaces extending $T^*: \hom(L, K) \to \mathrm{bihom}(H \times H', K)$.*

In the above situation, by abuse of language we may say simply that $T: H \times H' \to L$ is a tensor product of H and H'.

Proposition 35. *Given finite-dimensional 2-Hilbert spaces H and H', there exists a tensor product $T: H \times H' \to L$. Given another tensor product $T': H \times H' \to L'$, there is an equivalence $F: L \to L'$ for which the following diagram commutes up to a specified natural isomorphism:*

$$\begin{tikzcd} & H \times H' \arrow[dl, "T"'] \arrow[dr, "T'"] & \\ L \arrow[rr, "F"'] & & L' \end{tikzcd}$$

Proof - Let $\{e_\lambda\}$ be a basis for H, and $\{f_\mu\}$ a basis for H'. Let L be the skeletal 2-Hilbert space with a basis of objects denoted by $\{e_\lambda \otimes f_\mu\}$, and with

$$\hom(e_\lambda \otimes f_\mu, e_\lambda \otimes f_\mu) = \hom(e_\lambda, e_\lambda) \otimes \hom(f_\mu, f_\mu)$$

as H*-algebras (using the obvious tensor product of H*-algebras). There is a unique bimorphism $T: H \times H' \to L$ with $T(e_\lambda, f_\mu) = e_\lambda \otimes f_\mu$. Given a 2-Hilbert space K one may check that $T^*: \hom(L, K) \to \mathrm{bihom}(H \times H', K)$ extends to an equivalence. Choosing such an equivalence for every K we obtain a tensor product of H and H'.

Given two tensor products as in the statement of the proposition, let $F: L \to L'$ be the image of T' under the chosen equivalence $\mathrm{bihom}(H \times H', L') \simeq \hom(L, L')$. One can check that L is an equivalence and that the above diagram commutes up to a specified natural isomorphism, much as in the usual proof that the tensor product of vector spaces is unique up to a specified isomorphism. $\Box$

Given a tensor product of the 2-Hilbert spaces H and H', we often write its underlying 2-Hilbert space as $H \otimes H'$. This notation may tempt one to speak of 'the' tensor product of H and H', which is is legitimate if one uses the generalized 'the' as advocated by Dolan [7]. In a set, when we speak of 'the' element with a given property, we implicitly mean that this element is unique. In a category, when we speak of 'the' object with a given property, we merely mean that this object is unique up to isomorphism — typically a specified isomorphism. Similarly, in a 2-category, when we speak of 'the' object with a given property, we mean that this object is unique up to equivalence — typically an equivalence that is specified up to a specified isomorphism. This is the sense in which we may refer to 'the' tensor product of H and H'. The generalized 'the' may be extended in an obvious recursive fashion to n-categories.

Suppose that H and H' are finite-dimensional 2-Hilbert spaces. Then for any pair of objects $x \in H$, $x' \in H'$, we can use the bimorphism $T\colon H \times H' \to H \otimes H'$ to define an object $x \otimes x' = T(x, x')$ in $H \otimes H'$. Similarly, given a morphism $f\colon x \to y$ in H and a morphism $f'\colon x' \to y'$, we obtain a morphism

$$f \otimes f' : x \otimes x' \to y \otimes y'$$

in $H \otimes H'$. We usually write

$$f \otimes x' : x \otimes x' \to y \otimes x'$$

for the morphism $f \otimes 1_{x'}$, and

$$x \otimes f' : x \otimes x' \to x \otimes y'$$

for the morphism $1_x \otimes f'$.

We expect that 2Hilb has the structure of a monoidal 2-category with the above tensor product as part of the monoidal structure. Kapranov and Voevodsky [17] have defined the notion of a weak monoidal structure on a strict 2-category, which should be sufficient for the purpose at hand. On the other hand the work of Gordon, Power and Street [14] gives a fully general notion of weak monoidal 2-category, namely a 1-object tricategory. This should also be suitable for studying the tensor product on 2Hilb, though it might be considered overkill. Both these sorts of monoidal 2-category involve various extra structures besides the tensor product of objects in 2Hilb. Most of these should arise from the universal property of the tensor product.

For example, suppose we are given a morphism $F\colon H \to H'$ and an object K in 2Hilb. Thus we have bimorphisms $T\colon H \times K \to H \otimes K$ and $T'\colon H \times K' \to H \otimes K'$, and T^* has some morphism

$$S\colon \mathrm{bihom}(H \times K, H' \otimes K) \to \mathrm{hom}(H \otimes K, H' \otimes K)$$

as inverse up to natural isomorphism. Applying S to the bimorphism given by the composite

$$H \times K \xrightarrow{F \times 1_K} H' \times K \xrightarrow{T'} H' \otimes K$$

we obtain a morphism we denote by

$$F \otimes K\colon H \otimes K \to H' \otimes K.$$

Similarly, given an object $H \in$ 2Hilb and a morphism $G\colon K \to K'$, we obtain a morphism

$$H \otimes G\colon H \otimes K \to H \otimes K'.$$

Moreover, we have:

Proposition 36. *Let $F\colon H \to H'$ and $G\colon K \to K'$ be morphisms in* 2Hilb. *Then the following diagram*

$$\begin{tikzcd} H \otimes K \arrow[r, "F\otimes K"] \arrow[d, "H\otimes G"'] & H' \otimes K \arrow[d, "H'\otimes G"] \\ H \otimes K' \arrow[r, "F\otimes K'"'] & H' \otimes K' \end{tikzcd}$$

commutes up to a specified natural isomorphism

$$\bigotimes_{F,G}\colon (F \otimes K)(H' \otimes G) \Rightarrow (H \otimes G)(F \otimes K').$$

Proof - Here we have fixed tensor products of all the 2-Hilbert spaces involved, so we have bimorphisms

$$T_{H,K}\colon H \times K \to H \otimes K$$

and so on. Applying the equivalence

$$\mathrm{bihom}(H \times K, H' \otimes K') \simeq \mathrm{hom}(H \otimes K, H' \otimes K')$$

coming from the definition of tensor product to the bimorphism given by the composite

$$H \times K \xrightarrow{F\times G} H' \times K' \xrightarrow{T_{H',K'}} H' \otimes K' \tag{1}$$

we obtain a morphism we denote by

$$F \otimes G\colon H \otimes K \to H' \otimes K'.$$

We shall construct a natural isomorphism from $(F\otimes K)(H'\otimes G)$ to $F\otimes G$. Composing this with an analogous natural isomorphism from $F \otimes G$ to $(H \otimes G)(F \otimes K')$ one obtains $\bigotimes_{F,G}$.

If we precompose $F \otimes G$ with $T_{H,K}$ we obtain a bimorphism naturally isomorphic to (1). If we precompose $(F \otimes K')(H' \otimes G)$ with $T_{H,K}$, we obtain a bimorphism naturally isomorphic to

$$H \times K \xrightarrow{F\times K} H' \times K \xrightarrow{T_{H',K}} H' \otimes K \xrightarrow{H'\otimes G} H' \otimes K' \tag{2}$$

Note also that in both cases, a *specified* natural isomorphism is given by the definition of tensor product. Since precomposition with $T_{H,K}$ is an equivalence between $\mathrm{bihom}(H\times K, H'\otimes K')$ and $\mathrm{hom}(H\otimes K, H'\otimes K')$, it thus suffices to exhibit a natural isomorphism between (1) and (2).

Factoring these by $F \times K$, it suffices to exhibit a natural isomorphism between

$$H' \times K \xrightarrow{H'\times G} H' \times K' \xrightarrow{T_{H',K'}} H' \otimes K'$$

and

$$H' \times K \xrightarrow{T_{H',K}} H' \otimes K \xrightarrow{H' \otimes G} H' \otimes K'$$

This arises from the definition of $H' \otimes G$. $\square$

The 2-morphism $\bigotimes_{F,G}$ is part of the structure one expects in a monoidal 2-category, and the fact that the diagram in Proposition 36 does not commute 'on the nose' is one of the key ways in which monoidal 2-categories differ from monoidal categories.

We expect a 2-categorical version of hom-tensor adjointness to hold for the tensor product defined in this section and the hom defined in section 4.3. In other words, given finite-dimensional 2-Hilbert space H, H', and K, the obvious functor from $\hom(H, \hom(H', K))$ to $\hom(H \otimes H', K)$ should be an equivalence. However, we shall not prove this here.

4.5 The braiding

The symmetry in Cat gives braiding morphisms in 2Hilb as follows. Let H and H' be 2-Hilbert spaces. We may take their tensor product in either order, obtaining tensor products $T\colon H \times H' \to H \otimes H'$ and $T'\colon H' \times H \to H' \otimes H$. By the universal property of the tensor product, the bimorphism given by the composite

$$H \times H' \xrightarrow{S_{H,H'}} H' \times H \xrightarrow{T'} H' \otimes H$$

defines a morphism, the *braiding*

$$R_{H,H'}\colon H \otimes H' \to H' \otimes H.$$

One can check that $R_{H,H'}$ is an equivalence.

We expect that 2Hilb has the structure of a braided monoidal 2-category with the above braiding morphisms. However, the existing notion of semistrict braided monoidal 2-category introduced by Kapranov and Voevodsky [17] and subsequently refined in HDA1 is insufficiently general to cover this example, since 2Hilb is not a semistrict monoidal 2-category. One should however be able to strictify 2Hilb, obtaining a semistrict braided monoidal 2-category. Alternatively, the work of Trimble [27] should give a fully general notion of weak braided monoidal 2-category, namely a tetracategory with one object and one morphism. This should apply to 2Hilb without strictification.

In any event, both semistrict and weak braided monoidal 2-categories involve various structures in addition to the braiding morphisms. Most of these should arise from the universal property of the tensor product together with the properties of the symmetry in Cat. For example, we have:

Proposition 37. *Let $F: H \to H'$ be a morphism and let K be an object in* 2Hilb*. Then the following diagram*

$$\begin{tikzcd} H \otimes K \arrow[r, "F \otimes K"] \arrow[d, "R_{H,K}"'] & H' \otimes K \arrow[d, "R_{H',K}"] \\ K \otimes H \arrow[r, "K \otimes F"'] & K \otimes H' \end{tikzcd}$$

commutes up to a specified natural isomorphism

$$R_{F,K}\colon (F \otimes K)R_{H',K} \Rightarrow R_{H,K}(K \otimes F).$$

Similarly, given an object H and a morphism $G: K \to K'$ in 2Hilb*, the following diagram*

$$\begin{tikzcd} H \otimes K \arrow[r, "H \otimes G"] \arrow[d, "R_{H,K}"'] & H \otimes K' \arrow[d, "R_{H,K'}"] \\ K \otimes H \arrow[r, "G \otimes H"'] & K' \otimes H \end{tikzcd}$$

commutes up to a specified natural isomorphism

$$R_{H,G}\colon (H \otimes G)R_{H,K'} \Rightarrow R_{H,K}(G \otimes H).$$

Proof - We only treat the first case as the second is analogous. Applying the equivalence

$$\mathrm{bihom}(H \times K, K' \otimes H') \simeq \mathrm{hom}(H \otimes K, K' \otimes H')$$

coming from the definition of tensor product to the bimorphism given by the composite

$$H \times K \xrightarrow{F \times K} H' \times K \xrightarrow{S_{H',K}} K \times H' \xrightarrow{T_{K,H'}} K \otimes H' \tag{3}$$

we obtain a morphism we denote by $A: H \otimes K \to K' \otimes H$. We shall construct a natural isomorphism from $(F \otimes K)R_{H',K}$ to A. Using the fact that (3) equals

$$H \times K \xrightarrow{S_{H,K}} K \times H \xrightarrow{K \times F} K \times H' \xrightarrow{T_{K,H'}} K \otimes H'$$

one can similarly obtain a natural isomorphism from A to $R_{H,K}(K \otimes F)$. The composite of these is $R_{F,K}$.

If we precompose A with $T_{H,K}$ we obtain a bimorphism naturally isomorphic to (3). If we precompose $(F \otimes K)R_{H',K}$ with $T_{H,K}$, we obtain a bimorphism naturally isomorphic to

$$H \times K \xrightarrow{F \times K} H' \times K \xrightarrow{T_{H',K}} H' \otimes K \xrightarrow{R_{H',K}} K' \otimes H \tag{4}$$

In both cases, a natural isomorphism is given by the definition of tensor product. It thus suffices to exhibit a natural isomorphism between (3) and (4). This may be constructed as in the proof of Proposition 36. □

4.6 The involutor

As indicated in Figure 1, for 2Hilb to be a stable 2-category it should possess an extra layer of structure after the tensor product and the braiding, namely the 'involutor'. Also, this structure should have an extra property making 2Hilb 'strongly involutory'. The involutor is a weakened form of the equation appearing in the definition of a symmetric monoidal category. Namely, while the braiding need not satisfy

$$R_{H,H'}R_{H',H} = 1_{H\otimes H'}$$

for all objects $H, H' \in$ 2Hilb, there should be a 2-isomorphism

$$I_{H,H'}\colon R_{H,H'}R_{H',H} \Rightarrow 1_{H\otimes H'},$$

the involutor.

We construct the involutor as follows. Choose tensor products $T\colon H\times H' \to H\otimes H'$ and $T'\colon H' \times H \to H' \otimes H$. Then by the universality of the tensor product, the commutativity of

$$\begin{tikzcd} & H' \times H \arrow[dr, "S_{H',H}"] & \\ H \times H' \arrow[ur, "S_{H,H'}"] \arrow[rr, "1_{H\times H'}"'] & & H \times H' \end{tikzcd}$$

implies that

$$\begin{tikzcd} & H' \otimes H \arrow[dr, "R_{H',H}"] & \\ H \otimes H' \arrow[ur, "R_{H,H'}"] \arrow[rr, "1_{H\otimes H'}"'] & & H \otimes H' \end{tikzcd}$$

commutes up to a specified natural transformation. This is the involutor

$$I_{H,H'}\colon R_{H,H'}R_{H',H} \Rightarrow 1_{H\otimes H'}.$$

In addition, for 2Hilb to be stable, or 'strongly involutory', the involutor should satisfy a special coherence law of its own, in analogy to how the braiding satisfies a special equation in a symmetric monoidal category. In HDA0 this equation was described in terms of $R_{H,H'}$ and a weak inverse thereof, but it turns out to be easier to give the equation by stating that the following horizontal composites agree:

$$I_{H,H'} \circ 1_{R_{H,H'}}\colon R_{H,H'}R_{H',H}R_{H,H'} \Rightarrow R_{H,H'}$$

and

$$1_{R_{H,H'}} \circ I_{H,H'}\colon R_{H,H'}R_{H',H}R_{H,H'} \Rightarrow R_{H,H'}$$

This is indeed the case, as one can show using the properties of the tensor product.

5 2-H*-algebras

Now we consider 2-Hilbert spaces with extra structure and properties, as listed in the second column of Figure 2.

Definition 38. *A* 2-H*-algebra H *is a 2-Hilbert space equipped with a* product *bimorphism* $\otimes\colon H \times H \to H$*, a* unit *object* $1 \in H$*, a unitary natural transformation* $a_{x,y,z}\colon (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ *called the* associator*, and unitary natural transformations* $\ell_x\colon 1 \otimes x \to x$*,* $r_x\colon x \otimes 1 \to x$ *called the* left *and* right unit laws*, making* H *into a monoidal category. We require also that every object* $x \in H$ *has a left dual.*

Recall that for H to be a monoidal category, one demands that the following pentagon commute:

$$\begin{tikzcd}
((x \otimes y) \otimes z) \otimes w \arrow[r, "a_{x\otimes y,z,w}"] \arrow[d, "a_{x,y,z}\otimes w"'] & (x \otimes y) \otimes (z \otimes w) \arrow[r, "a_{x,y,z\otimes w}"] & x \otimes (y \otimes (z \otimes w)) \\
(x \otimes (y \otimes z)) \otimes w \arrow[rr, "a_{x,y\otimes z,w}"] & & x \otimes ((y \otimes z) \otimes w) \arrow[u, "x\otimes a_{y,z,w}"']
\end{tikzcd}$$

as well as the following diagram involving the unit laws:

$$\begin{tikzcd}
(1 \otimes x) \otimes 1 \arrow[rr, "a_{1,x,1}"] \arrow[d, "\ell_x \otimes 1"'] & & 1 \otimes (x \otimes 1) \arrow[d, "1 \otimes r_x"] \\
x \otimes 1 \arrow[r, "r_x"] & x & 1 \otimes x \arrow[l, "\ell_x"']
\end{tikzcd}$$

Mac Lane's coherence theorem [22] says that every monoidal category is equivalent, as a monoidal category, to a *strict* monoidal category, that is, one for which the associators and unit laws are all identity morphisms. Sometimes we will use this to streamline formulas by not parenthesizing tensor products and not writing the associators and unit laws. Such formulas apply literally only to the strict case, but one can always use Mac Lane's theorem to apply them to general monoidal categories. In practice, this amounts to parenthesizing tensor products however one likes, and inserting associators and unit laws when needed to make the formulas make sense.

A *left dual* of an object x in a monoidal category is an object y together with morphisms

$$e\colon y \otimes x \to 1$$

and

$$i\colon 1 \to x \otimes y,$$

called the *unit* and *counit*, such that the following diagrams commute:

$$\begin{tikzcd}
x \arrow[rr, "1_x"] \arrow[dr, "i \otimes x"'] & & x \\
& x \otimes y \otimes x \arrow[ur, "x \otimes e"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
y \arrow[rr, "1_y"] \arrow[dr, "y \otimes i"'] & & y \\
& y \otimes x \otimes y \arrow[ur, "e \otimes y"'] &
\end{tikzcd}$$

(These diagrams apply literally only when the monoidal category is strict.) In this situation we also say that x is a *right dual* of y, and that (x, y, i, e) is an *adjunction*. All adjunctions having x as right dual are uniquely isomorphic in the following sense:

Proposition 39. *Given an adjunction* (x, y, i, e) *in a monoidal category and an isomorphism* $f: y \to y'$*, there is an adjunction* (x, y', i', e') *given by:*

$$i' = i(x \otimes f), \qquad e' = (f^{-1} \otimes x)e.$$

Conversely, given two adjunctions (x, y, i, e) *and* (x, y', i', e')*, there is a unique isomorphism* $f: y \to y'$ *for which* $i' = i(x \otimes f)$ *and* $e' = (f^{-1} \otimes x)e$*. This is given in the strict case by the composite*

$$y = y \otimes 1 \xrightarrow{y \otimes i'} y \otimes x \otimes y' \xrightarrow{e \otimes y'} 1 \otimes y' = y'$$

Proof - This result is well-known and the proof is a simple calculation. □

Similarly, any two adjunctions having a given object as right dual are canonically isomorphic. We may thus speak of 'the' left or right dual of a given object, using the generalized 'the', as described in Section 4.4. Note that duality at the morphism level of a 2-H*-algebra allows us to turn left duals into right duals, and vice versa, at the object level:

Proposition 40. *Suppose that* H *is a 2-H*-algebra. Then* (x, x^*, i, e) *is an adjunction if and only if* (x^*, x, e^*, i^*) *is an adjunction.*

Proof - The proof is analogous to that of Proposition 21. □

Next we turn to braided and symmetric 2-H*-algebras. A good example of a braided 2-H*-algebra is the category of tilting modules of a quantum group when the parameter q is a suitable root of unity [4]. Categories very similar to our braided 2-H*-algebras have been studied by Fröhlich and Kerler [12] under the name 'C*-quantum categories'; our definitions differ only in some fine points. A good example of a symmetric 2-H*-algebra is the category of finite-dimensional continuous unitary representations of a compact topological group. Doplicher and Roberts [8] have studied categories very similar to our symmetric 2-H*-algebras.

Definition 41. *A* braided 2-H*-algebra *is a 2-H*-algebra* H *equipped with a unitary natural isomorphism* $B_{x,y}: x \otimes y \to y \otimes x$ *making* H *into a braided monoidal category.*

Definition 42. *A* symmetric 2-H*-algebra *is a 2-H*-algebra for which the braiding is a symmetry.*

Recall that for H to be a braided monoidal category, the following two hexagons must commute:

$$\begin{tikzcd}
x \otimes (y \otimes z) \arrow[r, "a^{-1}_{x,y,z}"] \arrow[d, "B_{x,y\otimes z}"'] & (x \otimes y) \otimes z \arrow[r, "B_{x,y}\otimes z"] & (y \otimes x) \otimes z \arrow[d, "a_{y,x,z}"] \\
(y \otimes z) \otimes x \arrow[r, "a_{y,z,x}"'] & y \otimes (z \otimes x) \arrow[r, "y\otimes B_{x,z}"'] & y \otimes (x \otimes z)
\end{tikzcd}$$

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{x,y,z}"] \arrow[d, "B_{x\otimes y,z}"'] & x \otimes (y \otimes z) \arrow[r, "x\otimes B_{y,z}"] & x \otimes (z \otimes y) \arrow[d, "a^{-1}_{x,z,y}"] \\
z \otimes (x \otimes y) \arrow[r, "a^{-1}_{z,x,y}"'] & (z \otimes x) \otimes y \arrow[r, "B_{x,z}\otimes y"'] & (x \otimes z) \otimes y
\end{tikzcd}$$

The braiding is a symmetry if $B_{x,y} = B^{-1}_{y,x}$ for all objects x and y.

5.1 The balancing

In the study of braided monoidal categories where objects have duals, it is common to introduce something called the 'balancing'. The balancing can treated in various ways [12, 16, 26]. For example, one may think of it as a choice of automorphism $b_x\colon x \to x$ for each object x, which is required to satisfy certain laws. While very important in topology, this extra structure seems somewhat ad hoc and mysterious from the algebraic point of view. We now show that braided 2-H*-algebras are automatically equipped with a balancing. The reason is that not only the objects, but also the morphisms, have duals. In fact, some of what follows would apply to any braided monoidal category in which both objects and morphisms have duals.

In any 2-H*-algebra, Proposition 40 gives a way to make any object x into the left dual of its left dual x^*. In a braided 2-H*-algebra, x also becomes the left dual of x^* in another way:

Proposition 43. *Let H be a braided 2-H*-algebra. Then (x, x^*, i, e) is an adjunction if and only if $(x^*, x, iB_{x,x^*}, B^{-1}_{x^*,x}e)$ is an adjunction.*

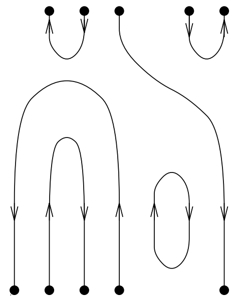
Proof - The proof is a simple computation. □

It follows from Proposition 39 that these two ways to make x into the left dual of x^* determine an automorphism of x. Simplifying the formula for this automorphism somewhat, we make the following definition:

Definition 44. *If H is a braided 2-H*-algebra and (x, x^*, i, e) is an adjunction in H, the* balancing *of the adjunction is the morphism $b\colon x \to x$ given in the strict case by the composite:*

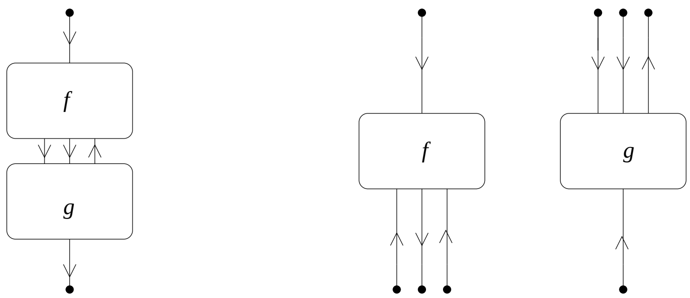
$$x \xrightarrow{e^*\otimes x} x^* \otimes x \otimes x \xrightarrow{x^*\otimes B_{x,x}} x^* \otimes x \otimes x \xrightarrow{e\otimes x} x$$

It is perhaps easiest to understand the significance of the balancing in terms of its relation to topology. We shall be quite sketchy about describing this, but the reader can fill in the details using the ideas described in HDA0 and the many references therein. Especially relevant is the work of Freyd and Yetter [11], Joyal and Street [15], and Reshetikhin and Turaev [26, 28]. We discuss this relationship more carefully in the Conclusions.



3. Typical tangle in 2 dimensions

The basic idea is to use tangles to represent certain morphisms in 2-H*-algebras. A typical oriented tangle in 2 dimensions is shown in Figure 3. If we fix an adjunction (x, x^*, i, e) in a strict 2-H*-algebra H, any such tangle corresponds uniquely to a morphism in H as follows. As shown in Figure 4, vertical juxtaposition of tangles corresponds to the composition of morphisms, while horizontal juxtaposition corresponds to the tensor product of morphisms.

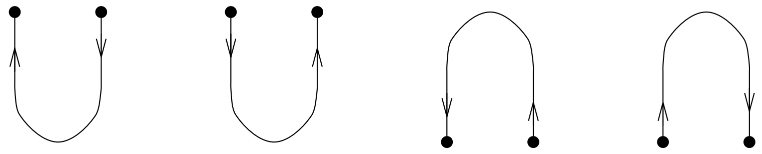


4. Composition and tensor product of tangles

Thus it suffices to specify the morphisms in H corresponding to certain basic tangles from which all others can be built up by composition and tensor product. These basic tangles are shown in Figures 5 and 6. A downwards-pointing segment corresponds to the identity on x, while an upwards-pointing segment corresponds to the identity on x^*.

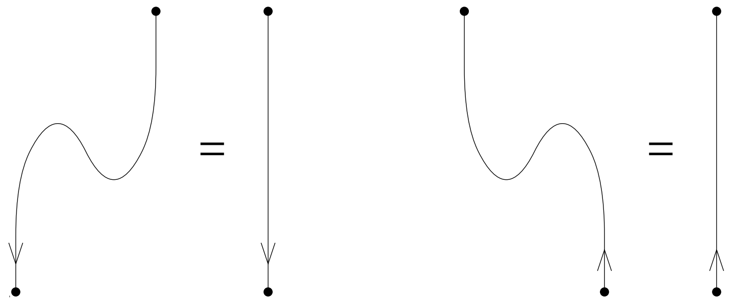
5. Tangles corresponding to 1_x and 1_{x^*}

The two oriented forms of a 'cup' tangle correspond to the morphisms e and i^*, while the two oriented forms of a 'cap' correspond to i and e^*.



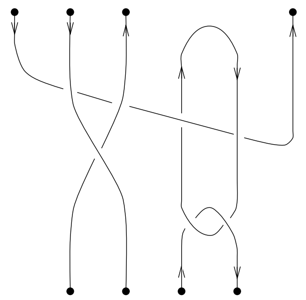
6. Tangles corresponding to e, i^*, i, and e^*

It then turns out that isotopic tangles correspond to the same morphism in H. The main thing to check is that the isotopic tangles shown in Figure 7 correspond to the same morphisms. This follows from the triangle diagrams in the definition of an adjunction. Similar equations with the orientation of the arrows reversed follow from Proposition 40.



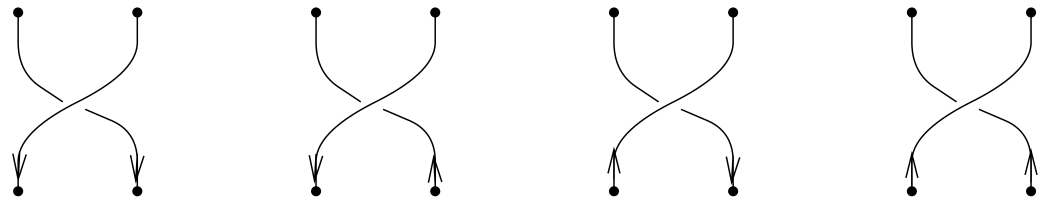
7. Tangle equations corresponding to the definition of adjunction

If H is braided, we can also map framed oriented tangles in 3 dimensions to morphisms in H. A typical such tangle is shown in Figure 8. We use the blackboard framing, in which each strand is implicitly equipped with a vector field normal to the plane in which the tangle is drawn.

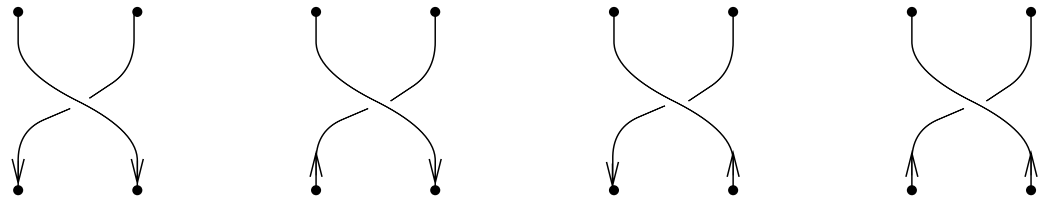


8. Typical tangle in 3 dimensions

We interpret the basic tangles in Figures 5 and 6 as we did before. Moreover, we let the tangles in Figure 9 correspond to the morphisms $B_{x,x}$, $B_{x^*,x}$, B_{x,x^*}, and B_{x^*,x^*}, and let the tangles in Figure 10 correspond to the morphisms $B^{-1}_{x,x}$, $B^{-1}_{x^*,x}$, B^{-1}_{x,x^*}, and $B^{-1}_{x^*,x^*}$.

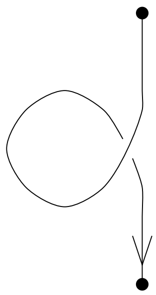


9. Tangles corresponding to $B_{x,x}$, $B_{x^*,x}$, B_{x,x^*}, and B_{x^*,x^*}



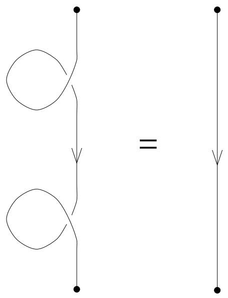
10. Tangles corresponding to $B^{-1}_{x,x}$, $B^{-1}_{x^*,x}$, B^{-1}_{x,x^*}, and $B^{-1}_{x^*,x^*}$

Now suppose we wish isotopic framed oriented tangles to correspond to the same morphism in H. Invariance under the 2nd and 3rd Reidemeister moves follows from the properties of the braiding, so it suffices to check invariance under the framed version of the 1st Reidemeister move. For this, note that the tangle shown in Figure 11 corresponds to the balancing of the adjunction (x, x^*, i, e). This tangle has a 2π twist in its framing.



11. Tangle corresponding to the balancing $b\colon x \to x$

The framed version of the 1st Reidemeister move, shown in Figure 12, represents the cancellation of two opposite 2π twists in the framing. Both tangles in this picture correspond to the same morphism in H precisely when the balancing $b\colon x \to x$ is unitary.

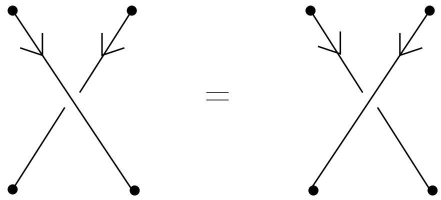


12. Tangle equation corresponding to unitarity of the balancing

In short, we obtain a map from isotopy classes of framed oriented tangles in 3 dimensions to morphisms in a braided 2-H*-algebra H whenever we choose an adjunction in H whose balancing is unitary. This motivates the following definition:

Definition 45. *An adjunction* (x, x^*, i, e) *in a braided 2-H*-algebra is* well-balanced *if its balancing is unitary.*

Similarly, given any well-balanced adjunction in a symmetric 2-H*-algebra H, we obtain a map from isotopy classes of framed oriented tangles in 4 dimensions to morphisms in H. We may draw tangles in 4 dimensions just as we draw tangles in 3 dimensions, but there is an extra rule saying that any right-handed crossing is isotopic to the corresponding left-handed crossing. One case of this rule is shown in Figure 13. Invariance under these isotopies follows directly from the fact that the braiding is a symmetry.



13. Tangle equation corresponding to symmetry

The important fact is that well-balanced adjunctions exist and are unique up to a unique unitary isomorphism. Moreover, all of them have the same balancing:

Theorem 46. *Suppose H is a braided 2-H*-algebra. For every object $x \in H$ there exists a well-balanced adjunction (x, y, i, e). Given well-balanced adjunctions (x, y, i, e) and (x, y', i', e'), there is a unique morphism $u\colon y \to y'$ such that*

$$i' = i(x \otimes u), \qquad e' = (u^{-1} \otimes x)e,$$

and this morphism is unitary.

Proof - To simplify notation we assume without loss of generality that H is strict. Suppose first that $x \in H$ is simple. Then for any adjunction (x, y, i, e), the balancing equals $\beta 1_x$ for some nonzero $\beta \in \mathbb{C}$. By Proposition 39 we may define a new adjunction $(x, y, |\beta|^{1/2} i, |\beta|^{-1/2} e)$. Since the balancing of this adjunction equals $\beta|\beta|^{-1} 1_x$, this adjunction is well-balanced.

Next suppose that $x \in H$ is arbitrary. Using Theorem 6 we can write x as an orthogonal direct sum of simple objects x_j, in the sense that there are morphisms

$$p_j \colon x \to x_j$$

with

$$p_j^* p_j = 1_{x_j}, \qquad \sum_j p_j p_j^* = 1_x.$$

Let y_j be a left dual of x_j, and define y to be an orthogonal direct sum of the objects y_j, with morphisms

$$q_j \colon y \to y_j$$

such that

$$q_j^* q_j = 1_{y_j}, \qquad \sum_j q_j q_j^* = 1_y.$$

Since the x_j are simple, there exist adjunctions (x_j, y_j, i_j, e_j) for which the balancings $b_j \colon x_j \to x_j$ are unitary. Define the adjunction (x, y, i, e) by

$$i = \sum_j i_j (p_j^* \otimes q_j^*), \qquad e = \sum_j (q_j \otimes p_j) e_j.$$

One can check that this is indeed an adjunction and that the balancing $b \colon x \to x$ of this adjunction is given by

$$b = \sum_j p_j b_j p_j^*,$$

and is therefore unitary.

Now suppose that (x, y', i', e') is any other well-balanced adjunction with x as right dual. Let b' denote the balancing of this adjunction. We shall prove that $b' = b$. By Propositions 8 and 39 there exists a unitary morphism $g \colon y \to y'$, and we have

$$\begin{aligned} b' &= (e'^* \otimes x)(y' \otimes B_{x,x})(e' \otimes x) \\ &= (e'^* (g^* \otimes x) \otimes x)(y \otimes B_{x,x})((g \otimes x) e' \otimes x). \end{aligned}$$

By Proposition 39, $(x, y, (g \otimes x) e', i(x \otimes g^{-1}))$ is an adjunction, so by the uniqueness up to isomorphism of right adjoints we have $(g \otimes x) e' = (y \otimes f) e$ for some isomorphism $f \colon x \to x$. We thus have

$$\begin{aligned} b' &= (e^* (y \otimes f^*) \otimes x)(y \otimes B_{x,x})((y \otimes f) e \otimes x) \\ &= f b f^*. \end{aligned}$$

We may write x as an orthogonal direct sum

$$x = \bigoplus_\lambda x_\lambda$$

where $\{e_\lambda\}$ is a basis of H and x_λ is a direct sum of some number of copies of e_λ. Then by our previous formula for b we have

$$b = \bigoplus_\lambda \beta_\lambda 1_{x_\lambda}$$

with $|\beta_\lambda| = 1$ for all λ. We also have

$$f = \bigoplus_\lambda f_\lambda$$

for some morphisms $f_\lambda : x_\lambda \to x_\lambda$. It follows that

$$b' = \bigoplus_\lambda \beta_\lambda f_\lambda {f_\lambda}^*$$

Since b and b' are unitary it follows that each morphism $f_\lambda {f_\lambda}^*$ is unitary. Since the only positive unitary operator is the identity, using Theorem 6 it follows that each $f_\lambda {f_\lambda}^*$ is the identity, so $b' = b$ as desired.

By Proposition 39, we know there is a unique isomorphism $u : y \to y'$ with

$$i' = i(x \otimes u), \qquad e' = (u^{-1} \otimes x)e,$$

and we need to show that u is unitary. Since $b' = b$, we have

$$(ib' \otimes y)(x \otimes B_{y,y}^{-1})(i^* \otimes y) = (ib \otimes y)(x \otimes B_{y,y}^{-1})(i^* \otimes y),$$

and if one simplifies this equation using the fact that

$$b = (e^* \otimes x)(y' \otimes B_{x,x})(e \otimes x)$$

and

$$b' = (e^*(u^{-1} \otimes x)^* \otimes x)(y' \otimes B_{x,x})((u^{-1} \otimes x)e \otimes x),$$

one finds that u is unitary. $\square$

Corollary 47. *In a braided 2-H*-algebra every well-balanced adjunction with x as right dual has the same balancing, which we call* the balancing *of x and denote as $b_x : x \to x$.*

Proof - This was shown in the proof above. $\square$

Note that for any simple object x in a braided 2-H*-algebra, the balancing b_x must equal 1_x times some unit complex number, the *balancing phase* of x. In physics, the balancing phase describes the change in the wavefunction of a particle that undergoes a 2π rotation. Note that in a symmetric 2-H*-algebra

$$\begin{aligned} b_x &= (e_x^* \otimes 1_x)(1_{x^*} \otimes B_{x,x})(e_x \otimes 1_x) \\ &= (e_x^* \otimes 1_x)(1_{x^*} \otimes B_{x,x}^*)(e_x \otimes 1_x) \\ &= b_x^*, \end{aligned}$$

so $b_x^2 = 1_x$. Thus in this case the balancing phase of any simple object must be ± 1. In physics, this corresponds to the fact that particles in 4-dimensional spacetime are either bosons and fermions depending on the phase they acquire when rotated by 2π, while in 3-dimensional spacetime other possibilities, sometimes called 'anyons', can occur [8, 12].

More generally, we make the following definition:

Definition 48. *If H is a symmetric 2-H*-algebra, an object $x \in H$ is* even *or* bosonic *if $b_x = 1$, and* odd *or* fermionic *if $b_x = -1$. We say H is* even *or* purely bosonic *if every object $x \in H$ is even.*

Note that if $x \oplus y$ is an orthogonal direct sum,

$$b_{x \oplus y} = b_x \oplus b_y,$$

so an object in any symmetric 2-H*-algebra is even (resp. odd) if and only if it is a direct sum of even (resp. odd) simple objects. Also, since

$$b_{x \otimes y} = (b_x \otimes b_y) B_{x,y} B_{y,x},$$

it follows that the tensor product of two even or two odd objects is even, while the tensor product of an even and an odd object is odd.

There is a way to turn any symmetric 2-H*-algebra into an even one, which will be useful in Section 6.

Proposition 49. (Doplicher-Roberts) *Suppose H is a symmetric 2-H*-algebra. Then there is a braiding $B^\flat$ on H given on simple objects $x, y \in H$ by*

$$B^\flat_{x,y} = (-1)^{|x|\,|y|} B_{x,y}$$

where $|x|$ equals 0 or 1 depending on whether x is even or odd, and similarly for $|y|$. Let $H^\flat$ denote H equipped with the new braiding $B^\flat$. Then $H^\flat$ is an even symmetric 2-H-algebra, the* bosonization *of H.*

Proof - This is a series of straightforward computations. One approach involves noting that for any objects $x, y \in H$,

$$B^\flat_{x,y} = \frac{1}{2}\, B_{x,y}(1_x \otimes 1_y + 1_x \otimes b_y + b_x \otimes 1_y - b_x \otimes b_y).$$

$\Box$

The above proposition is essentially due to Doplicher and Roberts, who proved it in a slightly different context [8]. However, our use of the term 'bosonization' is borrowed from Majid [24], who uses it to denote a related process that turns a super-Hopf algebra into a Hopf algebra.

5.2 Trace and dimension

The notion of the 'dimension' of an object in a braided 2-H*-algebra will be very important in Section 6. First we introduce the related notion of 'trace'.

Definition 50. *If H is a braided 2-H*-algebra and $f\colon x \to x$ is a morphism in H, for any well-balanced adjunction (x, x^*, i, e) we define the* trace *of f*, $\mathrm{tr}(f) \in \mathrm{end}(1)$, *by*

$$\mathrm{tr}(f) = e^*(x^* \otimes f)e.$$

The trace is independent of the choice of well-balanced adjunction, by Theorem 46. Also, one can show that an obvious alternative definition of the trace is actually equivalent:

$$\mathrm{tr}(f) = i(f \otimes x^*)i^*.$$

Definition 51. *If H is a braided 2-H*-algebra, we define the dimension of x*, $\dim(x)$, *to be* $\mathrm{tr}(1_x)$.

Note that if x, y are objects in a braided 2-H*-algebra, we have

$$\dim(x \oplus y) = \dim(x) + \dim(y), \qquad \dim(x \otimes y) = \dim(x)\dim(y), \qquad \dim(x^*) = \dim(x).$$

Moreoever, we have:

Proposition 52. *If H is a symmetric 2-H*-algebra and $x \in H$ is any object, then the spectrum of* $\dim(x)$ *is a subset of* $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Proof - We follow the argument of Doplicher and Roberts [8]. For any $n \geq 0$, the group algebra of the symmetric group S_n acts as endomorphisms of $x^{\otimes n}$, and the morphisms p_S, p_A corresponding to complete symmetrization and complete antisymmetrization, respectively, are self-adjoint projections in the H*-algebra $\mathrm{end}(x^{\otimes n})$. It follows that $\mathrm{tr}(p_S), \mathrm{tr}(p_A) \geq 0$. If x is even, a calculation shows that

$$\mathrm{tr}(p_A) = \frac{1}{n!}\, \dim(x)(\dim(x) - 1) \cdots (\dim(x) - n + 1)$$

For this to be nonnegative for all n, the spectrum of $\dim(x)$ must lie in $\mathbb{N}$. Similarly, if x is odd, a calculation shows that

$$\mathrm{tr}(p_S) = \frac{1}{n!}\dim(x)(\dim(x)-1)\cdots(\dim(x)-n+1)$$

so again the spectrum of $\dim(x)$ lies in $\mathbb{N}$. In general, any object $\dim(x)$ is a sum of simple objects, which are either even or odd, so by the additivity of dimension, the spectrum of $\dim(x)$ again lies in $\mathbb{N}$. $\square$

For any 2-H*-algebra, the Eckmann-Hilton argument shows that $\mathrm{end}(1)$ is a commutative H*-algebra, and thus isomorphic to a direct sum of copies of $\mathbb{C}$. (See HDA0 or HDA1 for an explanation of the Eckmann-Hilton argument.)

Definition 53. *A 2-H*-algebra H is* connected *if the unit object $1 \in H$ is simple.*

In a connected 2-H*-algebra, $\mathrm{end}(1) \cong \mathbb{C}$. The dimension of any object in a connected symmetric 2-H*-algebra is thus a nonnegative integer.

In addition to the above notion of dimension it is also interesting to consider the 'quantum dimension'. Here our treatment most closely parallels that of Majid [24].

Definition 54. *If H is a braided 2-H*-algebra and $f\colon x \to x$ is a morphism in H, for any well-balanced adjunction (x, x^*, i, e) we define the* quantum trace *of f, $\mathrm{qtr}(f) \in \mathrm{end}(1)$, by*

$$\mathrm{qtr}(f) = \mathrm{tr}(b_x f).$$

We define the quantum dimension *of x, $\mathrm{qdim}(x)$, to be $\mathrm{qtr}(1_x)$.*

In the case of a symmetric 2-H*-algebra, the quantum trace is also called the 'supertrace'. Suppose H is a connected symmetric 2-H*-algebra and x is a simple object. Then $\mathrm{qdim}(x) \geq 0$ if x is even and $\mathrm{qdim}(x) \leq 0$ if x is odd. The idea of odd objects as negative-dimensional is implicit in Penrose's work on negative-dimensional vector spaces [25].

5.3 Homomorphisms and 2-homomorphisms

There is a 2-category with 2-H*-algebras as objects and 'homomorphisms' and '2-homomorphisms' as morphisms and 2-morphisms, respectively. This is also true for braided 2-H*-algebras and symmetric 2-H*-algebras.

Definition 55. *Given 2-H*-algebras H and H', a* homomorphism *$F\colon H \to H'$ is a morphism of 2-Hilbert spaces that is also a monoidal functor. If H and H' are braided, we say that F is a* homomorphism of braided 2-H*-algebras *if F is additionally a braided monoidal functor. If H and H' are symmetric, we say that F is a* homomorphism of symmetric 2-H*-algebras *if F is a morphism of 2-Hilbert spaces that is also a symmetric monoidal functor.*

Recall that a functor $F\colon C \to C'$ between monoidal categories is monoidal if it is equipped with a natural isomorphism $\Phi_{x,y}\colon F(x)\otimes F(y) \to F(x\otimes y)$ making the following diagram commute for any objects $x, y, z \in C$:

$$\begin{tikzcd}
(F(x)\otimes F(y))\otimes F(z) \arrow[r, "\Phi_{x,y}\otimes 1_{F(z)}"] \arrow[d, "a_{F(x),F(y),F(z)}"'] & F(x\otimes y)\otimes F(z) \arrow[r, "\Phi_{x\otimes y,z}"] & F((x\otimes y)\otimes z) \arrow[d, "F(a_{x,y,z})"] \\
F(x)\otimes (F(y)\otimes F(z)) \arrow[r, "1_{F(x)}\otimes \Phi_{y,z}"] & F(x)\otimes F(y\otimes z) \arrow[r, "\Phi_{x,y\otimes z}"] & F(x\otimes (y\otimes z))
\end{tikzcd}$$

together with an isomorphism $\phi\colon 1_{C'} \to F(1_C)$ making the following diagrams commute for any object $x \in C$:

$$\begin{tikzcd}
1\otimes F(x) \arrow[r, "\ell_{F(x)}"] \arrow[d, "\phi\otimes 1_{F(x)}"'] & F(x) \\
F(1)\otimes F(x) \arrow[r, "\Phi_{1,x}"] & F(1\otimes x) \arrow[u, "F(\ell_x)"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(x)\otimes 1 \arrow[r, "r_{F(x)}"] \arrow[d, "1_{F(x)}\otimes\phi"'] & F(x) \\
F(x)\otimes F(1) \arrow[r, "\Phi_{x,1}"] & F(x\otimes 1) \arrow[u, "F(r_x)"']
\end{tikzcd}$$

If C and C' are braided, we say that F is braided if additionally it makes the following diagram commute for all $x, y \in C$:

$$\begin{tikzcd}
F(x)\otimes F(y) \arrow[r, "B_{F(x),F(y)}"] \arrow[d, "\Phi_{x,y}"'] & F(y)\otimes F(x) \arrow[d, "\Phi_{y,x}"] \\
F(x\otimes y) \arrow[r, "F(B_{x,y})"] & F(y\otimes x)
\end{tikzcd}$$

A symmetric monoidal functor is simply a braided monoidal functor that happens to go between symmetric monoidal categories! No extra condition is involved here.

Note that if $F\colon H \to H'$ is a homomorphism of braided 2-H*-algebras, F maps any well-balanced adjunction in H to one in H'. Thus it preserves dimension in the following sense:

$$\dim(F(x)) = F(\dim(x))$$

for any object $x \in H$. In particular, if H and H' are connected, so that we can identify the dimension of objects in either with numbers, we have simply $\dim(F(x)) = \dim(x)$.

Definition 56. *If H and H' are 2-H*-algebras, possibly braided or symmetric, and $F, G\colon H \to H'$ are homomorphisms of the appropriate sort, a* 2-homomorphism $\alpha\colon F \Rightarrow G$ *is a monoidal natural transformation.*

Suppose that (F, Φ, ϕ) and (G, Γ, γ) are monoidal functors from the monoidal category C to the monoidal category D. Then a natural transformation $\alpha\colon F \to G$ is monoidal if the diagrams

$$\begin{tikzcd}
F(x) \otimes F(y) \arrow[r, "\alpha_x \otimes \alpha_y"] \arrow[d, "\Phi_{x,y}"'] & G(x) \otimes G(y) \arrow[d, "\Gamma_{x,y}"] \\
F(x \otimes y) \arrow[r, "\alpha_{x \otimes y}"] & G(x \otimes y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[d, "\phi"'] \arrow[dr, "\gamma"] & \\
F(1) \arrow[r, "\alpha_1"] & G(1)
\end{tikzcd}$$

commute. There are no extra conditions required of 'braided monoidal' or 'symmetric monoidal' natural transformations.

Finally, when we speak of two 2-H*-algebras H and H', possibly braided or symmetric, being *equivalent*, we always mean the existence of homomorphisms $F\colon H \to H'$ and $G\colon H' \to H$ of the appropriate sort that are inverses up to a 2-isomorphism.

6 Reconstruction Theorems

In this section we give a classification of symmetric 2-H*-algebras. Doplicher and Roberts proved a theorem which implies that connected even symmetric 2-H*-algebras are all equivalent to categories of compact group representations [8, 9]. Here and in all that follows, by a 'representation' of a compact group we mean a finite-dimensional continuous unitary representation. Given a compact group G, let $\mathrm{Rep}(G)$ denote the category of such representations of G. This becomes a connected even symmetric 2-H*-algebra in an obvious way. While Doplicher and Roberts worked using the language of 'C*-categories', their result can be stated as follows:

Theorem 57. (Doplicher-Roberts) *Let H be a connected even symmetric 2-H*-algebra. Then there exists a homomorphism of symmetric 2-H*-algebras $T: H \to \mathrm{Hilb}$, unique up to a unitary 2-homomorphism. Let $U(T)$ be the group of unitary 2-homomorphisms $\alpha: T \Rightarrow T$, given the topology in which a net $\alpha_\lambda \in U(T)$ converges to α if and only if $(\alpha_\lambda)_x \to \alpha_x$ in norm for all $x \in H$. Then $U(T)$ is compact, each Hilbert space $T(x)$ becomes a representation of $U(T)$, and the resulting homomorphism $\tilde{T}: H \to \mathrm{Rep}(U(T))$ extends to an equivalence of symmetric 2-H*-algebras.*

Note that any continuous homomorphism $\rho: G \to G'$ between compact groups determines a homomorphism of symmetric 2-H*-algebras,

$$\rho^*: \mathrm{Rep}(G') \to \mathrm{Rep}(G),$$

sending each representation σ of G' to the representation $\sigma \circ \rho$ of G. The above theorem yields a useful converse to this construction:

Corollary 58. (Doplicher-Roberts) *Let $F: H' \to H$ be a homomorphism of connected even symmetric 2-H*-algebras. Let $T: H \to \mathrm{Hilb}$ be a homomorphism of symmetric 2-H*-algebras. Then there exists a continuous group homomorphism*

$$F^*: U(T) \to U(FT)$$

such that $F^*(\alpha)$ equals the horizontal composite $F \circ \alpha$. Moreover, $(F^*)^*$ equals F up to a unitary 2-homomorphism.

Dolan [7] has noted that a generalization of the Doplicher-Roberts theorem to even symmetric 2-H*-algebras — not necessarily connected — amounts to a categorification of the Gelfand-Naimark theorem. The spectrum of a commutative H*-algebra H is a set $\mathrm{Spec}(H)$ whose points are homomorphisms from H to $\mathbb{C}$. The Gelfand-Naimark theorem implies that H is isomorphic to the algebra of functions from $\mathrm{Spec}(H)$ to $\mathbb{C}$. Similarly, we may define the 'spectrum' of an even symmetric 2-H*-algebra H to be the groupoid $\mathrm{Spec}(H)$ whose objects are homomorphisms from H to Hilb, and whose morphisms are unitary 2-homomorphisms between these. Moreover, we shall show that H is equivalent to a symmetric 2-H*-algebra whose objects are 'representations' of $\mathrm{Spec}(H)$ — certain functors from $\mathrm{Spec}(H)$ to Hilb. Indeed, our proof of this uses an equivalence

$$\hat{\ }: H \to \mathrm{Rep}(\mathrm{Spec}(H))$$

that is just the categorified version of the 'Gelfand transform' for commutative H*-algebras.

In fact, there is no need to restrict ourselves to symmetric 2-H*-algebras that are even. To treat a general symmetric 2-H*-algebra H we need objects of $\mathrm{Spec}(H)$ to be homomorphisms from H to a symmetric 2-H*-algebra of 'super-Hilbert spaces'. The spectrum will then be a 'supergroupoid' — though not at all the most general sort of thing one could imagine calling a supergroupoid. (Perhaps a better term would be 'balanced groupoid'.)

Definition 59. *Define* SuperHilb *to be the category whose objects are $\mathbb{Z}_2$-graded (finite-dimensional) Hilbert spaces, and whose morphisms are linear maps preserving the grading.*

The category SuperHilb can be made into a symmetric 2-H*-algebra where the $*$-structure is the ordinary Hilbert space adjoint, the product is the usual tensor product of $\mathbb{Z}_2$-graded Hilbert spaces, and the braiding is given on homogeneous elements $v \in x$, $w \in y$ by

$$B_{x,y}(v \otimes w) = (-1)^{\deg v \deg w}\, w \otimes v.$$

Definition 60. *If H is a symmetric 2-H*-algebra, define* $\mathrm{Spec}(H)$ *to be the category whose objects are symmetric 2-H*-algebra homomorphisms $F: H \to$* SuperHilb *and whose morphisms are unitary 2-homomorphisms between these.*

Definition 61. *A* topological groupoid *is a groupoid for which the* hom*-sets are topological spaces and the groupoid operations are continuous. A* compact groupoid *is is a topological groupoid with compact Hausdorff* hom*-sets and finitely many isomorphism classes of objects.*

Definition 62. *A* supergroupoid *is a groupoid G equipped with a natural transformation $\beta: 1_G \Rightarrow 1_G$, the* balancing, *with $\beta^2 = 1$. A* compact supergroupoid *is a supergroupoid that is also a compact groupoid.*

Let H be a symmetric 2-H*-algebra. Then $\mathrm{Spec}(H)$ becomes a topological groupoid if for any $S, T: H \to \mathrm{Hilb}$ we give $\hom(S,T)$ the topology in which a net α_λ converges to α if and only if $(\alpha_\lambda)_x \to \alpha_x$ in norm for any $x \in H$. We shall show that $\mathrm{Spec}(H)$ is a compact groupoid. Also, $\mathrm{Spec}(H)$ becomes a supergroupoid if for any object $T \in \mathrm{Spec}(H)$ we define $\beta_T: T \Rightarrow T$ by

$$(\beta_T)_x = b_{T(x)} = T(b_x)$$

for any object $x \in H$. One can check that $\beta: 1_{\mathrm{Spec}(H)} \Rightarrow 1_{\mathrm{Spec}(H)}$ is a natural transformation, and $\beta^2 = 1$ because the balancing for H satisfies $b_x^2 = 1$ for any $x \in H$.

Definition 63. *Given a compact supergroupoid G, a (continuous, unitary, finite-dimensional)* representation *of G is a functor $F: G \to$* SuperHilb *such that $F(g)$ is* *unitary for every morphism g in G, $F\colon \hom(x,y) \to \hom(F(x),F(y))$ is continuous for all objects $x,y \in G$, and*

$$F(\beta_x) = b_{F(x)}$$

for every object $x \in G$. We define $\mathrm{Rep}(G)$ to be the category having representations of G as objects and natural transformations between these as morphisms.

Let G be a compact supergroupoid. Then the category $\mathrm{Rep}(G)$ becomes an even symmetric 2-H*-algebra in a more or less obvious way as follows. Given objects $F, F' \in \mathrm{Rep}(G)$, we make $\hom(F,F')$ into a Hilbert space with the obvious linear structure and the inner product given by

$$\langle \alpha, \beta \rangle = \sum_x \mathrm{tr}({\alpha_x}^* \beta_x)$$

where the sum is taken over any maximal set of nonisomorphic objects of G. This makes $\mathrm{Rep}(G)$ into a Hilb-category. Moreover, $\mathrm{Rep}(G)$ becomes a 2-Hilbert space if we define the dual of $\alpha\colon F \Rightarrow F'$ by $(\alpha^*)_x = (\alpha_x)^*$. We define the tensor product of objects $F, F' \in \mathrm{Rep}(G)$ by

$$(F \otimes F')(x) = F(x) \otimes F'(x), \qquad (F \otimes F')(f) = F(f) \otimes F'(f)$$

for any object $x \in G$ and morphism f in G. It is easy to define a tensor product of morphisms and associator making $\mathrm{Rep}(G)$ into a monoidal category, and to check that $\mathrm{Rep}(G)$ is then a 2-H*-algebra. Finally, $\mathrm{Rep}(G)$ inherits a braiding from the braiding in SuperHilb, making $\mathrm{Rep}(G)$ into a symmetric 2-H*-algebra.

Now suppose H is an even symmetric 2-H*-algebra. Then there is a functor

$$\hat{\ }\colon H \to \mathrm{Rep}(\mathrm{Spec}(H)),$$

the *categorified Gelfand transform*, given as follows. For every object $x \in H$, $\hat{x}$ is the representation with

$$\hat{x}(T) = T(x)$$

for all $T \in \mathrm{Spec}(H)$, and

$$\hat{x}(\alpha) = \alpha_x$$

for all $\alpha\colon T \Rightarrow T'$, where $T, T' \in \mathrm{Spec}(H)$. For every morphism $f\colon x \to y$ in H, $\hat{f}\colon \hat{x} \Rightarrow \hat{y}$ is the natural transformation with

$$\hat{f}(T) = T(f)$$

for all $T \in \mathrm{Spec}(H)$. Our generalized Doplicher-Roberts theorem states:

Theorem 64. *Suppose that H is a symmetric 2-H*-algebra. Then $\mathrm{Spec}(H)$ is a compact supergroupoid and $\hat{\ }\colon H \to \mathrm{Rep}(\mathrm{Spec}(H))$ extends to an equivalence of symmetric 2-H*-algebras.*

Proof - We have described how $\mathrm{Spec}(H)$ is a supergroupoid. To see that it is compact, note that for any $S, T \in \mathrm{Spec}(H)$ the hom-set $\hom(S,T)$ is a compact Hausdorff space, by Tychonoff's theorem. We also need to show that $\mathrm{Spec}(H)$ has finitely many isomorphism classes of objects. The unit object 1_H is the direct sum of finitely many nonisomorphic simple objects e_i, the kernels of the minimal projections p_i in the commutative H*-algebra $\mathrm{end}(1_H)$. Any object $x \in H$ is thus a direct sum of objects $x_i = e_i \otimes x$, and any morphism $f\colon x \to y$ is a direct sum of morphisms $f_i\colon x_i \to y_i$. In short, H is, in a fairly obvious sense, the direct sum of finitely many connected symmetric 2-H*-algebras H_i. Any homomorphism $T\colon H \to \mathrm{SuperHilb}$ induces a homomorphism from $\mathrm{end}(1_H)$ to $\mathrm{end}(1_{\mathrm{SuperHilb}}) \cong \mathbb{C}$, which must annihilate all but one of the projections p_i, so T sends one of the objects x_i to $1_{\mathrm{SuperHilb}}$ and the rest to 0. Thus $\mathrm{Spec}(H)$ is, as a groupoid, equivalent to the disjoint union of the groupoids $\mathrm{Spec}(H_i)$, and hence has finitely many isomorphism classes of objects.

To show that the categorified Gelfand transform is an equivalence, first suppose that H is even and connected. Then the supergroupoid $\mathrm{Spec}(H)$ has $\beta = 1$, so every representation $F\colon \mathrm{Spec}(H) \to \mathrm{SuperHilb}$ factors through the inclusion $\mathrm{Hilb} \hookrightarrow \mathrm{SuperHilb}$. Moreover, by Theorem 57 all the objects of $\mathrm{Spec}(H)$ are isomorphic, so $\mathrm{Spec}(H)$ is equivalent, as a groupoid, to the group $\mathrm{U}(T)$ for any $T \in \mathrm{Spec}(H)$. We thus obtain an equivalence of symmetric 2-H*-algebras between $\mathrm{Rep}(\mathrm{Spec}(H))$ and $\mathrm{Rep}(\mathrm{U}(T))$ as defined in Theorem 57. Using this, the fact that $\tilde{T}\colon H \to \mathrm{Rep}(\mathrm{U}(T))$ is an equivalence translates into the fact that $\hat{\ }\colon H \to \mathrm{Rep}(\mathrm{Spec}(H))$ is an equivalence.

Next, suppose that H is even but not connected. Then H is a direct sum of the even connected symmetric 2-H*-algebras H_i as above, and $\mathrm{Rep}(\mathrm{Spec}(H))$ is similarly the direct sum of the $\mathrm{Rep}(\mathrm{Spec}(H_i))$. Because the categorified Gelfand transform $\hat{\ }\colon H_i \to \mathrm{Rep}(\mathrm{Spec}(H_i))$ is an equivalence for all i, $\hat{\ }\colon H \to \mathrm{Rep}(\mathrm{Spec}(H))$ is an equivalence.

Finally we treat the general case where H is an arbitrary symmetric 2-H*-algebra. Note that if H and K are symmetric 2-H*-algebras, a symmetric 2-H*-algebra homomorphism $F\colon H \to K$ gives rise to a symmetric 2-H*-algebra homomorphism $F^\flat\colon H^\flat \to K^\flat$ between their bosonizations, where $F^\flat$ is the same as F on objects and morphisms. Note also that F is an equivalence of symmetric 2-H*-algebras if and only if $F^\flat$ is. Thus to show that $\hat{\ }\colon H \to \mathrm{Rep}(\mathrm{Spec}(H))$ is an equivalence, it suffices to show $\hat{\ }^\flat\colon H^\flat \to \mathrm{Rep}(\mathrm{Spec}(H))^\flat$ is an equivalence.

For this, note that any supergroupoid G has a *bosonization* $G^\flat$, in which the underlying compact groupoid of G is equipped with the trivial balancing $\beta = 1$. Moreover, there is a homomorphism of symmetric 2-H*-algebras

$$\mathrm{Rep}(G)^\flat \xrightarrow{X} \mathrm{Rep}(G^\flat)$$

sending any representation $F \in \mathrm{Rep}(G)^\flat$ to the representation $X(F) \in \mathrm{Rep}(G^\flat)$ given by the commutative square

$$\begin{tikzcd} G^\flat \arrow[r, "X(F)"] \arrow[d, "I"'] & \mathrm{SuperHilb} \\ G \arrow[r, "F"] & \mathrm{SuperHilb} \arrow[u, "E"'] \end{tikzcd}$$

Here $I\colon G^\flat \to G$ is the identity on the underlying groupoids, while the 2-H*-algebra homomorphism $E\colon \mathrm{SuperHilb} \to \mathrm{SuperHilb}$ maps any super-Hilbert space to the even super-Hilbert space with the same underlying Hilbert space, and acts as the identity on morphisms. One may check that $X(F)$ is really a compact supergroupoid representation. Similarly, given a morphism $\alpha\colon F \Rightarrow F'$ in $\mathrm{Rep}(G)^\flat$, we define $X(\alpha)$ to be the horizontal composite $I \circ \alpha \circ E$. In fact, X is an equivalence, for given any representation F of $G^\flat$ we can turn it back into a representation of G by equipping each Hilbert space $F(x)$, $x \in G$ with the grading $F(\beta_x)$, where β is the balancing of G.

Similarly, for any symmetric 2-H*-algebra H there is an equivalence

$$\mathrm{Spec}(H)^\flat \xrightarrow{Y} \mathrm{Spec}(H^\flat)$$

sending any object $T \in \mathrm{Spec}(H)^\flat$ to the object $Y(T) \in \mathrm{Spec}(H^\flat)$ given by the commutative square

$$\begin{tikzcd} H^\flat \arrow[r, "Y(T)"] \arrow[d, "I"'] & \mathrm{SuperHilb} \\ H \arrow[r, "T"] & \mathrm{SuperHilb} \arrow[u, "E"'] \end{tikzcd}$$

where $I\colon H^\flat \to H$ is the identity on the underlying 2-H*-algebras, while E is given as above.

We thus have equivalences

$$\mathrm{Rep}(\mathrm{Spec}(H))^\flat \xrightarrow{\sim} \mathrm{Rep}(\mathrm{Spec}(H)^\flat) \xrightarrow{\sim} \mathrm{Rep}(\mathrm{Spec}(H^\flat))$$

and their composite gives a diagram commuting up to natural isomorphism:

$$\begin{tikzcd} H^\flat \arrow[r, "\hat{\ }^\flat"] \arrow[dr, "\hat{\ }"'] & \mathrm{Rep}(\mathrm{Spec}(H))^\flat \arrow[d] \\ & \mathrm{Rep}(\mathrm{Spec}(H^\flat)) \end{tikzcd}$$

It follows that $\hat{\ }^\flat\colon H^\flat \to \mathrm{Rep}(\mathrm{Spec}(H))^\flat$ is an equivalence, as was to be shown. $\square$

Presumably what Theorem 64 is trying to tell us is that there are 2-functors Rep and Spec going both ways between the 2-category of compact supergroupoids and the 2-category of symmetric 2-H*-algebras, and that these extend to a 2-equivalence of 2-categories. We shall not try to prove this here. However, it is worth noting that for any compact supergroupoid G, there is a functor

$$\check{\ }\colon G \to \mathrm{Spec}(\mathrm{Rep}(G))$$

given as follows. For every object $x \in G$, $\hat{x}$ is the object of Spec(Rep(G)) with

$$\hat{x}(F) = F(x)$$

for all $F \in \mathrm{Rep}(G)$, and

$$\hat{x}(\alpha) = \alpha_x$$

for all $\alpha\colon F \to F'$, where $F, F' \in \mathrm{Rep}(G)$. For every morphism $g\colon x \to y$ in G, $\check{g}\colon \check{x} \Rightarrow \check{y}$ is the natural transformation with

$$\check{g}(F) = F(g)$$

for all $F \in \mathrm{Rep}(G)$. Presumably $\check{\ }\colon G \to \mathrm{Spec}(\mathrm{Rep}(G))$ is in some sense an equivalence of compact supergroupoids.

6.1 Compact abelian groups

The representation theory of compact abelian groups is rendered especially simple by the use of Fourier analysis, as generalized by Pontryagin. Suppose that T is a compact abelian group. Then its dual $\hat{T}$ is defined as the set of equivalence classes of irreducible representations ρ of T. The dual becomes a discrete abelian group with operations given as follows:

$$[\rho][\rho'] = [\rho \otimes \rho'],$$
$$[\rho]^{-1} = [\rho^*]$$

Then the Fourier transform is a unitary isomorphism

$$f\colon L^2(T) \to L^2(\hat{T})$$

given by

$$f(\chi_\rho) = \delta_{[\rho]}$$

where χ_ρ is the character of the representation ρ, and $\delta_{[\rho]}$ is the function on $\hat{T}$ which equals 1 at $[\rho]$ and 0 elsewhere.

The Fourier transform has an interesting categorification. Note that the ordinary Fourier transform has as its domain the infinite-dimensional Hilbert space $L^2(T)$, which has a basis given by the characters of irreducible representations of T. The categorified Fourier transform will have as domain the 2-Hilbert space $\mathrm{Rep}(T)$, which has a basis given by the irreducible representations themselves. (Taking the character of a representation is a form of 'decategorification'.) Similarly, just as the ordinary Fourier transform has as its codomain an infinite-dimensional Hilbert space of $\mathbb{C}$-valued functions on $\hat{T}$, the categorified Fourier transform will have as its codomain a 2-Hilbert space of Hilb-valued functions on $\hat{T}$.

More precisely, define $\mathrm{Hilb}[G]$ for any discrete group G to be the category whose objects are G-graded Hilbert spaces for which the total dimension is finite, and whose morphisms are linear maps preserving the grading. Alternatively, we can think of $\mathrm{Hilb}[G]$ as the category of hermitian vector bundles over G for which the sum of the dimensions of the fibers is finite. We may write any object $x \in \mathrm{Hilb}[G]$ as a G-tuple $\{x(g)\}_{g\in G}$ of Hilbert spaces. The category $\mathrm{Hilb}[G]$ becomes a 2-H*-algebra in an obvious way with a product modelled after the convolution product in the group algebra $\mathbb{C}[G]$:

$$(x \otimes y)(g) = \bigoplus_{\{g',g''\in G\,:\, g'g''=g\}} x(g') \otimes y(g'').$$

If G is abelian, $\mathrm{Hilb}[G]$ becomes a symmetric 2-H*-algebra.

Now suppose that T is a compact abelian group. Given any object $x \in \mathrm{Rep}(T)$, we may decompose x into subspaces corresponding to the irreducible representations of T:

$$x = \bigoplus_{g\in\hat{T}} x(g).$$

We define the *categorified Fourier transform*

$$F\colon \mathrm{Rep}(T) \to \mathrm{Hilb}[\hat{T}]$$

as follows. For any object $x \in \mathrm{Rep}(T)$, we set

$$F(x) = \{x(g)\}_{g\in\hat{T}}.$$

Moreoever, any morphism $f\colon x \to y$ in $\mathrm{Rep}(T)$ gives rise to linear maps $f(g)\colon x(g) \to y(g)$ and thus a morphism $F(f)$ in $\mathrm{Hilb}[\hat{T}]$. One can check that F is not only 2-Hilbert space morphism but actually a homomorphism of symmetric 2-H*-algebras.

This is the categorified analog of how the ordinary Fourier transform sends pointwise multiplication to convolution. Note that, in analogy to the formula

$$f(\chi_\rho) = \delta_{[\rho]}$$

satisfied by the ordinary Fourier transform, for any irreducible representation ρ of T the categorified Fourier transform $F(\rho)$ is a hermitian vector bundle that is 1-dimensional at $[\rho]$ and 0-dimensional elsewhere.

Theorem 65. *If T is a compact abelian group, the categorified Fourier transform $F\colon \mathrm{Rep}(T) \to \mathrm{Hilb}(\hat{T})$ is an equivalence of symmetric 2-H*-algebras.*

Proof - There is a homomorphism $G\colon \mathrm{Hilb}[\hat{T}] \to \mathrm{Rep}(T)$ sending each object $\{x(g)\}_{g\in\hat{T}}$ in $\mathrm{Hilb}[\hat{T}]$ to a representation of T which is a direct sum of spaces $x(g)$ transforming according to the different isomorphism classes $g \in \hat{T}$ of irreducible representations of T. One can check that FG and GF are naturally isomorphic to the identity. $\square$

6.2 Compact classical groups

The representation theory of a 'classical' compact Lie group has a different flavor from that of general compact Lie groups. The representation theory of general compact Lie groups heavily involves the notions of maximal torus, Weyl group, roots and weights. We hope to interpret this theory in terms of 2-Hilbert spaces in a future paper. However, the representation theory of a classical group can also be studied using Young diagrams [29]. This approach relies on the fact that its category of representations have simple universal properties. These universal properties can be described in the language of symmetric 2-H*-algebras, and a description along these lines represents a distilled version of the Young diagram theory.

For example, consider the group $\mathrm{U}(n)$. The fundamental representation of $\mathrm{U}(n)$ on $\mathbb{C}^n$ is the 'universal n-dimensional representation'. In other words, for a group to have a (unitary) representation on $\mathbb{C}^n$ is precisely for it to have a homomorphism to $\mathrm{U}(n)$. This universal property can also be expressed as a universal property of $\mathrm{Rep}(\mathrm{U}(n))$. Suppose that G is a compact group. Then any n-dimensional representation $y \in \mathrm{Rep}(G)$ is isomorphic to a representation of the form $\rho\colon G \to \mathrm{U}(n)$. The representation ρ gives rise to a homomorphism

$$\rho^*\colon \mathrm{Rep}(\mathrm{U}(n)) \to \mathrm{Rep}(G),$$

and letting x denote the fundamental representation of $\mathrm{U}(n)$, we have $\rho^*(x) = \rho$. Since ρ and y are isomorphic, there is a unitary 2-homomorphism from ρ^* to a homomorphism

$$F\colon \mathrm{Rep}(\mathrm{U}(n)) \to \mathrm{Rep}(G)$$

with $F(x) = y$.

In short, for any n-dimensional object $y \in \mathrm{Rep}(G)$ there is a homomorphism $F\colon \mathrm{Rep}(\mathrm{U}(n)) \to \mathrm{Rep}(G)$ of symmetric 2-H*-algebras with $F(x) = y$. On the other hand, suppose $F'\colon \mathrm{Rep}(\mathrm{U}(n)) \to \mathrm{Rep}(G)$ is any other homomorphism with $F'(x) = y$. We claim that there is a unitary 2-homomorphism from F to F'. By Corollary 58, there exists a homomorphism $\rho'\colon G \to \mathrm{U}(n)$ with a unitary 2-homomorphism from F' to ρ'^*. On the other hand, by construction there is a unitary 2-homomorphism from F to ρ^* for some $\rho\colon G \to \mathrm{U}(n)$. To show there is a unitary 2-homomorphism from F to F', it thus suffices to show that ρ and ρ' are isomorphic in $\mathrm{Rep}(G)$. This holds because $\rho \cong y = F'(x) \cong \rho'^*(x) = \rho'$.

Now, since any connected even symmetric 2-H*-algebra is unitarily equivalent to $\mathrm{Rep}(G)$ for some compact G by Theorem 57, we may restate these results as follows. Suppose H is a connected even symmetric 2-H*-algebra and let y be an n-dimensional object of H. Then there exists a homomorphism $F\colon \mathrm{Rep}(\mathrm{U}(n)) \to H$ with $F(x) = y$. Moreover, this is unique up to a unitary 2-homomorphism. Furthermore, we can drop the assumption that H is even by working with the full subcategory whose objects are all the even objects of H.

We may thus state the universal property of $\mathrm{Rep}(\mathrm{U}(n))$ as follows:

Theorem 66. $\mathrm{Rep}(\mathrm{U}(n))$ *is the free connected symmetric 2-H*-algebra on an even object x of dimension n. That is, given any even n-dimensional object y of a connected symmetric 2-H*-algebra H, there exists a homomorphism of symmetric 2-H*-algebras $F\colon \mathrm{Rep}(U(n)) \to H$ with $F(x) = y$, and F is unique up to a unitary 2-homomorphism.*

Let $\Lambda^n x$ denote the cokernel of $p_A\colon x^{\otimes n} \to x^{\otimes n}$ (complete antisymmetrization), and let $S^n x$ denote the cokernel of $p_S\colon x^{\otimes n} \to x^{\otimes n}$ (complete symmetrization). We can describe the category of representations of $\mathrm{SU}(n)$ as follows:

Theorem 67. $\mathrm{Rep}(\mathrm{SU}(n))$ *is the free connected symmetric 2-H*-algebra on an even object x with $\Lambda^n x \cong 1$.*

Proof - Suppose that G is a compact group and the object $y \in \mathrm{Rep}(G)$ has $\Lambda^n y \cong 1$. It follows that y is n-dimensional by the computation in Proposition 52, and the isomorphism $\Lambda^n y \cong 1$ determines a G-invariant volume form on the representation y. Thus y is isomorphic to a representation of the form $\rho\colon G \to \mathrm{SU}(n)$. The rest of the proof follows that of Theorem 66. □

Here we can see in a simple context how our theory is a distillation of the theory of Young diagrams. (The Young diagram approach to representation theory is more familiar for $\mathrm{SU}(n)$ than for $\mathrm{U}(n)$.) In heuristic terms, the above theorem says that every representation of $\mathrm{SU}(n)$ is generated from the fundamental representation x using the operations present in a symmetric 2-H*-algebra — the $*$-structure, direct sums, cokernels, tensor products, duals, and the symmetry — with no relations other than those implied by the axioms for a connected symmetric 2-H*-algebra and the fact that x is even and $\Lambda^n x \cong 1$. The theory of Young diagrams makes this explicit by listing the irreducible representations of $\mathrm{SU}(n)$ in terms of minimal projections $p: x^{\otimes x} \to x^{\otimes k}$, or in other words, Young diagrams with k boxes. The symmetric 2-H*-algebra of representations of a subgroup $G \subset \mathrm{SU}(n)$, such as $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$, is a quotient of $\mathrm{Rep}(\mathrm{SU}(n))$. We may describe this quotienting process by giving extra relations as in Theorems 69 and 70 below. These extra relations give identities saying that different Young diagrams correspond to the same representation of G.

The classical groups $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$ are related to the concept of self-duality. Given adjunctions (x, x^*, i_x, e_x) and (y, y^*, i_y, e_y) in a monoidal category C, for any morphism $f: x \to y$ there is a morphism $f^\dagger: y^* \to x^*$, given in the strict case by the composite:

$$y^* = y^* \otimes 1 \xrightarrow{y^* \otimes i_x} y^* \otimes x \otimes x^* \xrightarrow{y^* \otimes f \otimes x^*} y^* \otimes y \otimes x^* \xrightarrow{e_y \otimes x^*} 1 \otimes x^* = x^*$$

(Our notation here differs from that of HDA0.) Since the left dual of an object in a 2-Hilbert space is also its right dual as in Proposition 40, given a morphism $f: x \to x^*$ we obtain another morphism $f^\dagger: x \to x^*$. Using this we may describe $\mathrm{Rep}(\mathrm{O}(n))$ and $\mathrm{Rep}(\mathrm{Sp}(n))$ as certain 'free connected symmetric 2-H*-algebras on one self-dual object':

Theorem 68. $\mathrm{Rep}(\mathrm{O}(n))$ *is the free connected symmetric 2-H*-algebra on an even object x of dimension n with an isomorphism $f: x \to x^*$ such that $f^\dagger = f$.*

Proof - Suppose that G is a compact group and the object $y \in \mathrm{Rep}(G)$ is n-dimensional and equipped with an isomorphism $f: x \to x^*$ with $f^\dagger = f$. Then there is a nondegenerate pairing $F: y \otimes y \to 1$ given by $F = (y \otimes f) i_y^*$. A calculation, given in the proof of Proposition 71, shows that F is symmetric. It follows that y is isomorphic to a representation of the form $\rho: G \to \mathrm{O}(n)$. The rest of the proof follows that of Theorem 66. $\square$

Theorem 69. $\mathrm{Rep}(\mathrm{Sp}(n))$ *is the free connected symmetric 2-H*-algebra on one even object x with $\Lambda^n x \cong 1$ and with an isomorphism $f: x \to x^*$ such that $f^\dagger = -f$.*

Proof - The proof is analogous to that of Theorem 68, except that the pairing F is skew-symmetric. $\square$

Following the proof of Theorem 67 we may also characterize $\mathrm{Rep}(\mathrm{SO}(n))$ as follows:

Theorem 70. $\mathrm{Rep}(\mathrm{SO}(n))$ *is the free connected symmetric 2-H*-algebra on an even object x with $\Lambda^n x \cong 1$ and with an isomorphism $f: x \to x^*$ such that $f^\dagger = f$.*

The conditions on the isomorphism $f: x \to x^*$ in Theorems 68 and 69 are quite reasonable, in the following sense:

Proposition 71. *Suppose that x is a simple object in a symmetric 2-H^*-algebra and that x is isomorphic to x^*. Then one and only one of the following is true: either there is an isomorphism $f: x \to x^*$ with $f = f^\dagger$, or there is an isomorphism $f: x \to x^*$ with $f = -f^\dagger$.*

Proof - Note that there is an isomorphism of complex vector spaces

$$\begin{array}{rcl} \hom(x, x^*) & \cong & \hom(x \otimes x, 1) \\ f & \mapsto & (1 \otimes f) i_x^*, \end{array}$$

and note that

$$\hom(x \otimes x, 1) \cong \hom(S^2 x, 1) \oplus \hom(\Lambda^2 x, 1).$$

Suppose $f: x \to x^*$ is an isomorphism and let $F = (1 \otimes f) i_x^*$. Since x is simple, f and thus F is unique up to a scalar multiple, so F must lie either in $\hom(S^2 x, 1)$ or $\hom(\Lambda^2 x, 1)$. In other words, $B_{x,x} F = \pm F$. Choose a well-balanced adjunction for x. Assuming without loss of generality that H is strict, we have

$$\begin{array}{rcl} f^\dagger & = & (x \otimes i_x)(x \otimes f \otimes x^*)(i_x^* \otimes x^*) \\ & = & (x \otimes i_x)(F \otimes x^*) \\ & = & \pm (x \otimes i_x)(B_{x,x} F \otimes x^*) \\ & = & \pm (f \otimes i_x)(B_{x^*,x} \otimes x^*)(i_x^* \otimes x^*) \\ & = & \pm f b_{x^*} \end{array}$$

Since $b_{x^*} = \pm 1_{x^*}$ depending on whether x, and thus x^*, is even or odd, we have $f^\dagger = \pm f$. $\Box$

This result is well-known if H is a category of compact group representations [13]. Here one may also think of the morphism $f: x \to x^*$ as a conjugate-linear intertwining operator $j: x \to x$. The condition that $f = \pm f^\dagger$ is then equivalent to the condition that $j^2 = \pm 1_x$. One says that x is a real representation if $j^2 = 1_x$ and a quaternionic representation if $j^2 = -1_x$, establishing the useful correspondence:

real : complex : quaternionic :: orthogonal : unitary : symplectic

The following alternate characterization of Rep(U(1)) is interesting because it emphasizes the relation between duals and inverses. Whenever T is a compact abelian group and $x \in \mathrm{Rep}(T)$, the dual x^* is also the inverse of x, in the sense that $x \otimes x^* \cong 1$. We have:

Theorem 72. Rep(U(1)) *is the free connected symmetric 2-H^*-algebra on an even object x with $x \otimes x^* \cong 1$.*

Proof - By Theorem 66 it suffices to show that an object x in a connected symmetric 2-H*-algebra is 1-dimensional if and only if $x \otimes x^* \cong 1$. On the one hand, by the multiplicativity of dimension, $x \otimes x^* \cong 1$ implies that $\dim(x) = 1$. On the other hand, suppose $\dim(x) = 1$. Then we claim $i_x : 1 \to x \otimes x^*$ and $i_x^* : x \otimes x^* \to 1$ are inverses. First, $i_x{}^* i_x$ is the identity since $\dim(x) = 1$. Second, $i_x{}^* i_x \in \mathrm{end}(x \otimes x^*)$ is idempotent since $\dim(x) = 1$. Since $x \otimes x^*$ is 1-dimensional, it is simple (by the additivity of dimension), so $i_x{}^* i_x$ must be the identity. $\square$

Finally, it is interesting to note that SuperHilb is the free connected symmetric 2-H*-algebra on an odd object x with $x \otimes x \cong 1$. This object x is the one-dimensional odd super-Hilbert space.

7 Conclusions

The reader will have noted that some of our results are slight reworkings of those in the literature. One advantage of our approach is that it immediately suggests generalizations to arbitrary n. While the general study of n-Hilbert spaces will require a deeper understanding of n-category theory, we expect many of the same themes to be of interest. With this in mind, let us point out some problems with what we have done so far.

One problem concerns the definition of the quantum-theoretic hierarchy. A monoid is a essentially a category with one object. More precisely, a category with one object x can be reconstructed from the monoid $\mathrm{end}(x)$, and up to isomorphism every monoid comes from a one-object category in this way. Comparing Figures 1 and 2, one might at first hope that by analogy an H*-algebra would be a one-dimensional 2-Hilbert space. Unfortunately, the way we have set things up, this is not the case.

If H is a one-dimensional 2-Hilbert space with basis given by the object x, then $\mathrm{end}(x)$ is an H*-algebra. However, $\mathrm{end}(x)$ is always isomorphic to $\mathbb{C}$; one does not get any other H*-algebras this way. The reason appears to be the requirement that a 2-Hilbert space has cokernels, so that if $\mathrm{end}(x)$ has nontrivial idempotents, x has subobjects. If we dropped this clause in the definition of a 2-Hilbert space, there would be a correspondence between H*-algebras and 2-Hilbert spaces, all of whose objects are direct sums of a single object x. Perhaps in the long run it will be worthwhile to modify the definition of 2-Hilbert space in this way. On the other hand, an H*-algebra is also an H*-category with one object. An H*-category has sums and differences of morphisms, but not of objects, i.e., it need not have direct sums and cokernels. Perhaps, therefore, a k-tuply monoidal n-Hilbert space should really be some sort of '$(n+k)$-H*-category' with one j-morphism for $j < k$, and sums and differences of j-morphisms for $j \geq k$.

A second problem concerns the program of getting an invariant of n-tangles in $(n + k)$-dimensions from an object in a k-tuply monoidal n-Hilbert space. Let us recall what is known so far here.

Oriented tangles in 2 dimensions are the morphisms in a monoidal category with duals, $C_{1,1}$. Here by 'monoidal category with duals' we mean a monoidal $*$-category in which every object has a left dual, the tensor product is a $*$-functor, and the associator is a unitary natural transformation. Suppose that X is any other monoidal category with duals, e.g. a 2-H*-algebra. Then any adjunction (x, x^*, i, e) in C uniquely determines a monoidal $*$-functor $F: C_{1,1} \to H$ up to monoidal unitary natural isomorphism. The functor F is determined by the requirement that it maps the positively oriented point to x, the negatively oriented point to x^*, and the appropriately oriented 'cup' and 'cap' tangles to e and i.

According to our philosophy we would prefer F to be determined by an object $x \in X$ rather than an adjunction. However F is not determined up to natural transformation by requiring that it map the positively oriented point to x. For example, take $X = \mathrm{Hilb}$ and let $x \in X$ be any object. We may let F send the negatively oriented point to the dual Hilbert space x^*, and send the cap and cup to the standard linear maps $e: x^* \otimes x \to \mathbb{C}$ and $i: \mathbb{C} \to x \otimes x^*$. Then $F(ii^*) = \dim(x) 1_x$. Alternatively we may let F send the cap and cup to $e' = \alpha^{-1} e$ and $i' = \alpha i$ for any nonzero $\alpha \in \mathbb{C}$. Then $F(ii^*) = |\alpha|^2 \dim(x) 1_x$. The problem is that while adjunctions in X are unique up to unique isomorphism, the isomorphism is not necessarily unitary.

In HDA0 we outlined a way to deal with this problem by 'strictifying' the notion of a monoidal category with duals. Roughly speaking, this amounts to equipping each object with a choice of left adjunction, and requiring the functor $F: C_{1,1} \to X$ to preserve this choice. Then F is determined up to monoidal unitary natural transformation by the requirement that it map the positively oriented point to a particular object $x \in X$. In this paper we have attempted to take the 'weak' rather than the 'strict' approach. Our point here is that the weak approach seems to make it more difficult to formulate the sense in which $C_{1,1}$ is the 'free' monoidal category with duals on one object.

In higher dimensions the balancing plays an interesting role in this issue. Framed oriented tangles in 3 dimensions form a braided monoidal category $C_{1,2}$ with duals. Here by 'braided monoidal category with duals' we mean a monoidal category with duals which is also braided, such that the braiding is unitary and every object x has a well-balanced adjunction (x, x^*, i, e). For any object x in a braided monoidal category X with duals, there is a braided monoidal $*$-functor $F: C_{1,2} \to X$ sending the positively oriented point to x. Moreover, because well-balanced adjunctions are unique up to unique *unitary* isomorphism, F is unique up to monoidal unitary natural isomorphism. This gives a sense in which $C_{1,2}$ is the free braided monoidal category with duals on one object.

Similarly, framed oriented tangles in 4 dimensions form a symmetric monoidal category with duals $C_{1,3}$, i.e., a braided monoidal category with duals for which the braiding is a symmetry. Again, for any object x in a symmetric monoidal category X with duals, there is a symmetric monoidal $*$-functor $F: C_{1,3} \to X$ sending the positively oriented point to x, and F is unique up to monoidal unitary natural isomorphism. (For an alternative 'strict' approach to the 3- and 4-dimensional cases, see HDA0.)

In short, we need to understand the notion of k-tuply monoidal n-Hilbert spaces more deeply, as well as the notion of 'free' k-tuply monoidal n-categories with duals.

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