Categorified Symplectic Geometry
and the Classical String
DRAFT VERSION ONLY

John C. Baez, Alexander E. Hoffnung, Christopher L. Rogers

Department of Mathematics, University of California,
Riverside, California 92521, USA

June 6, 2007

Abstract

A Lie 2-algebra is a 'categorified' version of a Lie algebra: that is, a category equipped with structures similar to those of a Lie algebra, but where the usual laws hold only up to isomorphism. It is well known that given a manifold equipped with a symplectic 2-form, the Poisson bracket gives rise to a Lie algebra of observables. Multisymplectic geometry generalizes the classical mechanics of point particles to n-dimensional field theories, decribing such a theory in terms of a 'phase space' that is a manifold equipped with a closed nondegenerate $(n+1)$-form. Here, given a manifold with a closed nondegenerate 3-form, we construct a Lie 2-algebra of observables. We then describe how this Lie 2-algebra can be used to describe dynamics in classical bosonic string theory.

1 Introduction

It is becoming clear that string theory can be viewed as a 'categorification' of particle physics, in which familiar algebraic and geometrical structures based in set theory are replaced by their category–theoretic analogues. The basic idea is simple. While a classical particle has a position nicely modelled by an element of a set, namely a point in space:

the position of a classical string is better modelled by a morphism in a category, namely an unparametrized path in space:

Similarly, while particles have worldlines in spacetime, which can be thought of as morphisms, strings have worldsheets, which can be thought of as 2-morphisms.

So far this viewpoint has been most fruitful in studying the string-theoretic generalizations of gauge theory [8]. The first clue was the B field in string theory. The electromagnetic field contributes to the change in phase of a charged particle as it moves through spacetime. This field is locally described by a 1-form A, which we integrate along the particle's worldline to compute a phase change. The B field contributes to the phase change of a charged string in a similar way: it is locally described by a 2-form, which we integrate over a string's worldsheet. When we seek a global description suitable for nontrivial spacetime topologies, the electromagnetic field is better thought of as a connection on a U(1) bundle. Similarly, the B-field is globally described by a connection on the categorified version of a U(1) bundle, namely a U(1) gerbe [11, 29].

Later it was found that connections on nonabelian gerbes also play a role in string theory [1, 2, 10]. Nonabelian gerbes are a special case of 2-bundles: that is, bundles with a smooth category rather than smooth manifold as fiber [9]. To understand connections on general 2-bundles, it was necessary to categorify the concepts of Lie group and Lie algebra, obtaining the notions of Lie 2-group [7, 6] and Lie 2-algebra [5, 23].

Still more recently, *iterated* categorification has become important in understanding the generalizations of gauge theory suitable for higher-dimensional membranes [24, 25]. It is clear by now that to understand the behavior of higher-dimensional membranes, we need to study n-connections on n-bundles: that is, structures analogous to connections that live on things like bundles with a smooth n-category as fiber. In the very simplest case — a topologically trivial n-bundle with the simplest nontrivial abelian 'n-group' playing the role of gauge group — an n-connection is just an n-form on the base space. In a straightforward generalization of electromagnetism, the integral of this n-form over the membrane's 'worldvolume' — that is, the surface it sweeps out in spacetime — constributes to its change in phase as it sweeps out this surface.

Given all this, we should expect that as we look deeper into the analogy between point particles, strings, and higher-dimensional membranes, we should find more examples of categorification. Perhaps the most obvious place to look is symplectic geometry. The reason is that symplectic geometry *also* uses a connection on a U(1) bundle to describe the change of phase of a point particle.

The simplest example is a free particle moving in some Lorentzian manifold M representing spacetime. If we keep track of the particle's momentum as well as its position, it traces out a path in the cotangent bundle $X = T^*M$. The cotangent bundle is equipped with a canonical 1-form α, and we can integrate α over this path to determine the particle's change of phase. This is not the historical reason why X is called a 'phase space', but the coincidence is a happy one.

The exterior derivative $\omega = d\alpha$ plays an important role in this story. First, by Stokes' theorem, the integral of this 2-form over any disc in X measures the change of phase of a particle as it moves around the boundary of the disc.

A deeper fact is that ω is a symplectic structure: that is, not only closed but nondegenerate. This lets us take any smooth function $F\colon X \to \mathbb{R}$ and find a unique vector field v_F such that

$$\iota_{v_F}\omega = dF$$

where ι stands for interior product. We should think of F as an 'observable' assigning a number to any state of the particle. In good situations, the vector field v_F will generate a one-parameter group of symmetries of X (that is, a flow). So, the symplectic nature of ω guarantees that *observables give rise to symmetries*. Moreover, by measuring how rapidly one observable changes under the one-parameter group of symmetries generated by another, we obtain a binary operation on observables, the Poisson bracket:

$$\{F, G\} = L_{v_F} G$$

where L stands for Lie derivative. This makes the space of observables into a Lie algebra.

Symplectic geometry generalizes this idea by replacing T^*M with a more general phase space X. We could simply let X be any manifold equipped with a 1-form α such that $\omega = d\alpha$ is symplectic. However, a 1-form is the same as a connection on a trivial U(1) bundle, and ω is then the curvature of this connection. Since physics is local, it makes more sense to equip X with a *locally* trivial U(1) bundle $P \to X$, together with a connection on P whose curvature 2-form ω is symplectic. This is the basic context for geometric quantization.

We can study symplectic geometry without assuming that the symplectic 2-form ω is the curvature of a connection on some U(1) bundle. In particular, we still obtain a Lie algebra of observables using the formulas above. But, some of the physical meaning of the symplectic structure is lost — namely, that the integral of ω over any disc in X measures the change of phase of a particle as it moves around the boundary of this disc. So, in geometric quantization the U(1) bundle is crucial; it will exist whenever ω defines an *integral* element of $H^2(X, \mathbb{R})$.

Now let us consider how all this generalizes when we move from point particles to strings.

As a first step towards understanding this, let us return to the point particle moving in a spacetime manifold M. We have said that the particle's phase changes in a way described by integrating the canonical 1-form α along its path in T^*M. However, in the presence of the electromagnetic field there is an additional phase change due to electromagnetism, at least when the particle is charged. To take this into account, we add to α the 1-form A describing the electromagnetic field, pulled back from M to T^*M. We then redefine the symplectic structure to be $\omega = d(\alpha + A)$. So, *electromagnetism affects the symplectic structure on the cotangent bundle of spacetime*. A more detailed account of this can be found in the book by Guillemin and Sternberg [17].

This suggests that when we pass from point particles to strings, and the electromagnetic field is replaced by the B field, we should correspondingly adjust our concept of 'symplectic structure'. Instead of a canonical 1-form, we should have some sort of canonical 2-form on phase space, so we can add the B field to this 2-form. But this in turn suggests that the analogue of the symplectic structure will be a 3-form!

This raises the puzzle: *how can we generalize symplectic geometry with a 3-form replacing the usual 2-form?*

Amusingly, the answer goes back to the work of DeDonder [14] and Weyl [27] in the 1930s. Their ideas have been more fully developed in the subject called 'multisymplectic geometry'. For an introduction, try for example the papers by Gotay, Isenberg, Marsden and Montgomery [16], Hélein and Kouneiher [18, 19], Kijowski [21], and Rovelli [22]. In particular, Gotay *et al* have already applied multisymplectic geometry to classical string theory. There are various way to do this. In this introduction we take a very naive approach, which will be corrected in Section 4.

To begin with, note that just as the position and velocity of a point particle in the spacetime M are given by a point in the tangent bundle TM, we could try to describe the position and velocity of a string by a point in $\Lambda^2 TM$ — that is, a point in M together with a tangent *bivector*. Similarly, just as the position and momentum of a particle are given by a point in T^*M, we could try to describe the position and momentum of a string by a point in $\Lambda^2 T^*M$.

Just as T^*M is equipped with a canonical 1-form, the generalized phase space $X = \Lambda^2 T^*M$ is equipped with a canonical 2-form α, as described in Example 2.2 below. The corresponding 3-form $\omega = d\alpha$ is 'multisymplectic', meaning that it is closed and also nondegenerate in the following sense:

$$\iota_v \omega = 0 \Rightarrow v = 0$$

for all vector fields v. This means that for any 1-form F, there is at most one vector field v_F such that

$$\iota_{v_F} \omega = dF.$$

This resembles the equation we have already seen in symplectic geometry, which associates symmetries to observables. But there is a difference: now v_F may not exist. So, it seems we should consider a 1-form F on X to be an observable only when there exists a vector field v_F satisfying the above equation.

We can then define a Poisson bracket of observables by the usual formula:

$$\{F, G\} = L_{v_F} G.$$

The result is always another observable. But, we do not obtain a Lie algebra of observables, because this Poisson bracket is only antisymmetric *up to an exact 1-form.* Exact 1-forms are always observables, but they play a special role, since they give rise to trivial symmetries: if F is exact, $v_F = 0$.

This suggests that in the stringy analogue of symplectic geometry we should seek, not a Lie algebra of observables, but a *Lie 2-algebra* of observables — that is, a category resembling a Lie algebra, with observables as objects. In this category two observables F and G will be deemed 'isomorphic' if they differ by an exact 1-form. This guarantees that they generate the same symmetries: $v_F = v_G$.

Indeed, such a Lie 2-algebra exists. In Thm. 3.3, we prove that for any manifold X equipped with a closed nondegenerate 3-form ω, there is a Lie 2-algebra for which:

- An object is a 1-form F on X for which there exists a vector field v_F with $\iota_{v_F}\omega = dF$.
- A morphism from F to G is a function f such that $F + df = G$.
- The bracket of objects F, G is $L_{v_F} G$.
- The bracket of morphisms vanishes.

On a more technical note, this Lie algebra is 'hemistrict' in the sense of Roytenberg [23]. This means that the Jacobi identity holds on the nose, but the skew-symmetry of the bracket holds only up to isomorphism. In Thm. 3.4 we construct another Lie 2-algebra with the same objects and morphisms, where the Lie bracket of observables is given instead by $\iota_{v_F}\iota_{v_G}\omega$. This Lie algebra is 'semistrict', meaning that the bracket is skew-symmetric, but the Jacobi identity holds only up to isomorphism. In Thm. 3.6 we show that these two Lie 2-algebras are isomorphic. This may seem surprising at first, but the notion of 'isomorphism' for Lie 2-algebra is sufficiently supple that superficially different Lie 2-algebras — one hemistrict, one semistrict — can be isomorphic.

In Section 4, we apply these ideas to the classical bosonic string propagating in Minkowski spacetime. This involves replacing the naive phase space $\Lambda^2 T^*M$ described above with something more sophisticated. Our treatment here is brief and we plan to expand it; in particular, we would like to include a nonzero B field. Finally, we list some open questions in Section 5.

2 Multisymplectic Geometry

The idea of multisymplectic geometry is simple and beautiful: associated to any n-dimensional classical field theory there is a finite-dimensional 'extended phase space' X equipped with a nondegenerate closed $(n+1)$-form ω. When $n = 1$, we are back to the classical mechanics of point particles and ordinary symplectic geometry. When $n = 2$, the examples include classical bosonic string theory, as explained in Section 4.

However, at this point an annoying terminological question intrudes: what do we call multisymplectic geometry for a fixed value of n? The obvious choice is 'n-symplectic geometry', but unfortunately, this term already means something else [13]. So, until a better choice comes along, we will use the term 'n-plectic geometry':

Definition 2.1. *An $(n+1)$-form ω on a C^∞ manifold X is* **multisymplectic**, *or more specificially an* **n-plectic structure**, *if it is both closed:*

$$d\omega = 0, \tag{1}$$

and nondegenerate:

$$\forall v \in T_x X, \ \iota_v \omega = 0 \Rightarrow v = 0 \tag{2}$$

where we use $\iota_v \omega$ *to stand for the interior product* $\omega(v, \cdot, \ldots, \cdot)$*. If* ω *is an* n*-plectic form on* X *we call the pair* (X, ω) *a* **multisymplectic manifold***, or* $\boldsymbol{n}$**-plectic manifold***.*

The references already provided contain many examples of multisymplectic manifolds. More examples, together with constraints on which manifolds can admit n-plectic structures, have been discussed by Cantrijn *et al* [12] and Ibort [20]. Here we give just two examples. Section 4 contains a more substantial one.

Example 2.2. *If* M *is a smooth manifold, let* $X = \Lambda^n T^* M$ *be the nth exterior power of the cotangent bundle of* M*. Then there is a canonical* n*-form* α *on* X *given as follows:*

$$\alpha(v_1, \ldots, v_n) = x(d\pi(v_1), \ldots, d\pi(v_n))$$

where $v_1, \ldots v_n$ *are tangent vectors at the point* $x \in X$*, and* $\pi \colon X \to M$ *is the projection from the bundle* X *to the base space* M*. Note that in this formula we are applying the* n*-form* $x \in \Lambda^n T^* M$ *to the* n*-tuple of tangent vectors* $d\pi(v_i)$ *at the point* $\pi(x)$*. The* $(n+1)$*-form*

$$\omega = d\alpha$$

is n*-plectic.*

Example 2.3. *If* G *is a compact simple Lie group, there is a 3-form* ω *on* G*, invariant under both left and right translations, defined by*

$$\omega(v_1, v_2, v_3) = \langle v_1, [v_2, v_3] \rangle$$

when v_i *are tangent vectors at the identity of* G *(that is, elements of the Lie algebra). This makes* (G, ω) *into a 2-plectic manifold.*

Example 2.3 is important in theory relating the Wess–Zumino–Witten model to loop groups and 2-groups [6]. In general it seems that the loop space of an n-plectic manifold is an *infinite-dimensional* $(n-1)$-plectic manifold, but we will not consider such infinite-dimensional examples here.

Next we quickly review how to generalize Hamiltonian mechanics with an n-plectic manifold playing the role of a phase space. Ordinary Hamiltonian mechanics corresponds to 1-plectic geometry, and in this case, observables are functions on phase space. In n-plectic geometry, observables will be $(n-1)$-forms — but not all $(n-1)$-forms, only certain 'Hamiltonian' ones:

Definition 2.4. *Let* (X, ω) *be an* n*-plectic manifold. An* $(n-1)$*-form* F *on* X *is* **Hamiltonian** *if there exists a vector field* v_F *on* X *such that*

$$dF = \iota_{v_F} \omega. \tag{3}$$

We say v_F *is the* **Hamiltonian vector field** *corresponding to* F*. The set of Hamiltonian* $(n-1)$ *forms on a multisymplectic manifold is a vector space and is denoted as* $\mathrm{Ham}\,(X)$*.*

The Hamiltonian vector field v_F is unique if it exists. However, except for the familiar case $n = 1$, there may be $(n-1)$-forms F having no Hamiltonian vector field. The reason is given an n-plectic form ω on X, this map:

$$\begin{array}{ccc} T_x X & \to & \Lambda^n T_x^* X \\ v & \mapsto & i_v \omega \end{array}$$

is one-to-one, but not necessarily onto unless $n = 1$.

The following proposition generalizes Liouville's Theorem:

Proposition 2.5. *If $F \in \mathrm{Ham}\,(X)$, then the Lie derivative $L_{\,v_F}\,\omega$ is zero.*

Proof. Since ω is closed, $L_{\,v_F}\,\omega = d\iota_{v_F}\omega = ddF = 0$. □

We can define a Poisson bracket of Hamiltonian $(n-1)$-forms in two ways:

Definition 2.6. *Given $F, G \in \mathrm{Ham}\,(X)$, the* **hemi-bracket** $\{F, G\}_{\mathrm{h}}$ *is the $(n-1)$-form given by*

$$\{F, G\}_{\mathrm{h}} = L_{\,v_F}\, G.$$

Definition 2.7. *Given $F, G \in \mathrm{Ham}\,(X)$, the* **semi-bracket** $\{F, G\}_{\mathrm{s}}$ *is the $(n-1)$-form given by*

$$\{F, G\}_{\mathrm{s}} = \iota_{v_F} \iota_{v_G} \omega.$$

The two brackets agree in the familiar case $n = 1$, but in general they differ by an exact form:

Proposition 2.8. *Given $F, G \in \mathrm{Ham}\,(X)$,*

$$\{F, G\}_{\mathrm{h}} = \{F, G\}_{\mathrm{s}} + d\iota_{v_F} G.$$

Proof. Since $L_v = \iota_v d + d\iota_v$,

$$\{F, G\}_{\mathrm{h}} = L_{\,v_F}\, G = \iota_{v_F} dG + d\iota_{v_F} G = \iota_{v_F} \iota_{v_G} \omega + d\iota_{v_F} G = \{F, F\}_{\mathrm{s}} + d\iota_{v_F} G.$$

□

Both brackets have nice properties:

Proposition 2.9. *Let $F, G, H \in \mathrm{Ham}\,(X)$ and let v_F, v_G, v_H be the respective Hamiltonian vector fields. The hemi-bracket $\{\cdot,\cdot\}_{\mathrm{h}}$ has the following properties:*

1. *The bracket of Hamiltonian forms is Hamiltonian:*

$$d\,\{F, G\}_{\mathrm{h}} = \iota_{v_{[v_F, v_G]}} \omega. \qquad (4)$$

so in particular we have

$$v_{\{F,G\}_{\mathrm{h}}} = [v_F, v_G].$$

2. *The bracket is antisymmetric up to an exact form:*

$$\{F, G\}_{\mathrm{h}} + dS_{F,G} = -\{G, F\}_{\mathrm{h}} \tag{5}$$

with $S_{F,G} = -(\iota_{v_F} G + \iota_{v_G} F)$.

3. *The bracket satisfies the Jacobi identity:*

$$\{F, \{G, H\}_{\mathrm{h}}\}_{\mathrm{h}} = \{\{F, G\}_{\mathrm{h}}, H\}_{\mathrm{h}} + \{G, \{F, H\}_{\mathrm{h}}\}_{\mathrm{h}}. \tag{6}$$

Proof. 1. If $F, G \in \mathrm{Ham}(X)$, then $d\{F, G\}_{\mathrm{h}} = \iota_{v_{[v_F, v_G]}}\omega + \iota_{v_G} L_{v_F}\omega$, by the identities relating the Lie derivative, exterior derivative, and interior product. Prop. 2.5 then gives the desired result.

2. Rewriting the Lie derivative in terms of d and ι gives

$$\begin{aligned} \{F, G\}_{\mathrm{h}} + \{G, F\}_{\mathrm{h}} &= \iota_{v_F}\iota_{v_G}\omega + \iota_{v_G}\iota_{v_F}\omega + d(\iota_{v_F} G + \iota_{v_G} F) \\ &= -dS_{F,G}. \end{aligned}$$

3. The definition of the bracket and property 1. give

$$\begin{aligned} \{\{F, G\}_{\mathrm{h}}, H\}_{\mathrm{h}} + \{G, \{F, H\}_{\mathrm{h}}\}_{\mathrm{h}} &= L_{v_{[v_F, v_G]}} H + L_{v_G} L_{v_F} H \\ &= L_{v_F} L_{v_G} H \\ &= \{F, \{G, H\}_{\mathrm{h}}\}_{\mathrm{h}}. \end{aligned}$$

□

Proposition 2.10. *Let* $F, G, H \in \mathrm{Ham}(X)$ *and let* v_F, v_G, v_H *be the respective Hamiltonian vector fields. The semi-bracket* $\{\cdot, \cdot\}_{\mathrm{s}}$ *has the following properties:*

1. *The bracket of Hamiltonian forms is Hamiltonian:*

$$d\{F, G\}_{\mathrm{s}} = \iota_{v_{[v_F, v_G]}}\omega. \tag{7}$$

so in particular we have

$$v_{\{F,G\}_{\mathrm{h}}} = [v_F, v_G].$$

2. *The bracket is antisymmetric:*

$$\{F, G\}_{\mathrm{s}} = -\{G, F\}_{\mathrm{s}} \tag{8}$$

3. *The bracket satisfies the Jacobi identity up to an exact form:*

$$\{F, \{G, H\}_{\mathrm{s}}\}_{\mathrm{s}} + dJ(F, G, H) = \{\{F, G\}_{\mathrm{s}}, H\}_{\mathrm{s}} + \{G, \{F, H\}_{\mathrm{s}}\}_{\mathrm{s}} \tag{9}$$

with $J(F, G, H) = -\iota_{v_F}\iota_{v_G}\iota_{v_H}\omega$.

Proof. 1. Eq. 2.8 and Prop. 2.9 imply $d\{F,G\}_{\mathrm{s}} = \iota_{v_{[F,G]}}\omega$.

2. The conclusion follows from the anti-symmetry of ω.

3. Eq. 2.8 implies

$$\{F,\{G,H\}_{\mathrm{s}}\}_{\mathrm{s}} = \{F,\{G,H\}_{\mathrm{h}}\}_{\mathrm{h}} - d\left(\iota_{v_F}\{G,H\}_{\mathrm{h}}\right),$$

$$\{\{F,G\}_{\mathrm{s}},H\}_{\mathrm{s}} = \{\{F,G\}_{\mathrm{h}},H\}_{\mathrm{h}} - d\left(\iota_{v_{[F,G]}}H\right),$$

and

$$\{G,\{F,H\}_{\mathrm{s}}\}_{\mathrm{s}} = \{G,\{F,H\}_{\mathrm{h}}\}_{\mathrm{h}} - d\left(\iota_{v_G}\{F,G\}_{\mathrm{h}}\right).$$

The identity $L_v = i_v d + d i_v$ then gives

$$d\left(\iota_{v_{[v_F,v_G]}}H\right) = L_{\,v_F}\, d\iota_{v_G}H - d\iota_{v_G}\{F,H\}_{\mathrm{h}}$$

which implies

$$\{\{F,G\}_{\mathrm{s}},H\}_{\mathrm{s}} + \{G,\{F,H\}_{\mathrm{s}}\}_{\mathrm{s}} = \{\{F,G\}_{\mathrm{h}},H\}_{\mathrm{h}} + \{G,\{F,H\}_{\mathrm{h}}\}_{\mathrm{h}} + L_{\,v_F}\, d\iota_{v_G}H.$$

Prop. 2.9 and the identities

$$L_{\,v_F}\, d\iota_{v_G}H = d\left(\iota_{v_F}\iota_{v_G}H\right)$$

and

$$d\left(\iota_{v_F}\{G,H\}_{\mathrm{h}}\right) = d\left(\iota_{v_F}\iota_{v_G}H\right) + d\left(\iota_{v_F}\iota_{v_G}dH\right)$$

then yield the desired result. □

In general, neither the hemi-bracket nor the semi-bracket makes $\mathrm{Ham}\,(X)$ into a Lie algebra, since each one satisfies one of the Lie algebra laws only *up to an exact* $(n-1)$*-form*. The exception is $n = 1$, the case of ordinary Hamiltonian mechanics. In this case both brackets equal the usual Poisson bracket. In what follows we consider the case $n = 2$.

3 Lie 2-Algebras

We begin with a quick review of the fully general Lie 2-algebras defined by Roytenberg [23]. It will be efficient to work with these using the language of chain complexes. A Lie 2-algebra is a category equipped with structures analogous to those of a Lie algebra. So, to begin with, it is a '2-vector space': a category where the set of objects and the set of morphisms are vector spaces, and all the category operations are linear. However, 2-vector spaces are equivalent to a 2-term chain complexes vector spaces:

$$L_0 \stackrel{d}{\leftarrow} L_1.$$

This will allow us to define a Lie 2-algebra as a 2-term chain complex equipped with a bracket operation satisfying the usual Lie algebra laws 'up to coherent chain homotopy'.

In particular, the bracket of 0-chains will be skew-symmetric up to a chain homotopy called the 'alternator':

$$[x, y] + dS_{x,y} = -[y, x]$$

while the Jacobi identity will hold up to a chain homotopy called the 'Jacobiator':

$$[x, [y, z]] + dJ_{x,y,z} = [[x, y], z] + [y, [x, z]].$$

Furthermore, these chain homotopies need to satisfy some laws of their own.

The definition below requires a few preliminary explanations. First, we use $L \otimes L'$ to denote the tensor product of 2-term chain complexes coming from the tensor product in the category of 2-vector spaces. This is *different than the usual tensor product of chain complexes*, where tensoring two 2-term chain complexes would give a 3-term chain complex. For details, see our previous work [5] and the useful clarification in Section 2.2 of Roytenberg's paper [23].

Second, with this tensor product, the category of 2-term chain complexes is symmetric monoidal, and we use

$$\sigma\colon L_1 \otimes L_2 \to L_2 \otimes L_1$$

to denote the symmetry or 'switch' map.

Third, given 0-chains x, y and a 1-chain T with $y = x + dT$, we write

$$T\colon x \to y.$$

We also write $1\colon x \to x$ in the case where the 1-chain T vanishes, and write $ST\colon x \to z$ for the 1-chain $S + T$, where $T\colon x \to y$ and $S\colon x \to z$. This notation alludes to how a 2-term chain complex can be thought of as a category. In this notation, the alternator in a Lie 2-algebra L gives a 1-chain

$$S_{x,y}\colon [x, y] \to -[y, x]$$

for every pair of 0-chains x, y, and the Jacobiator gives a 1-chain

$$J_{x,y,z}\colon [x, y, z]] \to [[x, y], z] + [y, [x, z]]$$

for every triple of 0-chains x, y, z.

Definition 3.1. *A* **Lie 2-algebra** *is a 2-term chain complex of vector spaces* $L = (L_0 \xleftarrow{d} L_1)$ *equipped with the following structure:*

- *a chain map* $[\cdot, \cdot]\colon L \otimes L \to L$ *called the bracket;*
- *an antisymmetric chain homotopy*

$$S\colon [\cdot, \cdot] \Rightarrow -[\cdot, \cdot] \circ \sigma,$$

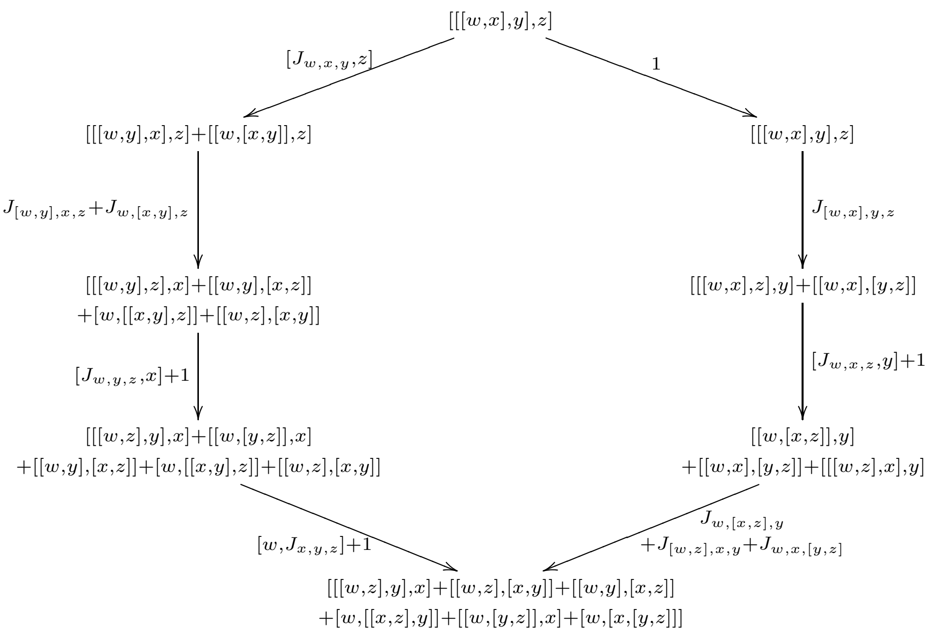
called the **alternator***;*

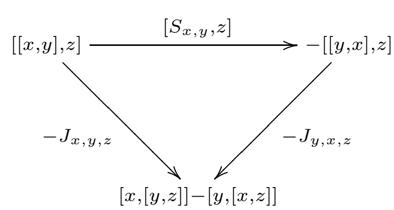
- *an antisymmetric chain homotopy*

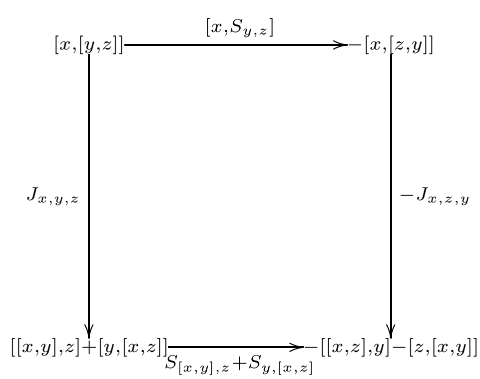
$$J\colon [\cdot,[\cdot,\cdot]] \Rightarrow [[\cdot,\cdot],\cdot] + [\cdot,[\cdot,\cdot]] \circ (\sigma \otimes 1),$$

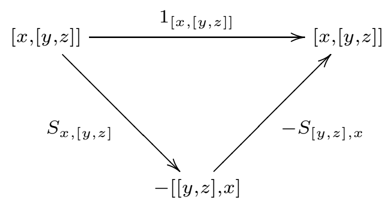
  called the **Jacobiator***.*

In addition, the following diagrams are required to commute:









Definition 3.2. *A Lie 2-algebra for which the Jacobiator is the identity chain homotopy is called* **hemistrict**. *One for which the alternator is the identity chain homotopy is called* **semistrict**.

Now suppose that (X, ω) is a 2-plectic manifold. We shall construct two Lie 2-algebras associated to (X, ω): one hemistrict and one semistrict. Then we shall prove these are isomorphic.

Both these Lie 2-algebras have the same underlying chain complex, namely:

$$\mathrm{Ham}\,(X) \xleftarrow{d} \mathrm{C}^{\infty}\,(X)$$

where d is the usual exterior derivative of functions. To see that this chain complex is well-defined, note that any exact form is Hamiltonian, with 0 as its Hamiltonian vector field.

The hemi-bracket $\{\cdot,\cdot\}_{\mathrm{h}}$ defined in Section 2 can then be seen as the degree 0 component of this chain map:

$$\begin{CD}
\mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) @<{d \times d}<< \mathrm{C}^{\infty}\,(X) \times \mathrm{C}^{\infty}\,(X) \\
@V{\{\cdot,\cdot\}_{\mathrm{h}}}VV @VV{0}V \\
\mathrm{Ham}\,(X) @<<{d}< \mathrm{C}^{\infty}\,(X)
\end{CD}$$

To see that this diagram commutes, again note that the the Hamiltonian vector field corresponding to an exact 1-form is zero, so $\{df, dg\}_{\mathrm{h}} = 0$. We will abuse notation and denote the above chain map by $\{\cdot,\cdot\}_{\mathrm{h}}$.

Theorem 3.3. *If (M, ω) is a 2-plectic manifold, there is a hemistrict Lie 2-algebra $L(M, \omega)_{\mathrm{h}}$ where:*

- *the space of 0-chains is* $\mathrm{Ham}\,(X)$,
- *the space of 1-chains is* $\mathrm{C}^{\infty}\,(X)$,
- *the differential is the exterior derivative* $d\colon \mathrm{C}^{\infty}\,(X) \to \mathrm{Ham}\,(X)$,
- *the bracket is* $\{\cdot,\cdot\}_{\mathrm{h}}$,
- *the alternator is the bilinear map* $S\colon \mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) \to \mathrm{C}^{\infty}\,(X)$ *defined by* $S_{F,G} = -(\iota_{v_F} G + \iota_{v_G} F)$, *and*

- *the Jacobiator is the identity, hence given by the trilinear map* $J\colon \mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) \to \mathrm{C}^\infty\,(X)$ *with* $J_{F,G,H} = 0$.

Proof. It follows from Prop. 2.9 that S is a chain homotopy with the correct source and target. Prop. 2.9 also says that the Jacobi identity holds, so the Jacobiator defined above is also a chain homotopy with the correct source and target. So, we just need to check that the Lie 2-algebra axioms hold. The first two diagrams commute since each edge is the identity. The commutativity of the third diagram is shown as follows:

$$\begin{array}{rcl} S_{\{F,G\}_{\mathrm{h}},H} + S_{G,\{F,H\}_{\mathrm{h}}} & = & -\iota_{v_{[F,G]}} H - \iota_{v_H} \{F,G\}_{\mathrm{h}} - \iota_{v_G} \{F,H\}_{\mathrm{h}} - \iota_{v_{[F,H]}} G \\ & = & -L_{\,v_F} \left(\iota_{v_G} H + \iota_{v_H} G\right) \\ & = & \{F, S_{G,H}\}_{\mathrm{h}} \end{array}$$

The last diagram commutes because

$$\begin{array}{rcl} S_{F,\{G,H\}_{\mathrm{h}}} - S_{\{G,H\}_{\mathrm{h}},F} & = & -\iota_{v_F} \{G,H\}_{\mathrm{h}} - \iota_{v_{[G,H]}} F + \iota_{v_{[G,H]}} F + \iota_{v_F} \{G,H\}_{\mathrm{h}} \\ & = & 0 \end{array}$$

□

Similarly, to define a semistrict Lie 2-algebra, we note that $\{\cdot,\cdot\}_{\mathrm{s}}$ is the degree 0 component of this chain map:

$$\begin{array}{ccc} \mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) & \xleftarrow{\;d \times d\;} & \mathrm{C}^\infty\,(X) \times \mathrm{C}^\infty\,(X) \\ \Big\downarrow {\scriptstyle \{\cdot,\cdot\}_{\mathrm{s}}} & & \Big\downarrow {\scriptstyle 0} \\ \mathrm{Ham}\,(X) & \xleftarrow[\;d\;]{} & \mathrm{C}^\infty\,(X) \end{array}$$

Theorem 3.4. *If* (M,ω) *is a 2-plectic manifold, there is a semistrict Lie 2-algebra* $L(M,\omega)_{\mathrm{s}}$ *where:*

- *the space of 0-chains is* $\mathrm{Ham}\,(X)$*,*
- *the space of 1-chains is* $\mathrm{C}^\infty\,(X)$*,*
- *the differential is the exterior derivative* $d\colon \mathrm{C}^\infty\,(X) \to \mathrm{Ham}\,(X)$*,*
- *the bracket is* $\{\cdot,\cdot\}_{\mathrm{s}}$*,*
- *the alternator is the identity, hence given by the bilinear map* $S\colon \mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) \to \mathrm{C}^\infty\,(X)$ *with* $S_{F,G} = 0$*, and*
- *the Jacobiator is the trilinear map* $J\colon \mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) \times \mathrm{Ham}\,(X) \to \mathrm{C}^\infty\,(X)$ *defined by* $J_{F,G,H} = \iota_{v_F} \iota_{v_G} \iota_{v_H} \omega$.

Proof. We note from Prop 2.10 that the semi-bracket is antisymmetric, so the alternator defined above is a chain homotopy with the right source and target. It follows from Prop. 2.10 that the Jacobiator is also a chain homotopy with the desired source and target. So again, we just need to check that the Lie 2-algebra axioms hold. The following identities can be checked by simple calculation, and the commutativity of the first diagram follows:

$$J_{\{K,F\}_s,G,H} = J_{\{H,K\}_s,F,G} - J_{\{F,H\}_s,G,K} - L_{\,v_G}\, J_{K,F,H}$$

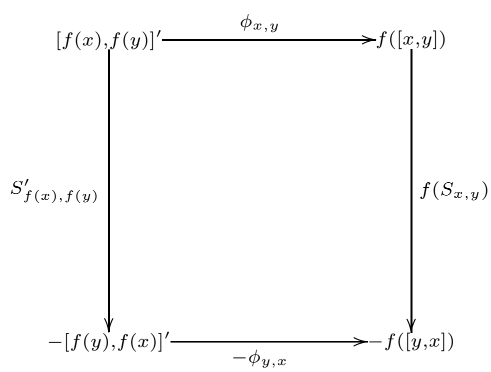
$$L_{\,v_G}\, J_{K,F,H} = J_{\{G,K\}_s,F,H} + J_{K,\{G,F\}_s,H} + J_{K,F,\{G,H\}_s}$$

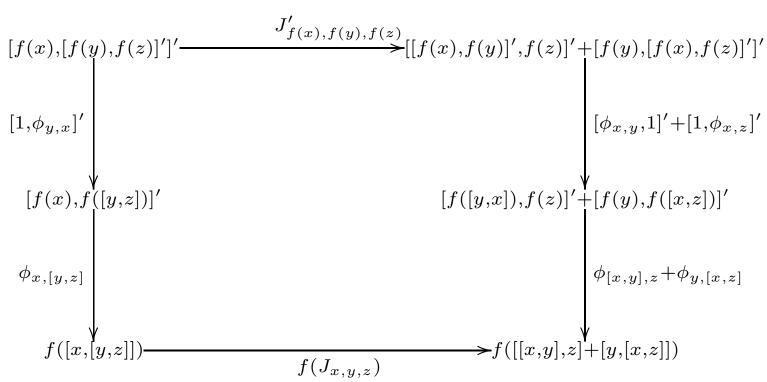
Since the Jacobiator is antisymmetric and the alternator is the identity, the second and third diagrams commute as well. The fourth diagram commutes because all the edges are identity morphisms. $\square$

Definition 3.5. *Given Lie 2-algebras* $L = (L_1 \stackrel{d}{\to} L_0)$ *and* $L' = (L_1' \stackrel{d}{\to} L_0')$ *with bracket, alternator and Jacobiator* $[\cdot,\cdot]$, S, J *and* $[\cdot,\cdot]'$, S', J' *respectively, a* **homomorphism** $F\colon L \to L'$ *consists of:*

- *a chain map* $f\colon L \to L'$, *and*
- *a chain homotopy* $\phi\colon [\cdot,\cdot]' \circ (f \otimes f) \Rightarrow f \circ [\cdot,\cdot]$

such that the following diagrams commute:





Roytenberg explains how to compose Lie 2-algebra homomorphisms [23], and we say a Lie 2-algebra homomorphism with an inverse is an **isomorphism**.

Theorem 3.6. *$L(M,\omega)_{\mathrm{h}}$ and $L(M,\omega)_{\mathrm{s}}$ are isomorphic as Lie 2-algebras.*

Proof. We show that the identity chain maps with appropriate chain homotopies define Lie 2-algebra homomorphisms and that their composites are the respective identity homomorphisms. There is a homomorphism $f\colon L(M,\omega)_{\mathrm{h}} \to L(M,\omega)_{\mathrm{s}}$ with the identity chain map and the chain homotopy given by $\phi(F,G) = \iota_{v_F} G$. That this is a chain homotopy follows from the bracket relation $\{F,G\}_{\mathrm{h}} = \{F,G\}_{\mathrm{s}} + d(\iota_{v_F} G)$ noted in Eq. 2.8. We check that this satisfies the axioms of a Lie 2-algebra homomorphism. Noting that the chain map is the identity, the commutativity of the first diagram is easily checked by recalling that $S_{F,G} = -(\iota_{v_G} F + \iota_{v_F} G)$ and that $S'_{F,G}$ is the identity. Noting that any edge given by the bracket of functions will be the identity and that $J_{F,G,H}$ is the identity, to check the commutativity of the second diagram we only need to check the following calculation.

$$\begin{aligned}
& J'_{F,G,H} + \phi(\{F,G\}_{\mathrm{h}}, H) + \phi(G, \{F,H\}_{\mathrm{h}}) - \phi(F, \{G,H\}_{\mathrm{h}}) \\
= \; & \iota_{v_F}\iota_{v_G}\iota_{v_H}\omega + \iota_{v_{[F,G]}} H + \iota_{v_G} L_{v_F} H - \iota_{v_F} L_{v_G} H \\
= \; & \iota_{v_F} L_{v_G} H - \iota_{v_F} d\iota_{v_G} H + \iota_{v_{[F,G]}} H + \iota_{v_G} L_{v_F} H - \iota_{v_F} L_{v_G} H \\
= \; & -\iota_{v_F} d\iota_{v_G} H + \iota_{v_{[F,G]}} H + \iota_{v_G} L_{v_F} H \\
= \; & -\iota_{v_F} d\iota_{v_G} H + L_{v_F}\iota_{v_G} H - \iota_{v_G} L_{v_F} H + \iota_{v_G} L_{v_F} H \\
= \; & -\iota_{v_F} d\iota_{v_G} H + L_{v_F}\iota_{v_G} H \\
= \; & d\iota_{v_F}\iota_{v_G} H - L_{v_F}\iota_{v_G} H + L_{v_F}\iota_{v_G} H \\
= \; & d\iota_{v_F}\iota_{v_G} H \\
= \; & 0
\end{aligned}$$

□

4 An Application to String Theory

Classical bosonic string theory is a theory of maps $\phi\colon \Sigma \to M$ where Σ is a surface and M is some manifold representing spacetime. For simplicity we will only consider the case where Σ is the cylinder $\mathbb{R} \times S^1$ and M is n-dimensional Minkowski spacetime, $\mathbb{R}^{(1,n-1)}$. A solution of classical bosonic string theory then consists of a map $\phi\colon \Sigma \to M$ which is a critical point of the area subject to certain boundary conditions.

Equivalently, by exploiting symmetries in the variational problem, one can describe solutions ϕ by equipping $\mathbb{R} \times S^1$ with its standard Lorentzian metric and then solving the $1+1$ dimensional field theory specified by the Lagrangian density

$$\ell = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \left(\frac{\partial \phi^i}{\partial \tau} \frac{\partial \phi^j}{\partial \tau} - \frac{\partial \phi^i}{\partial \sigma} \frac{\partial \phi^j}{\partial \sigma} \right),$$

where $\eta = \mathrm{diag}(1, -1, \ldots, -1)$ is the Minkowski metric and $\{\phi^i\}_{i=1}^n$ are the coordinates of the map ϕ in $\mathbb{R}^{(1,n-1)}$. The corresponding Euler–Lagrange equations are

$$\frac{\partial^2 \phi^i}{\partial \tau^2} - \frac{\partial^2 \phi^i}{\partial \sigma^2} = 0.$$

We next describe this theory using multisymplectic geometry, following Hélein [18]. (The work of Gotay *et al* [16] focuses instead on the Polyakov approach, where the metric on Σ is taken as an independent variable.)

Let $E \xrightarrow{p} \Sigma \times M$ be the vector bundle over $\Sigma \times M$ with fiber $T^*_{(\tau,\sigma)}\Sigma \otimes T_x M$ over the point $(\tau, \sigma, x) \in \Sigma \times M$. Then the Lagrangian density for the string can be defined as a smooth function on E:

$$\ell = \frac{1}{2} \sum_{i,j=1}^n \eta_{ij} \left(u^i_\tau u^j_\tau - u^i_\sigma u^j_\sigma \right),$$

which depends in this example only on the fiber coordinates $\{u^i_\tau, u^i_\sigma\}_{i=1}^n$ on $T^*_{(\tau,\sigma)}\Sigma \times TM$. From this Lagrangian, the 'de Donder–Weyl Hamiltonian' h can be constructed via a Legendre transform. It is a smooth function on the dual vector bundle E^*:

$$\begin{aligned} h &= \sum_{i=1}^n \left(p^\tau_i u^i_\tau + p^\sigma_i u^i_\sigma \right) - \ell \\ &= \frac{1}{2} \sum_{i,j=1}^n \eta^{ij} \left(p^\tau_i p^\tau_j - p^\sigma_i p^\sigma_j \right), \end{aligned}$$

where u^i_τ and u^i_σ are defined implicitly by $p^\tau_i = \partial\ell/\partial u^i_\tau$, $p^\sigma_i = \partial\ell/\partial u^i_\sigma$, and $\{p^\tau_i, p^\sigma_i\}_{i=1}^n$ are coordinates on the fiber $T_{(\tau,\sigma)}\Sigma \otimes T^*M$. Note that h differs from the standard (non-covariant) Hamiltonian density ε for a field theory, which can also be written as a smooth function on E^*:

$$\begin{aligned} \varepsilon &= \sum_{i=1}^n p^\tau_i u^i_\tau - \ell \\ &= \frac{1}{2} \sum_{i,j=1}^n \eta^{ij} \left(p^\tau_i p^\tau_j + p^\sigma_i p^\sigma_j \right). \end{aligned}$$

Let π be a smooth section of E^* with coordinates $\{\pi^\tau_i, \pi^\sigma_i\}_{i=1}^n$. It is then straightfoward to show that $\phi\colon \Sigma \to M$ is a solution of the equations of motion (Eq. 10) if and only if ϕ and π satisfy the following system of equations:

$$\begin{aligned} \sum_{i=1}^n \frac{\partial \pi^\tau_i}{\partial \tau} + \frac{\partial \pi^\sigma_i}{\partial \sigma} &= -\sum_{i=1}^n \left. \frac{\partial h}{\partial x^i} \right|_{x=\phi, p=\pi} && (10) \\ \frac{\partial \phi^i}{\partial \tau} &= \left. \frac{\partial h}{\partial p^\tau_i} \right|_{x=\phi, p=\pi} && (11) \\ \frac{\partial \phi^i}{\partial \sigma} &= \left. \frac{\partial h}{\partial p^\sigma_i} \right|_{x=\phi, p=\pi}. && (12) \end{aligned}$$

This system of equations is a generalization of Hamilton's equations for a classical point particle.

We take as our extended phase space $X = E^* \times \mathbb{R}$. A point in X has coordinates $(\tau, \sigma, x^i, p_i^\tau, p_i^\sigma, e)$. Define the 2-form θ on X as follows:

$$\theta = e\ d\tau \wedge d\sigma + \sum_{i=1}^{n} \left(p_i^\tau dx^i \wedge d\sigma - p_i^\sigma dx^i \wedge d\tau \right).$$

Here the variable e should be considered as the canonical conjugate to the area form $d\tau \wedge d\sigma$. Let

$$\omega = d\theta.$$

If we assume $\omega(v, \cdot, \cdot) = 0$, then the linear independence of the basis in $\Omega^2(M)$ implies $v = 0$. Hence ω is nondegenerate and therefore a 2-plectic structure on X.

As before let $\phi\colon \Sigma \to M$ be a smooth map and π be a smooth section of E^*. Consider the submanifold S of X with coordinates:

$$(\tau, \sigma, \phi^i(\tau,\sigma), \pi_i^\tau(\tau,\sigma), \pi_i^\sigma(\tau,\sigma), -h).$$

Note that S is constructed from the graphs of ϕ and π and from the constraint $e+h = 0$. This constraint is analogous to the one that is used in finding constant energy solutions in the extended phase space approach to classical mechanics. At each point in S, a tangent bivector $v = v_\tau \wedge v_\sigma \in \Gamma(\Lambda^2 TX)$ can be defined as

$$v_\tau = \frac{\partial}{\partial \tau} + \sum_{i=1}^{n} \left(\frac{\partial \phi^i}{\partial \tau} \frac{\partial}{\partial x^i} + \frac{\partial \pi_i^\tau}{\partial \tau} \frac{\partial}{\partial p_i^\tau} + \frac{\partial \pi_i^\sigma}{\partial \tau} \frac{\partial}{\partial p_i^\sigma} \right)$$
$$v_\sigma = \frac{\partial}{\partial \sigma} + \sum_{i=1}^{n} \left(\frac{\partial \phi^i}{\partial \sigma} \frac{\partial}{\partial x^i} + \frac{\partial \pi_i^\tau}{\partial \sigma} \frac{\partial}{\partial p_i^\tau} + \frac{\partial \pi_i^\sigma}{\partial \sigma} \frac{\partial}{\partial p_i^\sigma} \right)$$

Explicit computation reveals that the submanifold S is generated by solutions to Hamilton's equations if and only if

$$\omega(v_\tau, v_\sigma, \cdot) = 0.$$

Quite generally, infinitesimal symmetries of the 2-form θ give rise to Hamiltonian 1-forms that generate these symmetries. For example, symmetry under time evolution lets us define a Hamiltonian. Consider the Lie derivative of θ along the coordinate vector field $\partial/\partial\tau$:

$$\begin{aligned} \mathcal{L}_{\frac{\partial}{\partial \tau}} \theta &= d\iota_{\frac{\partial}{\partial \tau}} \theta + \iota_{\frac{\partial}{\partial \tau}} \omega \\ &= d\left(e\ d\sigma - de \wedge d\sigma + \textstyle\sum_{i=1}^{n} \left(p_i^\sigma dx^i - dp_i^\sigma \wedge dx^i \right)\right) \\ &= 0. \end{aligned}$$

Hence θ is invariant with respect to infinitesimal displacements along the τ coordinate. If we define a 1-form H by

$$\begin{aligned} H &= -\iota_{\frac{\partial}{\partial \tau}} \theta \\ &= -e\ d\sigma - \textstyle\sum_{i=1}^{n} p_i^\sigma dx^i. \end{aligned}$$

then Eq. 4 implies that $dH = \iota_{\frac{\partial}{\partial \tau}} \omega$. Hence H belongs to set of Hamiltonian 1-forms $\mathrm{Ham}\,(X)$, and the Hamiltonian vector field v_H describes time evolution.

One may wonder how this Hamiltonian 1-form H is related to the usual concept of energy. To understand this, consider the solution submanifold Σ as defined above. Let $\Sigma_{\tau_0} \subset \Sigma$ be a 1-dimensional curve on Σ at constant 'time' $\tau = \tau_0$. Denote the restriction of H to Σ_{τ_0} as H_{τ_0}. A computation yields

$$\begin{aligned} H_{\tau_0} &= h\ d\sigma - \sum_{i=1}^{n} \pi_i^{\sigma} d\phi^i \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \eta^{ij} \left(\pi_i^{\tau} \pi_j^{\tau} - \pi_i^{\sigma} \pi_j^{\sigma} \right) d\sigma - \sum_{i=1}^{n} \pi_i^{\sigma} d\phi^i. \end{aligned}$$

On Σ_{τ_0}, $d\tau = 0$. Hence $d\phi^i = \frac{\partial \phi^i}{\partial \sigma} d\sigma$. Also since ϕ satisfies Eq. 12:

$$\begin{aligned} \pi_i^{\tau} &= \sum_{j=1}^{n} \eta_{ij} \frac{\partial \phi^j}{\partial \tau}, \\ \pi_i^{\sigma} &= - \sum_{j=1}^{n} \eta_{ij} \frac{\partial \phi^j}{\partial \sigma}. \end{aligned}$$

Therefore the expression for H_{τ_0} becomes:

$$\begin{aligned} H_{\tau_0} &= \frac{1}{2} \sum_{i,j=1}^{n} \eta^{ij} \left(\pi_i^{\tau} \pi_j^{\tau} + \pi_i^{\sigma} \pi_j^{\sigma} \right) d\sigma \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \left(\frac{\partial \phi^i}{\partial \tau} \frac{\partial \phi^j}{\partial \tau} + \frac{\partial \phi^i}{\partial \sigma} \frac{\partial \phi^j}{\partial \sigma} \right) d\sigma \\ &= \varepsilon\ d\sigma. \end{aligned}$$

Hence H_{τ_0} is the Hamiltonian 1-form that corresponds to the energy density of the string at τ_0, and the total energy of the string at $\tau = \tau_0$ is simply:

$$\int_{\Sigma_{\tau_0}} H_{\tau_0}.$$

5 Conclusions

The work presented here raises many questions. Here are three obvious ones:

- Does an n-plectic manifold give rise to a Lie n-algebra when $n > 2$? There is not yet a definition of weak or hemistrict Lie n-algebras for $n > 2$, but a semistrict Lie n-algebra is just an n-term chain complex equipped with the structure of an L_∞-algebra. So, it would be easiest to start by considering a generalization of the semi-bracket, and see if this can be used to construct a semistrict Lie n-algebra.

- Does the Lie 2-algebra of observables in 2-plectic geometry extend to something like a Poisson algebra? It is far from clear how to define a product for Hamiltonian 1-forms, and the usual product of a Hamiltonian 1-form and a smooth function is not Hamiltonian.

  It seems that the free loop space LX of a 2-plectic manifold (X, ω) is an infinite-dimensional symplectic manifold on which the symplectic 2-form $\tilde{\omega}$ is defined as follows:

$$\tilde{\omega}(v, w) = \int_0^{2\pi} \omega(u(\gamma), v(\gamma(\sigma)), w(\gamma(\sigma))\; d\sigma$$

  where v, w are tangent vectors at the loop $\gamma \in LX$, $v(\gamma(\sigma))$ and $w(\gamma(\sigma))$ are the corresponding tangent vectors at the point $\gamma(\sigma) \in X$, and u is the vector field generating the circle action on LX. Similarly, any Hamiltonian 1-form F on X determines a function $\tilde{F}$ on LX by:

$$\tilde{F}(\gamma) = \int_0^{2\pi} F(\gamma(\sigma))\; d\sigma.$$

  Even if we are unable to multiply Hamiltonian 1-forms, we can multiply these functions. So, there is a map from $\mathrm{Ham}\,(X)$ to the Poisson algebra of functions on LX. This map has already been exploited by Kijowski [21] and also Hélein and Kouneiher [18, 19]. But, it would be nice to define more directly some sort of 'Poisson 2-algebra' of observables in 2-plectic geometry.

- When a symplectic structure ω on a manifold X defines an integral class in $H^2(X, \mathbb{R})$, there is a U(1) bundle over X equipped with a connection whose curvature is ω. As mentioned in the Introduction, this plays a fundamental role in the geometric quantization of X. Similarly, when a 2-plectic structure ω on a manifold X defines an integral class in $H^3(X, \mathbb{R})$, there is a U(1) gerbe over X equipped with a connection whose curvature is ω [11]. Is there an analogue of geometric quantization that applies in this case?

  Following the ideas of Freed [15], we might hope that geometrically quantizing this gerbe will give a '2-Hilbert space' of states. However, Freed's work only treats Schrödinger quantization, and that only in the special case where the resulting 2-Hilbert space is finite-dimensional. Finite-dimensional 2-Hilbert spaces are by now well-understood [3], but the infinite-dimensional ones are still being developed [4, 28]. Geometric quantization for gerbes is an even greater challenge. However, we expect the problem of geometrically quantizing a U(1) gerbe on X to be closely related to the better-understood problem of geometrically quantizing the corresponding U(1) bundle on the loop space of X.

References

[1] P. Aschieri, L. Cantini, and B. Jurčo, Nonabelian bundle gerbes, their differential geometry and gauge theory, *Comm. Math. Phys.* **254** (2005), 367–400. Also available as arXiv:hep-th/0312154.

[2] P. Aschieri and B. Jurčo, Gerbes, M5-brane anomalies and E_8 gauge theory, *JHEP* **10** (2004) 068. Also available as arXiv:hep-th/0409200.

[3] J. Baez, Higher-dimensional algebra II: 2-Hilbert spaces, *Adv. Math.* **127** (1997), 125–189. Also available as arXiv:q-alg/9609018.

[4] J. Baez, A. Baratin, L. Freidel and D. Wise, Representations of 2-groups on higher Hilbert spaces, manuscript in preparation.

[5] J. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, *TAC* **12**, (2004), 492–528. Also available as arXiv:math/0307263.

[6] J. Baez, A. Crans, D. Stevenson and U. Schreiber, From loop groups to 2-groups, *HHA* **9** (2007), 101–135. Also available as arXiv:math/0504123.

[7] J. Baez and A. Lauda, Higher-dimensional algebra V: 2-groups, *TAC* **12** (2004), 423–491. Also available as arXiv:math.QA/0307200.

[8] J. Baez and U. Schreiber, Higher gauge theory, in *Categories in Algebra, Geometry and Mathematical Physics*, eds. A. Davydov *et al*, *Contemp. Math.* **431**, AMS, Providence, Rhode Island, 2007, pp. 7–30. Also available as arXiv:math/0511710.

[9] T. Bartels, Higher gauge theory: 2-bundles, available as arXiv:math/0410328.

[10] L. Breen and W. Messing, Differential geometry of gerbes, *Adv. Math.* **198** (2005), 732–846. Also available as arXiv:math/0106083.

[11] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhauser, Boston, 1993.

[12] F. Cantrijn, A. Ibort, and M. De Leon, On the geometry of multisymplectic manifolds, *J. Austral. Math. Soc. (Series A)* **66** (1999), 303–330.

[13] D. Cartin, Generalized symplectic manifolds, available as arXiv:dg-ga/9710027.

[14] T. DeDonder, *Theorie Invariantive du Calcul des Variations*, Gauthier–Villars, Paris, 1935.

[15] D. Freed, Higher algebraic structures and quantization, *Comm. Math. Phys.* **159** (1994), 343–398. Also available as arXiv:hep-th/9212115.

[16] M. Gotay, J. Isenberg, J. Marsden, and R. Montgomery, Momentum maps and classical relativistic fields. Part I: covariant field theory, available as arXiv:physics/9801019.

[17] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge U. Press, Cambridge, 1984.

[18] F. Hélein, Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory, in *Noncompact Problems at the Intersection of Geometry*, eds. A. Bahri *et al*, AMS, Providence, Rhode Island, 2001, pp. 127–148. Also available as arXiv:math-ph/0212036.

[19] F. Hélein and J. Kouneiher, The notion of observable in the covariant Hamiltonian formalism for the calculus of variations with several variables, *Adv. Theor. Math. Phys.* **8** (2004), 735–777. Also available as arXiv:math-ph/0401047.

[20] A. Ibort, Multisymplectic geometry: generic and exceptional, in *Proceedings of the IX Fall Workshop on Geometry and Physics, Vilanova i la Geltrú, 2000*, eds. X. Grácia *et al*, Publicaciones de la RSME vol. 3, Real Sociedad Matemática Española, Madrid, 2001, pp. 79–88.

[21] J. Kijowski, A finite-dimensional canonical formalism in the classical field theory, *Commun. Math. Phys.* **30** (1973), 99–128.

[22] C. Rovelli, Covariant Hamiltonian formalism for field theory: Hamilton-Jacobi equation on the space G, arXiv:gr-qc/0207043.

[23] D. Roytenberg, On weak Lie 2-algebras, available as arXiv:0712.3461.

[24] H. Sati, U. Schreiber and J. Stasheff, L_∞-algebra connections and applications to String- and Chern–Simons n-transport, available as arXiv:0801.3480.

[25] H. Sati, U. Schreiber and J. Stasheff, Fivebrane structures, available as arXiv:0801.3480.

[26] U. Schreiber, *From Loop Space Mechanics to Nonabelian Strings*, Ph.D. thesis, Universität Duisburg-Essen, 2005. Also available as arXiv:hep-th/0509163.

[27] H. Weyl, Geodesic fields in the calculus of variation for multiple integrals, *Ann. Math.* **36** (1935), 607–629.

[28] D. Yetter, Measurable categories, *Appl. Cat. Str.* **13** (2005), 469–500. Also available as arXiv:math.CT/0309185.

[29] Y. Zunger, p-Gerbes and extended objects in string theory, arXiv:hep-th/0002074.