2-Tangles

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Abstract

Just as links may be algebraically described as certain morphisms in the category of tangles, compact surfaces smoothly embedded in $\mathbf{R}^4$ may be described as certain 2-morphisms in the 2-category of '2-tangles in 4 dimensions'. In this announcement we give a purely algebraic characterization of the 2-category of unframed unoriented 2-tangles in 4 dimensions as the 'free semistrict braided monoidal 2-category with duals on one unframed self-dual object'. A forthcoming paper will contain a proof of this result using the movie moves of Carter, Rieger and Saito. We comment on how one might use this result to construct invariants of 2-tangles.

1 Introduction

Recent work on 'quantum invariants' of knots, links, tangles, and 3-manifolds depends crucially on a purely algebraic characterization of tangles in 3-dimensional space. It follows from work of Freyd and Yetter, Turaev, and Shum [13, 18, 19, 20] that isotopy classes of framed oriented tangles in 3 dimensions are the morphisms of a certain very special category: the 'free braided monoidal category with duals on one object'. It follows that we can easily obtain functors from this category to other braided monoidal categories with duals, such as the category of representations of a quantum group. Any such functor gives an invariant of tangles, and therefore of knots and links.

The 'tangle hypothesis' [2] suggests a vast generalization of this result, applicable to n-manifolds smoothly embedded in $(n+k)$-dimensional space. It says that framed oriented n-tangles in $n+k$ dimensions are the n-morphisms of the 'free k-tuply monoidal weak n-category with duals on one object'. The hope is that a precise formulation and proof of this hypothesis will open the door to constructing quantum invariants of n-dimensional submanifolds of $\mathbf{R}^{n+k}$.

Unfortunately the tangle hypothesis involves concepts from topology and n-category theory that have so far only been worked out in certain low-dimensional cases. As a kind of warmup, we wish to prove a version of this hypothesis in the case $n = k = 2$. So far we have only completed work on the unframed, unoriented case, which allows us to take maximal advantage of the recent work of Carter, Rieger and Saito [7].

Since the theory of k-tuply monoidal weak n-categories is not yet well developed for $n = k = 2$, we use the better-understood 'semistrict' ones as a kind of stopgap. These are also known as 'semistrict braided monoidal 2-categories'. The result announced here is thus that the 2-category of unframed unoriented 2-tangles in 4 dimensions is the 'free semistrict braided monoidal 2-category with duals on one unframed self-dual object'. The proof appears in Langford's dissertation [17], and will be published as part of the 'Higher-Dimensional Algebra' series [4].

To appreciate this result, one needs some feeling for the topology and algebra involved: that is, for higher-dimensional tangles and higher-dimensional category theory. Thus we begin with a brief sketch of both, concentrating on the situation at hand. We omit many details, which can be found in the references.

2 Preliminaries

What are 2-tangles in 4 dimensions? Roughly speaking, they are surfaces in 4 dimensions going from one tangle in 3 dimensions to another tangle in 3 dimensions. Many ways of visualizing and representing them can be found in the work of Carter, Rieger, and Saito [7] and the references therein. To describe them more precisely, we equip the space $[0,1]^4$ with standard coordinates (x, y, z, t). A 2-tangle $\alpha\colon f \Rightarrow g$ in 4 dimensions is a compact 2-dimensional surface, smoothly embedded in $[0,1]^4$ in such a way that its intersections with the hyperplanes $\{t = 0\}$ and $\{t = 1\}$ are the tangles f and g in 3 dimensions. The tangles f, g may only touch the boundary of the cube $[0,1]^3$ at the top and bottom, i.e., at the planes $\{z = 0\}$ and $\{z = 1\}$. Near these planes, f and g must have a product structure. In other words, they must look like straight vertical lines. Similarly, the surface α may only touch the boundary of $[0,1]^4$ at the hyperplanes $\{t = 0\}$, $\{t = 1\}$, $\{z = 0\}$, and $\{z = 1\}$, and near these hyperplanes it must have a product structure. The surface α can have a boundary, but this boundary must lie in the hyperplanes where either t or z is 0 or 1. The surface can also have right-angled corners, but only at those points where both t and z are either 0 or 1. Finally, we assume without loss of generality that α is in 'general position' in a certain precise sense.

Perhaps the most interesting examples of 2-tangles are compact surfaces without boundary smoothly embedded in the interior of $[0,1]^4$. Topologists call these 'knotted surfaces'. The advantage of working with more general 2-tangles is that it allows us to build complicated knotted surfaces by gluing together simple 2-tangles. We can glue together 2-tangles in two basic ways. First, given 2-tangles $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, we can form the 2-tangle $\alpha \cdot \beta\colon f \Rightarrow h$ by gluing together α and β along a hyperplane of constant t, namely, the hyperplane $t = 1$ for α and the hyperplane $t = 0$ for β. Second, given 2-tangles $\alpha\colon f \Rightarrow g$ and $\beta\colon f' \Rightarrow g'$ such that the composite tangles ff' and gg' are well-defined, we can form the 2-tangle $\alpha \circ \beta\colon ff' \Rightarrow gg'$ by gluing α and β along a hyperplane of constant z.

With a little work, we obtain an algebraic structure known as a 2-category, having objects, morphisms between objects, and 2-morphisms between morphisms. We denote the 2-category of 2-tangles in 4 dimensions by $\mathcal{T}$. The objects of $\mathcal{T}$ are certain equivalence classes of finite sets of points with distinct y coordinates in the square $\{0 \le x, y \le 1\}$. The morphisms in $\mathcal{T}$ are certain equivalence classes of tangles, where it is crucial that tangles differing by a Reidemeister move or by changing the relative heights of crossings, maxima, and minima are not regarded as equivalent. The 2-morphisms in $\mathcal{T}$ are suitably defined ambient isotopy classes of 2-tangles.

In any 2-category one can compose morphisms $f\colon A \to B$ and $g\colon B \to C$ to obtain a morphism $fg\colon A \to C$. (Note our ordering convention here.) In $\mathcal{T}$ this operation corresponds to composition of tangles. Also in any 2-category there are two ways to compose 2-morphisms α and β, written $\alpha \cdot \beta$ and $\alpha \circ \beta$. In $\mathcal{T}$ these operations work as described above. In the 2-categorical literature $\alpha \cdot \beta$ is usually called the 'vertical composite' of α and β, while $\alpha \circ \beta$ is called the 'horizontal composite'. In what follows, we write $1_f \circ \alpha$ simply as $f\alpha$, and $\alpha \circ 1_f$ as αf.

Of course, there is a list of axioms to check [15] in order to show that $\mathcal{T}$ is a 2-category. Actually, Kharlamov and Turaev [16] have already constructed a 2-category of 2-tangles in 4 dimensions. For technical reasons ours is not quite the same, but we expect that it is 'equivalent' in the sense of 2-category theory.

The reader may wonder why we did not consider a third basic way to glue together 2-tangles: namely, along a hyperplane of constant y. In fact, while the details are rather technical, this form of gluing equips $\mathcal{T}$ with the the structure of a semistrict monoidal 2-category. In a semistrict monoidal 2-category, one can tensor objects with objects, morphisms or 2-morphisms, and there is an object I serving as the unit for the tensor product. The reason for the term 'semistrict' is that certain equations which held strictly in a monoidal category are weakened to 2-isomorphisms. Most importantly, in a monoidal category the equation $(A \otimes g)(f \otimes B') = (f \otimes B)(A' \otimes g)$ holds for any morphisms $f\colon A \to A'$ and $g\colon B \to B'$, while in a semistrict monoidal 2-category this is replaced by a specified 2-isomorphism, the 'tensorator':

$$\bigotimes_{f,g}\colon (A \otimes g)(f \otimes B') \Rightarrow (f \otimes B)(A' \otimes g).$$

Again there are various axioms that must hold. These were first explicitly listed by Kapranov and Voevodsky [14], and later expressed more tersely in the language of 2-category theory [5, 11]. While the details are rather lengthy, we can equip $\mathcal{T}$ with unit object, tensor products, and tensorator, and check that $\mathcal{T}$ becomes a semistrict monoidal 2-category. (As noted by Kharlamov and Turaev, Fischer's paper on 2-tangles [12] has serious flaws, such as not discussing the tensorator.)

One can also consider gluing 2-tangles together along a hyperplane of constant x. By the Eckmann-Hilton argument [2, 5], this form of gluing makes $\mathcal{T}$ into a semistrict braided monoidal 2-category. This means, first of all, that for any objects A and B there is a morphism $R_{A,B}\colon A \otimes B \to B \otimes A$, called the 'braiding'. This morphism must be a invertible up to a 2-isomorphism. In addition, for any morphisms $f\colon A \to A'$ and

$g\colon B \to B'$, there are braiding 2-isomorphisms

$$R_{f,B}\colon (f \otimes B)R_{A',B} \Rightarrow R_{A,B}(B \otimes f)$$

and

$$R_{A,g}\colon (A \otimes g)R_{A,B'} \Rightarrow R_{A,B}(g \otimes A).$$

Now, in a strict braided monoidal category, for any objects A, B, and C we have $(R_{A,B} \otimes C)(B \otimes R_{A,C}) = R_{A,B\otimes C}$ and $(A \otimes R_{B,C})(R_{A,C} \otimes B) = R_{A\otimes B,C}$. In fact, these equations are crucial for proving the Yang-Baxter equation. In a semistrict braided monoidal 2-category these equations are weakened to specified 2-isomorphisms

$$\tilde{R}_{(A|B,C)}\colon (R_{A,B} \otimes C)(B \otimes R_{A,C}) \Rightarrow R_{A,B\otimes C}$$

and

$$\tilde{R}_{(A,B|C)}\colon (A \otimes R_{B,C})(R_{A,C} \otimes B) \Rightarrow R_{A\otimes B,C}.$$

There is also a list of axioms that must hold. The first definition of braided monoidal 2-category was given by Kapranov and Voevodsky [14]. Later this definition was modified in various ways by Baez and Neuchl [5]. These modifications are necessary for the proper treatment of 2-tangles, and especially for an unambiguous statement of the Zamolodchikov tetrahedron equation, as had been noted by Breen [6]. Subsequently Day and Street [11] re-expressed the Baez-Neuchl definition in a more compact way, and Crans [10] added some axioms governing the braiding of the unit object. In what follows we use the definition given by Crans. It turns out that one can equip $\mathcal{T}$ with braiding morphisms and 2-isomorphisms and check that it is a semistrict braided monoidal 2-category in this sense.

There is a very special object Z in $\mathcal{T}$, corresponding to a *single point* in the square $\{0 \le x, y \le 1\}$. Our result makes precise the sense in which $\mathcal{T}$ is freely generated by this object. For this we need to use the duality structure of $\mathcal{T}$. The study of duality in n-categories is only beginning, so before stating our result we need to define braided monoidal 2-categories 'with duals'. Since we are working with unframed unoriented tangles, the object Z has special properties: it is 'self-dual' and 'unframed'. The concept of a self-dual object is straightforward, but the concept of an 'unframed' object is rather subtle. In the study of framed tangles, a twist in the framing is often represented as a certain morphism $b_A\colon A \to A$ known as the 'balancing'. In our situation, an 'unframed object' is not an object for which the balancing is the identity, but one for which the balancing is *isomorphic* to the identity via a 2-isomorphism that satisfies a highly nontrivial equation of its own.

Finally, the study of universal properties for n-categories is also just beginning, so we must clarify what is meant by the 'free' braided monoidal 2-category with duals on one unframed self-dual object. We do so by means of a universal property.

In what follows, we take for granted our basic result that there exists a semistrict braided monoidal 2-category $\mathcal{T}$ whose 2-morphisms are ambient isotopy classes of smooth 2-tangles in 4 dimensions. In particular, letting 1_I denote the identity morphism of the unit object of $\mathcal{T}$, the 2-morphisms $\alpha\colon 1_I \Rightarrow 1_I$ in $\mathcal{T}$ are precisely the ambient isotopy classes of compact surfaces without boundary smoothly embedded in $[0,1]^4$, or equivalently, in $\mathbf{R}^4$. In what follows we give an algebraic characterization of $\mathcal{T}$, and thus of these 'knotted surfaces'.

3 Statement of Theorem

In what follows, by monoidal and braided monoidal 2-categories we mean 'semistrict' ones as defined in reference [5], but with the braided monoidal 2-categories satisfying the extra axioms introduced by Crans [10], which say that $R_{\cdot,\cdot}$, $\tilde{R}_{(\cdot|\cdot,\cdot)}$, and $\tilde{R}_{(\cdot,\cdot|\cdot)}$ are the identity whenever one of the arguments is the unit object I.

Definition 1. *A* monoidal 2-category with duals *is, to begin with, a monoidal 2-category equipped with the following structures:*

1. For every 2-morphism $\alpha\colon f \Rightarrow g$ there is a 2-morphism $\alpha^*\colon g \Rightarrow f$ called the *dual* of α.

2. For every morphism $f\colon A \to B$ there is a morphism $f^*\colon B \to A$ called the *dual* of f, and 2-morphisms $i_f\colon 1_A \Rightarrow ff^*$ and $e_f\colon f^*f \Rightarrow 1_B$, called the *unit* and *counit* of f, respectively.

3. For any object A, there is a object A^* called the *dual* of A, morphisms $i_A\colon I \to A \otimes A^*$ and $e_A\colon A^* \otimes A \to I$ called the *unit* and *counit* of A, respectively, and a 2-morphism $T_A\colon (i_A \otimes A)(A \otimes e_A) \Rightarrow 1_A$ called the *triangulator* of A.

We say that a 2-morphism α is unitary *if it is invertible and $\alpha^{-1} = \alpha^*$. Given a 2-morphism $\alpha\colon f \Rightarrow g$, we define the* adjoint *$\alpha^\dagger\colon g^* \Rightarrow f^*$ by*

$$\alpha^\dagger = (g^* i_f) \cdot (g^* \alpha f^*) \cdot (e_g f^*).$$

In addition, the structures above are also required to satisfy the following conditions:

1. $X^{**} = X$ for any object, morphism or 2-morphism X.

2. $1_X^* = 1_X$ for any object or morphism X.

3. For all objects A, B, morphisms f, g, and 2-morphisms α, β for which both sides of the following equations are well-defined, we have

$$(\alpha \cdot \beta)^* = \beta^* \cdot \alpha^*,$$

$$(\alpha \circ \beta)^* = \alpha^* \circ \beta^*,$$

$$(fg)^* = g^* f^*,$$

$$(A \otimes \alpha)^* = A \otimes \alpha^*, \qquad (\alpha \otimes A)^* = \alpha^* \otimes A,$$
$$(A \otimes f)^* = A \otimes f^*, \qquad (f \otimes A)^* = f^* \otimes A,$$

and

$$(A \otimes B)^* = B^* \otimes A^*.$$

4. For all morphisms f and g, the 2-morphism $\bigotimes_{f,g}$ is unitary.

5. For any object or morphism X we have $i_{X^*} = e_X^*$ and $e_{X^*} = i_X^*$.

6. For any object A, the 2-morphism T_A is unitary.

7. If I is the unit object, $T_I = 1_{1_I}$.

8. For any objects A and B we have

$$i_{A\otimes B} = i_A(A \otimes i_B \otimes A^*),$$

$$e_{A\otimes B} = (B^* \otimes e_A \otimes B)e_B,$$

and

$$T_{A\otimes B} = [(i_A \otimes A \otimes B)(A \otimes \bigotimes\nolimits_{i_B,e_A}^{-1} \otimes B)(A \otimes B \otimes e_B)] \cdot [(T_A \otimes B) \circ (A \otimes T_B)].$$

9. For any object A and morphism f we have

$$i_{A\otimes f} = A \otimes i_f, \qquad i_{f\otimes A} = i_f \otimes A,$$

$$e_{A\otimes f} = A \otimes e_f, \qquad e_{f\otimes A} = e_f \otimes A.$$

10. For any morphisms f and g, $i_{fg} = i_f \cdot (f i_g f^*)$ and $e_{fg} = (g^* e_f g) \cdot e_g$.

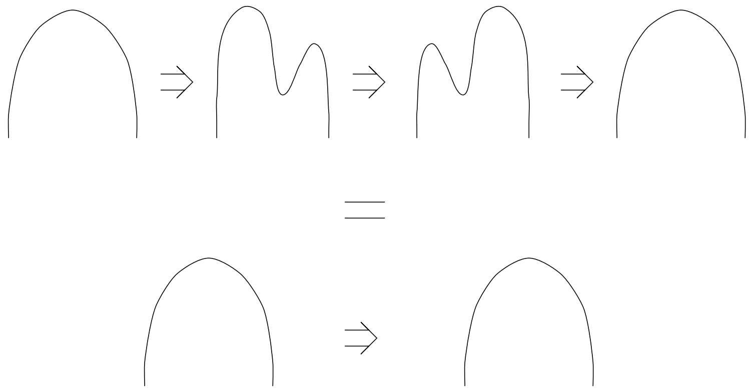
11. For any morphism f, $i_f f \cdot f e_f = 1_f$ and $f^* i_f \cdot e_f f^* = 1_{f^*}$.

12. For any 2-morphism α, $\alpha^{\dagger *} = \alpha^{*\dagger}$.

13. For any object A we have

$$[i_A(A \otimes T_{A^*}^\dagger)] \cdot [\bigotimes\nolimits_{i_A,i_A}^{-1}(A \otimes e_A \otimes A^*)] \cdot [i_A(T_A \otimes A^*)] = 1_{i_A}.$$

The final equation has the following geometrical interpretation in terms of movie moves (see [7]):



1. Equation satisfied by the triangulator

Note that the equations in clause 5 of the definition above allow us to express the counit in terms of the unit (or vice versa) using duality. This allows us to avoid mentioning the counit at certain points in some definitions below.

Definition 2. *A* braided monoidal 2-category with duals *is a monoidal 2-category with duals that is also a braided monoidal 2-category for which the braiding is unitary in the sense that:*

1. *For any objects A, B, the 2-morphisms $i_{R_{A,B}}$ and $e_{R_{A,B}}$ are unitary.*
2. *For any object A and morphism f, the 2-morphisms $R_{A,f}$ and $R_{f,A}$ are unitary.*
3. *For any objects A, B, C, the 2-morphisms $\tilde{R}_{(A,B|C)}$ and $\tilde{R}_{(A|B,C)}$ are unitary.*

In general the above definition does not deal adequately with the subtle issue of framings (or algebraically speaking, balancings). We can use this definition here because we are mainly interested in $\mathcal{T}$, which is generated by an 'unframed self-dual' object in the sense defined below. Geometrically, this object is simply a point embedded in the unit square.

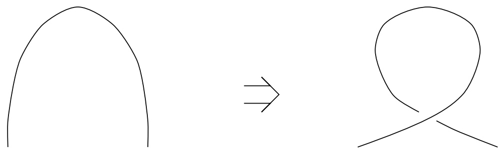
In previous work [1], it was shown how the balancing arises naturally in any braided monoidal category with duals. The same idea applies to braided monoidal 2-categories with duals. Explicitly, for any object A in a braided monoidal 2-category with duals, the balancing $b_A\colon A \to A$ is given by:

$$b_A = (e_A^* \otimes A)(A^* \otimes R_{A,A})(e_A \otimes A).$$

For an 'unframed' object A, the 1st Reidemeister move corresponds to a 2-morphism $V_A\colon b_A \Rightarrow 1_A$. However, the connection to the movie moves of Carter, Rieger and Saito becomes a bit clearer if we work not with the balancing but with the closely related morphism $i_{A^*} R_{A^*,A}\colon 1 \to A \otimes A^*$. The 1st Reidemeister move then corresponds to a 2-isomorphism

$$W_A\colon i_A \Rightarrow i_{A^*} R_{A^*,A}$$

which we call the 'writhing', with the following geometrical interpretation:



2. The writhing

One can construct a 2-isomorphism $V_A\colon b_A \Rightarrow 1_A$ given a 2-isomorphism $W_A\colon i_A \Rightarrow i_{A^*}R_{A^*,A}$, and conversely. In what follows we only study the writhing for a self-dual object A.

Definition 3. *A* self-dual object *in a braided monoidal 2-category with duals is an object A with $A^* = A$.*

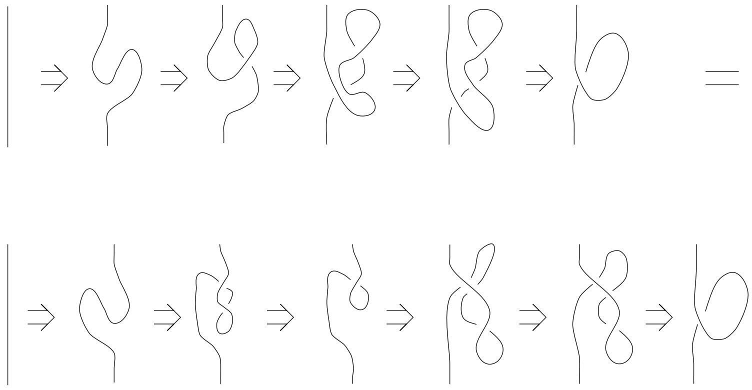
Definition 4. *A self-dual object A in a braided monoidal 2-category with duals is* unframed *if $\tilde{R}_{(A|A,A)} = 1$, $\tilde{R}_{(A,A|A)} = 1$, and it is equipped with a unitary 2-morphism*

$$W_A\colon i_A \Rightarrow i_A R_{A,A}$$

called the writhing*, satisfying the equation:*

$$\begin{gathered}
T_A^\dagger \cdot ((A\otimes W_A)(e_A\otimes A)) \cdot ((A\otimes (i_A R_{A,A}))R^\dagger_{A,i_A})\cdot \\
((A\otimes (i_A R_{A,A}))\tilde{R}^\dagger_{(A|A,A)}(A\otimes e_A)) \cdot ((A\otimes (i_A i^*_{R_{A,A}}))(R^*_{A,A}\otimes A)(A\otimes e_A)) \\
= T_A^{-1}\cdot ((i_A\otimes A)(A\otimes (i_{R_{A,A}}e_A)))\cdot ((i_A\otimes A)(A\otimes (R_{A,A}W_A^\dagger)))\cdot \\
(R^{\dagger\,-1}_{e_A,A}(A\otimes (R_{A,A}e_A)))\cdot ((A\otimes i_A)\tilde{R}^\dagger_{(A,A|A)}(A\otimes (R_{A,A}e_A)))\cdot \\
((A\otimes i_A)(R^*_{A,A}\otimes A)(A\otimes (e_{R_{A,A}}e_A)))
\end{gathered}$$

The rather terrifying equation above has the following geometrical interpretation in terms of 'movie moves':



3. Equation satisfied by the writhing

The conditions $\tilde{R}_{(A|A,A)} = 1$ and $\tilde{R}_{(A|A,A)} = 1$ have nothing to do with framing per se; they actually amount to a kind of 'strictification' of the braiding as far as the object A is concerned. We include them in the definition of 'unframed object' merely to simplify the exposition in what follows, and will remove them in our forthcoming more detailed treatment.

Definition 5. *We say a braided monoidal 2-category is* generated *by an unframed self-dual object Z if:*

1. Every object is a tensor product of copies of Z.
2. Every morphism can be obtained by composition from:
   (a) 1_Z,
   (b) i_Z,
   (c) $R_{Z,Z}$,
   (d) tensor products of arbitrary objects with the above morphisms,
   (e) duals of the above morphisms.
3. Every 2-morphism can be obtained by horizontal and vertical composition from:
   (a) 2-morphisms 1_f for arbitrary morphisms f,
   (b) 2-morphisms $\bigotimes_{f,g}$ for arbitrary morphisms f and g,
   (c) 2-morphisms $R_{Z,f}$ and $R_{f,Z}$ for arbitrary morphisms f,
   (d) 2-morphisms i_f for arbitrary morphisms f,
   (e) T_Z,
   (f) W_Z,
   (g) tensor products of arbitrary objects with the above 2-morphisms, and
   (h) duals of the above 2-morphisms.

Definition 6. *For monoidal 2-categories $\mathcal{C}$ and $\mathcal{D}$, a* strict monoidal 2-functor *$F\colon \mathcal{C} \to \mathcal{D}$ is a 2-functor such that $F(1) = 1$, $F(A \otimes X) = F(A) \otimes F(X)$, $F(X \otimes A) = F(X) \otimes F(A)$ and $F(\bigotimes_{f,g}) = \bigotimes_{F(f),F(g)}$, for any object A, object, morphism or 2-morphism X, and morphisms f and g.*

Definition 7. *If $\mathcal{C}, \mathcal{D}$ are braided monoidal 2-categories with duals and $\mathcal{C}$ is generated by the unframed self-dual object Z, we say a monoidal 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ mapping Z to an unframed self-dual object in $\mathcal{D}$* preserves braiding and duals strictly on the generator *if:*

(a) $F(X^*) = F(X)^*$ for every object, morphism, or 2-morphism X,

(b) $F(i_f) = i_{F(f)}$ for every morphism f,

(c) $F(i_Z) = i_{F(Z)}$,

(d) $F(T_Z) = T_{F(Z)}$,

(e) $F(W_Z) = W_{F(Z)}$

(f) $F(R_{Z,Z}) = R_{F(Z),F(Z)}$,

(g) $F(R_{Z,f}) = R_{F(Z),F(f)}$ and $F(R_{f,Z}) = R_{F(f),F(Z)}$ for f equal to $R_{Z,Z}, R^*_{Z,Z}, i_Z$, and e_Z.

Theorem 8. *There is a braided monoidal 2-category with duals $\mathcal{T}$ for which there is an explicit one-to-correpondence between 2-morphisms of $\mathcal{T}$ and smooth ambient isotopy classes of unframed unoriented smooth 2-tangles in 4 dimensions. $\mathcal{T}$ is generated by an unframed self-dual object Z, and for any braided monoidal 2-category with duals $\mathcal{C}$ and unframed self-dual object $A \in \mathcal{C}$ there is a unique strict monoidal 2-functor $F\colon \mathcal{T} \to \mathcal{C}$ with $F(Z) = A$ that preserves braiding and duals strictly on the generator.*

Thus we say that $\mathcal{T}$ is the *free braided monoidal 2-category with duals on one unframed self-dual object.*

4 Conclusions

Our result implies that we obtain an invariant of unframed unoriented 2-tangles in 4 dimensions from any braided monoidal 2-category with duals containing a self-dual unframed object. Similarly, we expect to be able to prove that the 2-category of framed oriented 2-tangles is the free braided monoidal 2-category with duals on one object. This would enable us to obtain invariants of framed oriented 2-tangles from arbitrary braided monoidal 2-categories with duals. However, given the rather lengthy definitions above, it is natural to wonder whether we can actually find invariants this way in practice: are there any interesting *examples* of braided monoidal 2-categories with duals?

We believe there are many examples and that the problem is mainly a matter of developing the machinery to get our hands on them. First, there is plenty of evidence [2, 3] suggesting that we can obtain braided monoidal 2-categories from the homotopy 2-types of double loop spaces. Second, Neuchl and the first author have shown how to obtain braided monoidal 2-categories from monoidal 2-categories by a 'quantum double' construction [5]. It seems plausible that applying this construction to a monoidal 2-category with duals will give a braided monoidal 2-category with duals. This reduces the question to obtaining monoidal 2-categories with duals. A good example of one of these should be the monoidal 2-category of unitary representations of a 2-groupoid, just as the monoidal category of unitary representations of a groupoid is an example of a monoidal category with duals [1]. Third, Crane and Frenkel have sketched a way to construct Hopf categories from Kashiwara and Lusztig's canonical bases for quantum groups [9]. There is reason to hope that the representation 2-categories of these Hopf categories are monoidal 2-categories with duals. Fourth, just as one can construct braided monoidal categories from solutions of the Yang-Baxter equation, one can construct braided monoidal 2-categories from solutions of the Zamolodchikov tetrahedron equations [14]. Many such solutions are known [8], so one may hope that some give braided monoidal 2-categories with duals. Finally, one expects 'braided monoidal 3-Hilbert spaces' to be interesting examples of braided monoidal 2-categories with duals [1].

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