MY FAVORITE NUMBERS:

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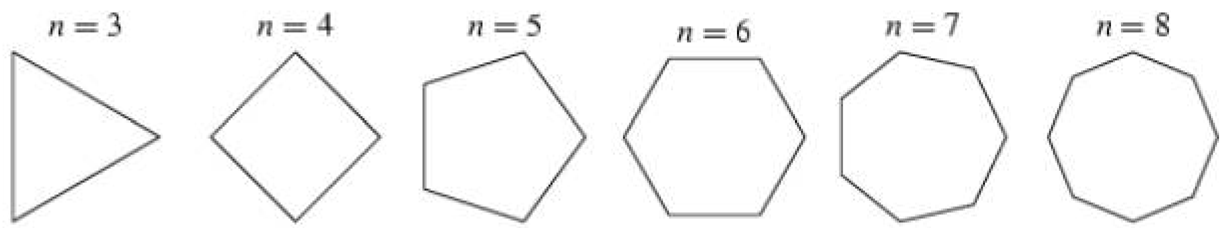
The Rankin Lectures

Supported by the Glasgow Mathematical Journal Trust

Different numbers have different 'personalities'.
The number 5 is difficult, angular, and awkward.

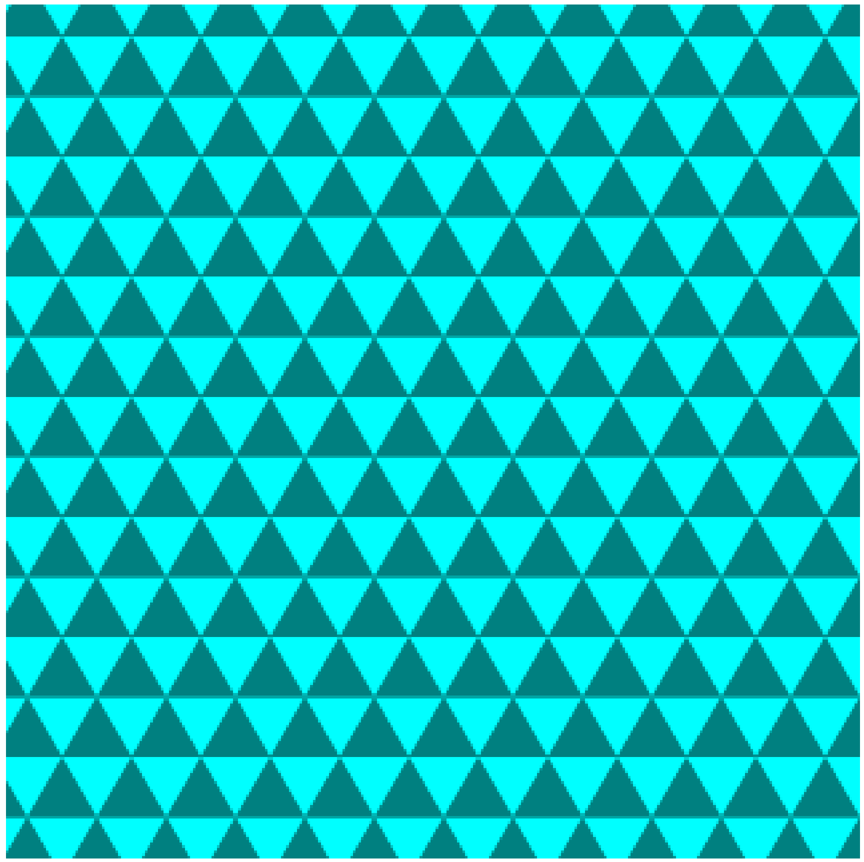
For example, there are regular polygons with any number of sides:



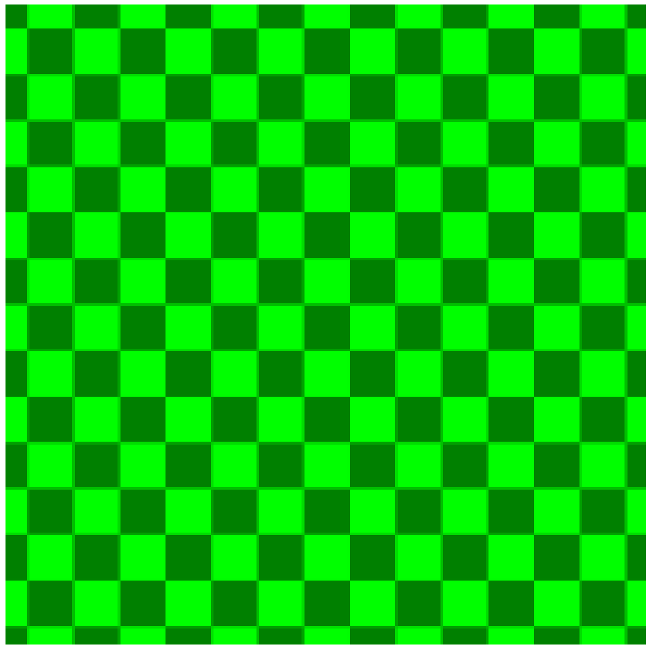
But 5 is the first that refuses to tile the plane.

Why?

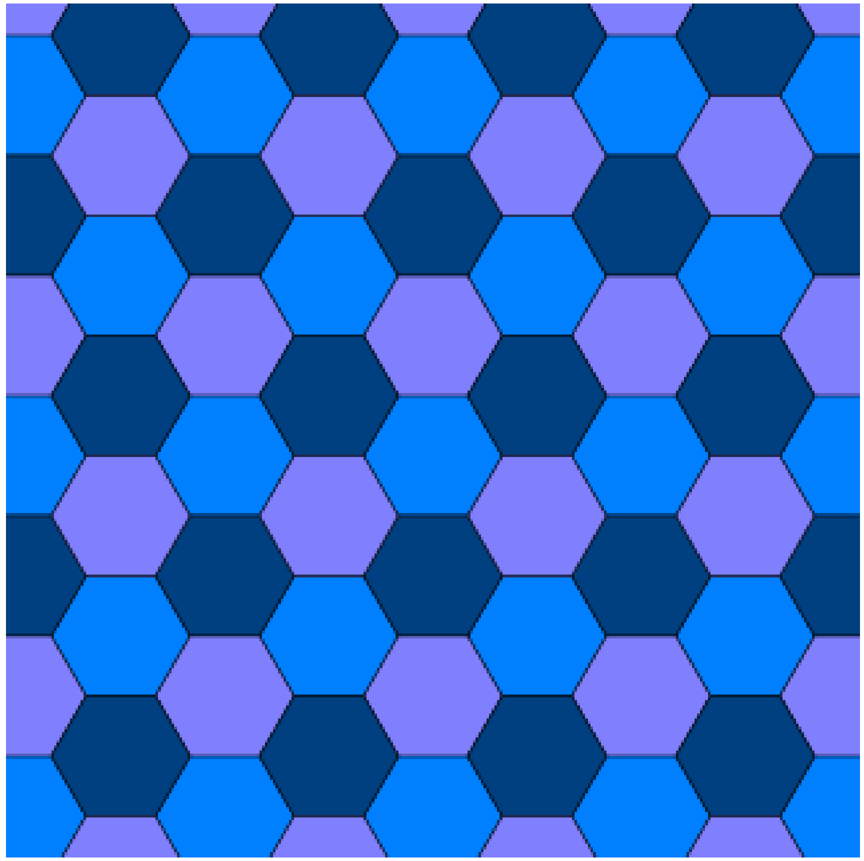
You can tile the plane with equilateral triangles:



You can also do it with squares:

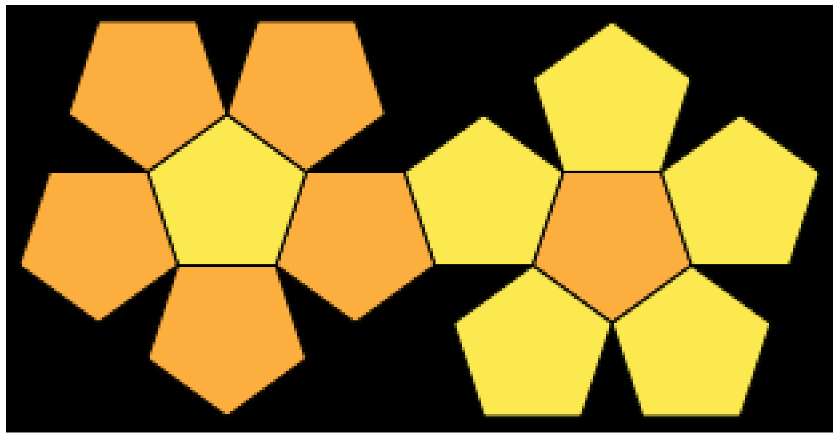


You can do it with regular hexagons:



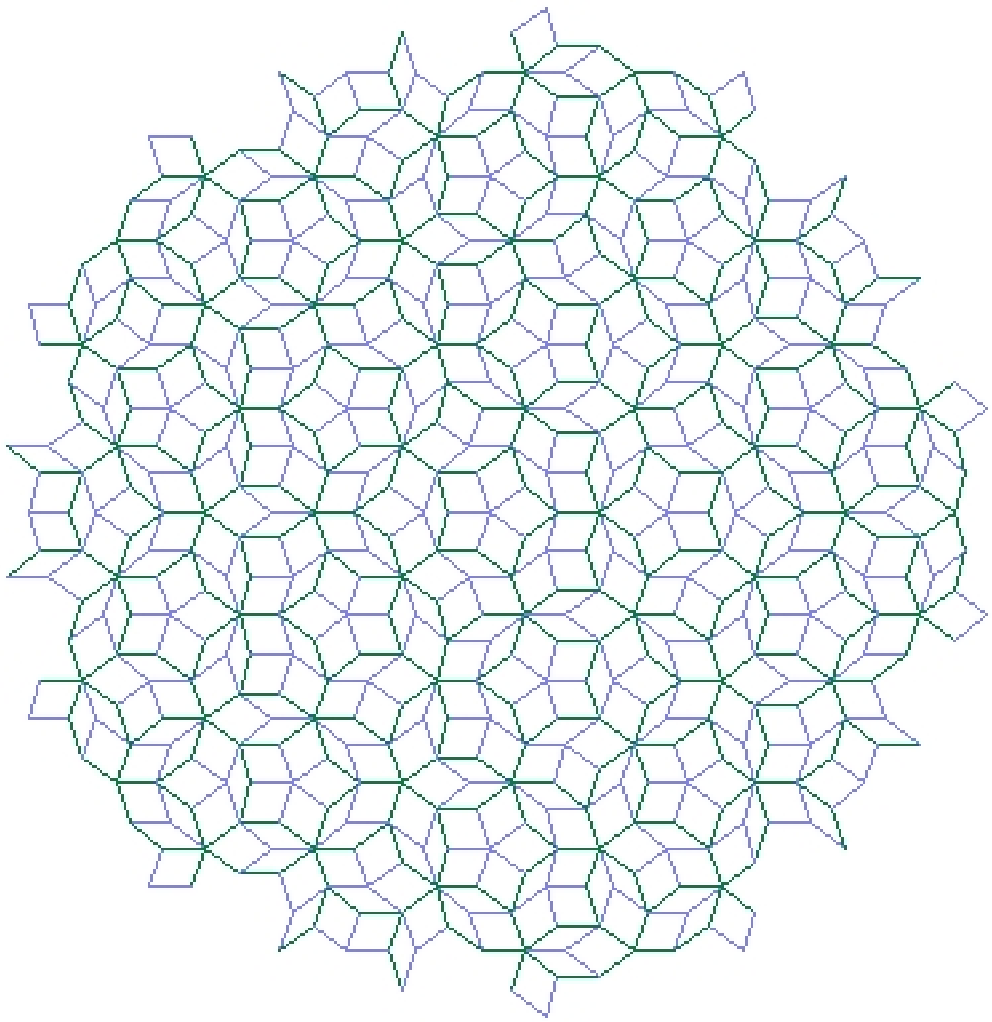
Regular polygons with 7 or more sides are clearly too fat, so they're excused.

But pentagons don't work for another reason!
There are small gaps:

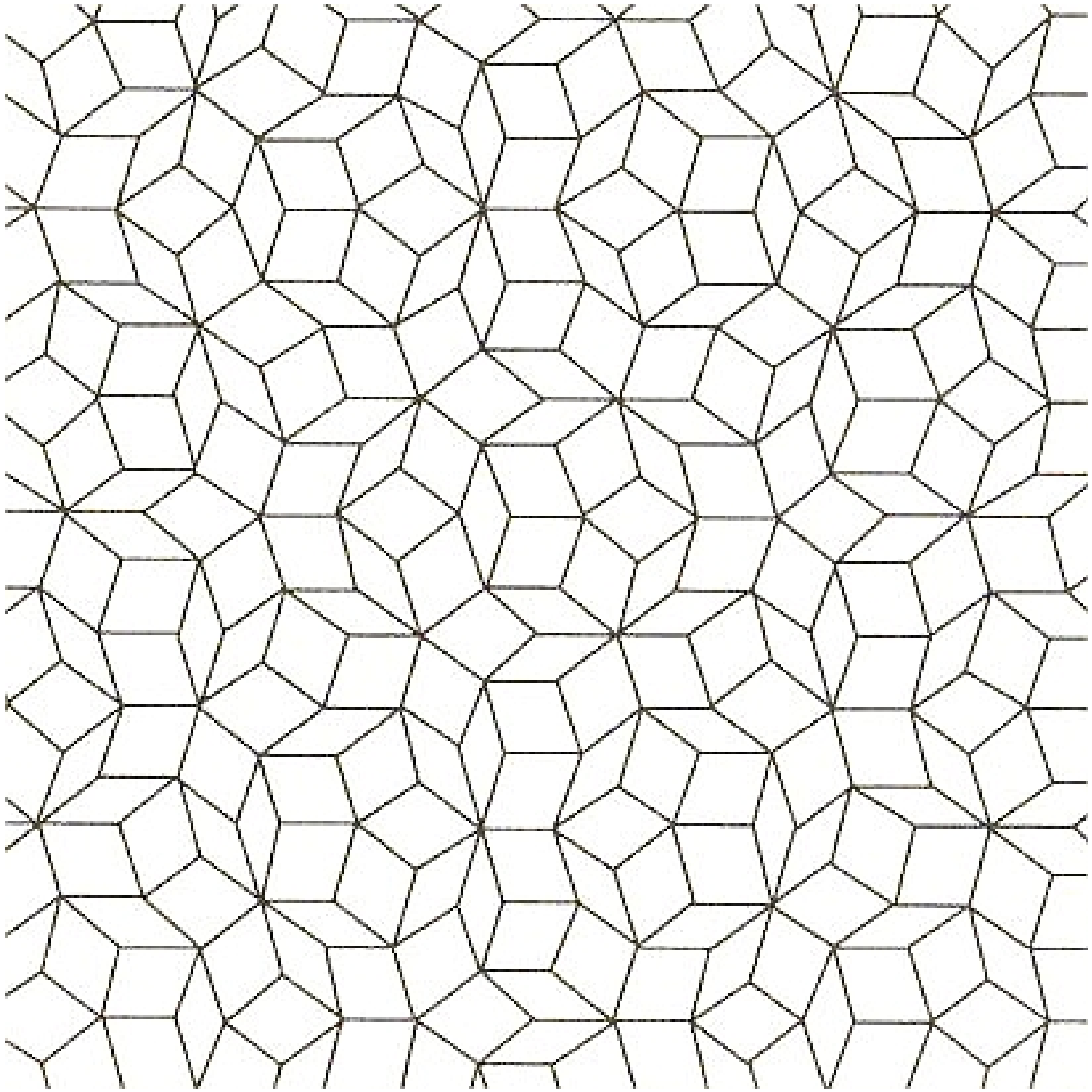


Indeed, a periodic tiling of the plane with 5-fold symmetry is impossible.

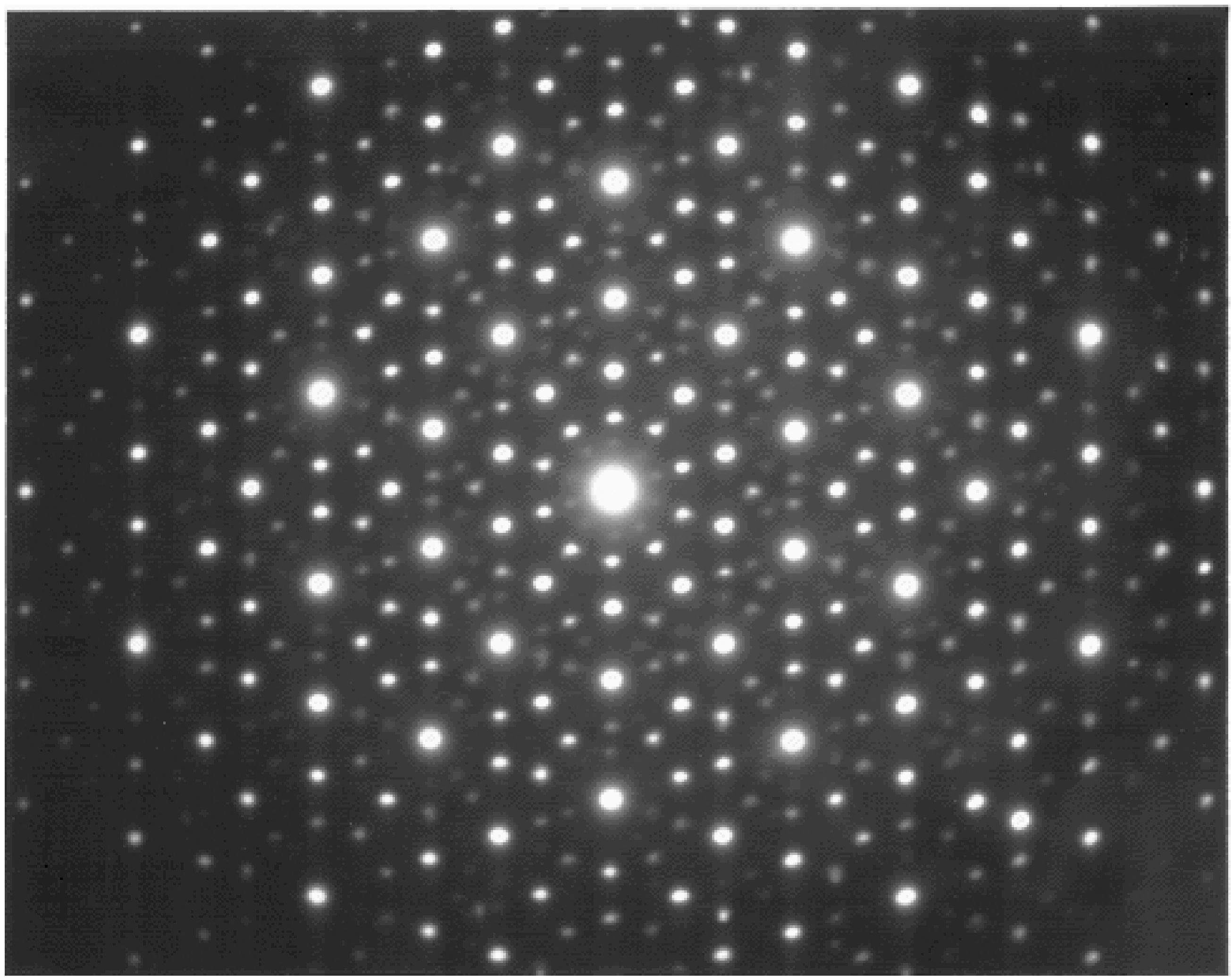
However, in 1972 Roger Penrose invented *nonperiodic* tilings with perfect 5-fold symmetry:



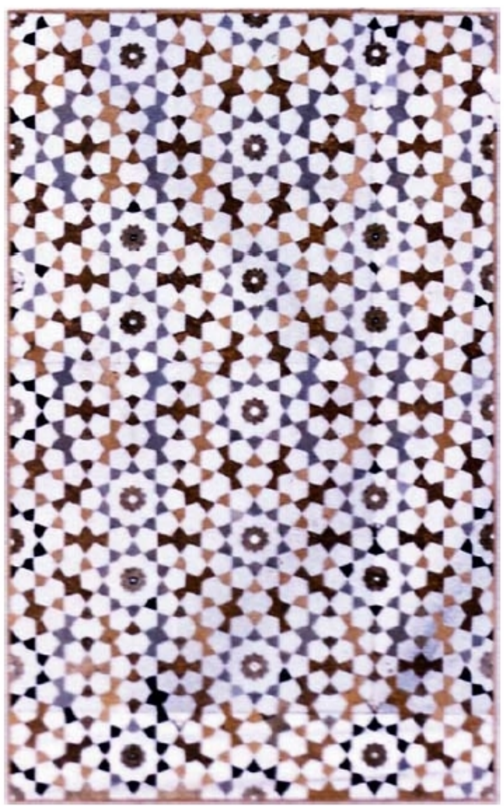
...or 'approximate' 5-fold symmetry:

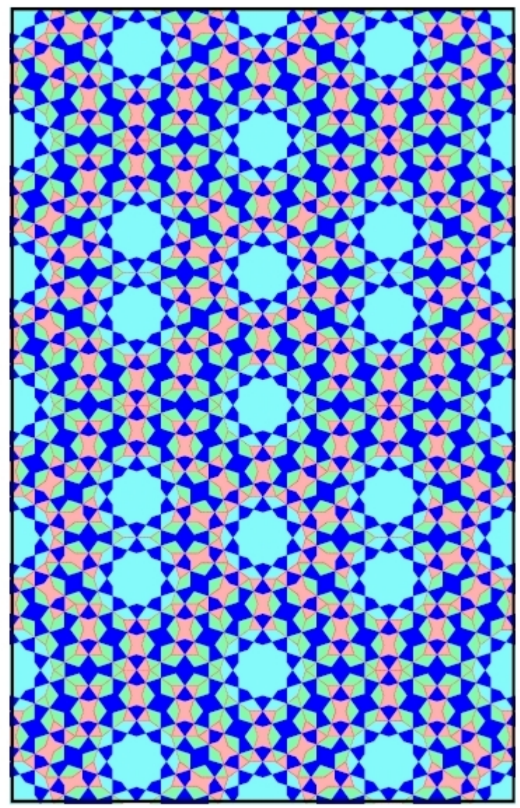


Periodic crystals with 5-fold symmetry
are impossible.
But in the 1980s, physicists and chemists discovered
'quasicrystals' with approximate 5-fold symmetry:

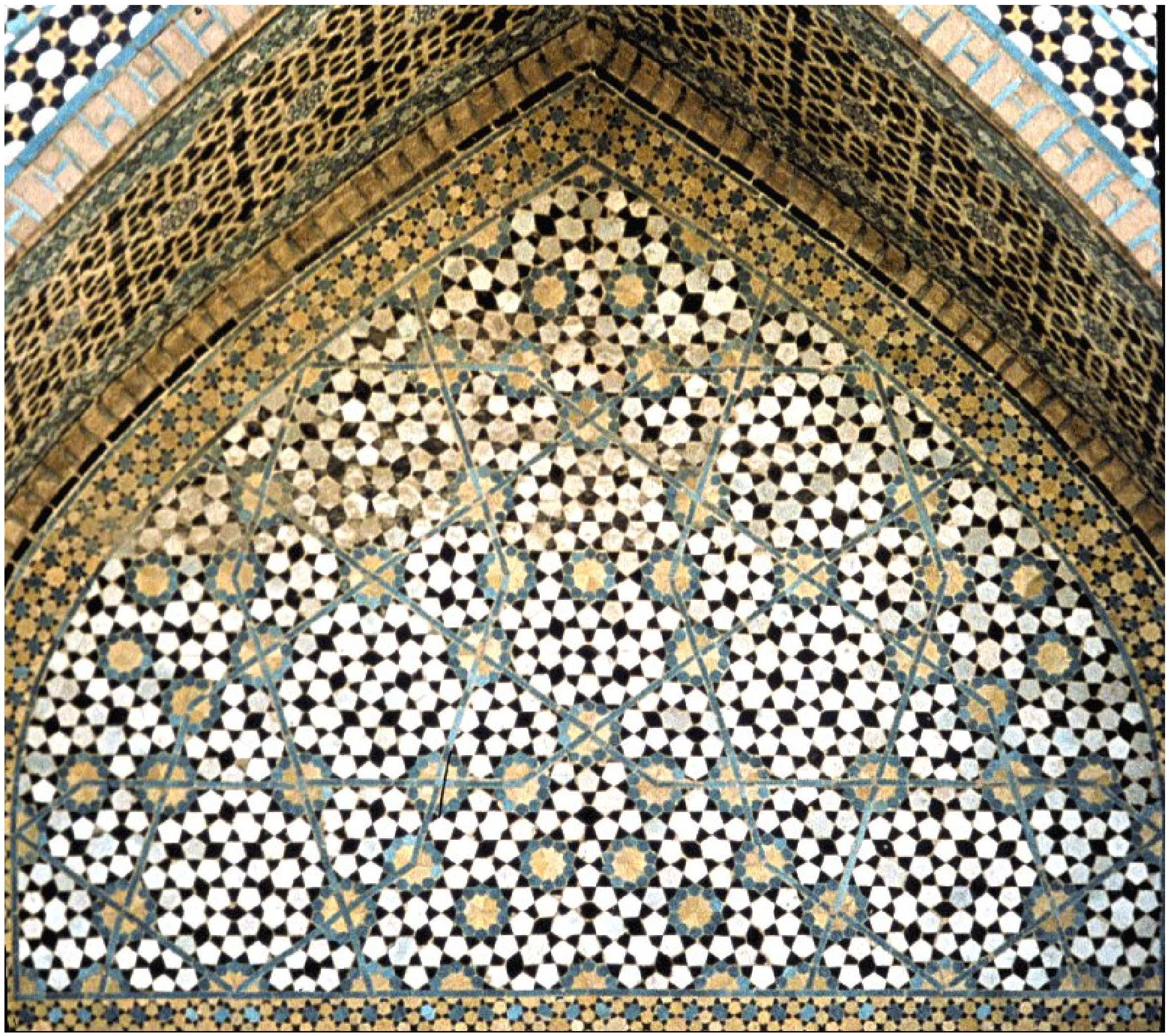


In 2007, it was noticed that some Islamic tile designs have 5-fold symmetry –
beating Penrose to the punch by several centuries!
From the I'timid al-Daula mausoleum in Agra, India, built in 1622:

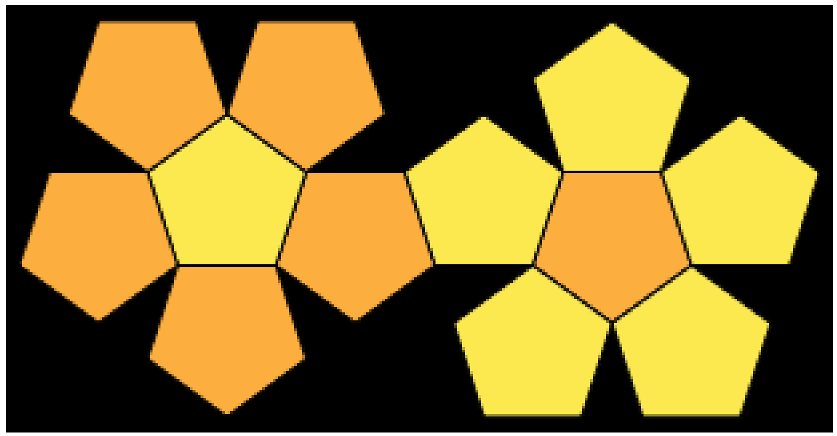




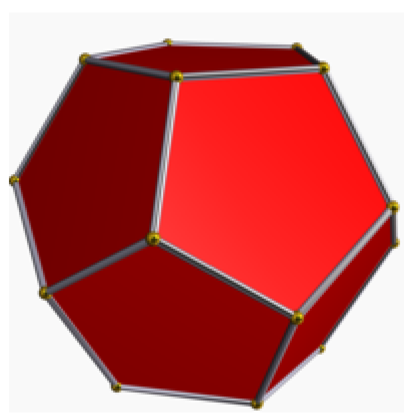
From the Darb-i Imam shrine in Isfahan, Iran, also built in the 1600s:



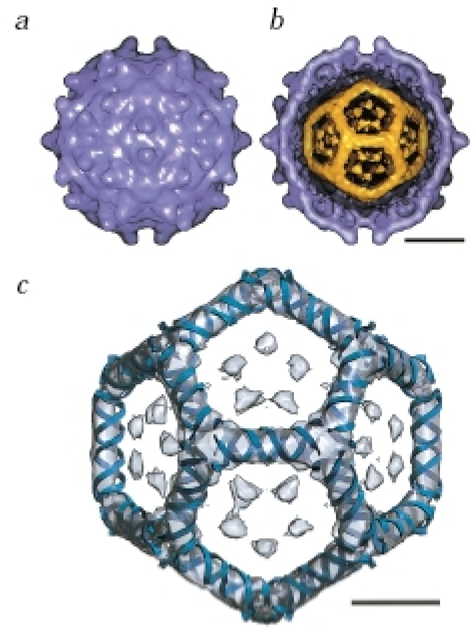
If we don't demand *periodic* patterns, 5-fold symmetry in 3 dimensions is easy. These pentagons:



curl up to form a *dodecahedron*, a 3d shape with 5-fold symmetry:

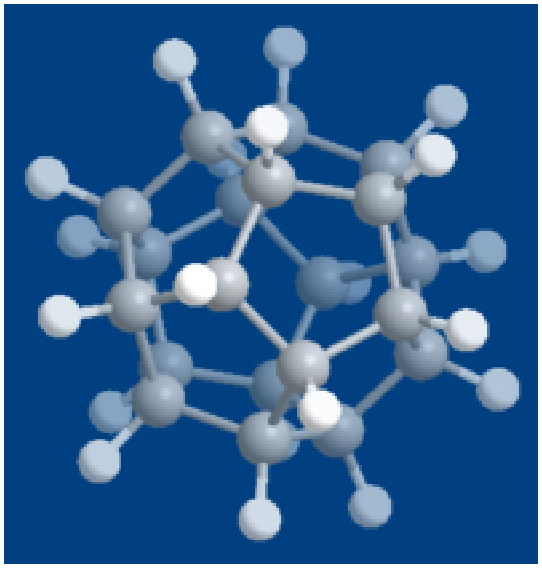


Nature takes advantage of this possibility.
The ‘Pariacoto virus’ contains a dodecahedron of RNA:



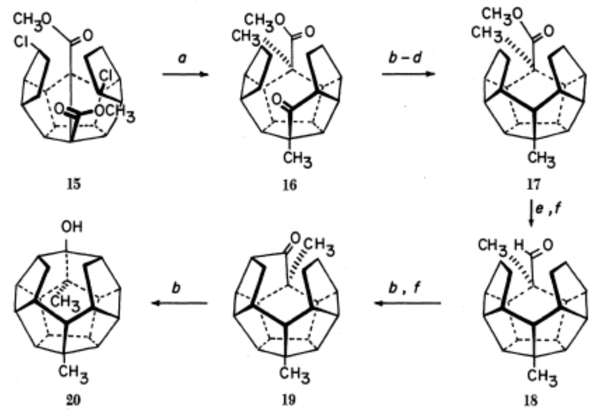
The first gray line is 10 nanometers long (10^{-8} meters).
The second is 5 nanometers.

The smallest known dodecahedron is
'dodecahedrane':



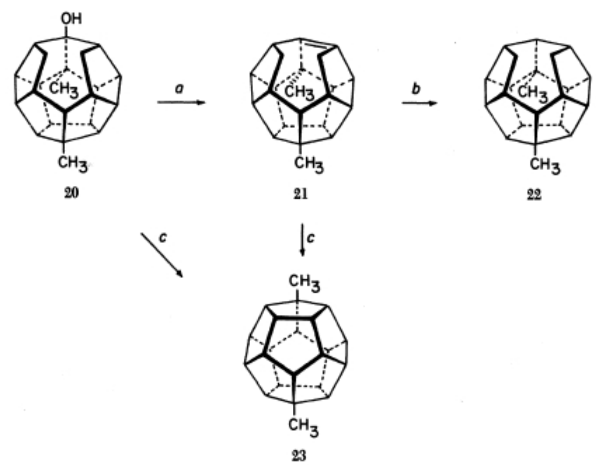
Chemical formula: $C_{20}H_{20}$

Dodecahedrane hasn't been found in nature, but it's recently been made in the lab:



a, Li/NH_3, then CH_3I; *b*, hν; *c*, TsOH, C_6H_6, *d*, HN═NH, CH_3OH; *e*, (*i*-Bu)$_2$AlH; *f*, PCC, CH_2Cl_2.

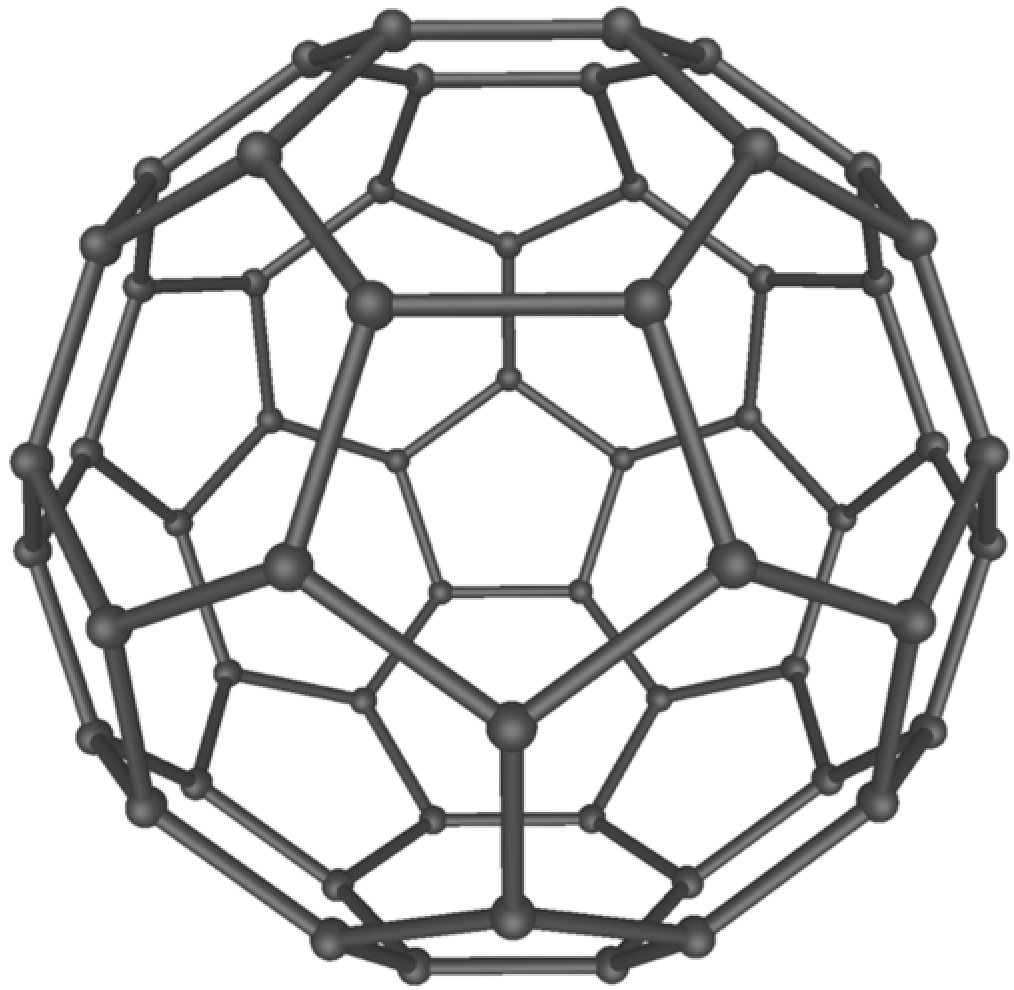
Scheme IV



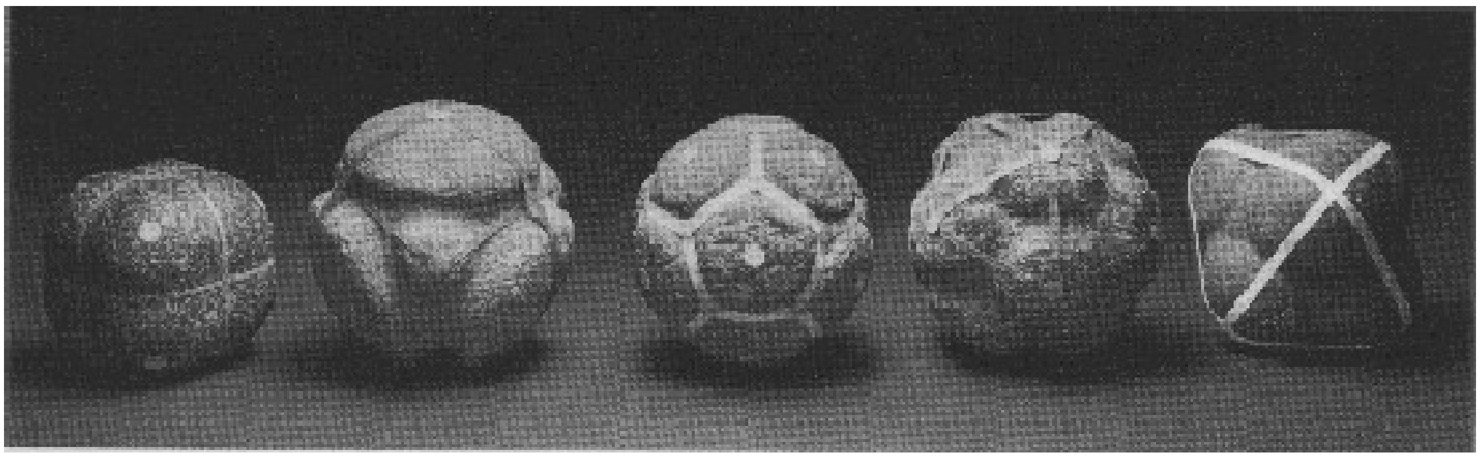
a, TsOH, C_6H_6; *b*, H_2NNH_2, H_2O_2; *c*, CF_3SO_3H, GH_2Cl_2.

Scheme V

Soot and interstellar dust are full of carbon in the form of 'buckyballs':



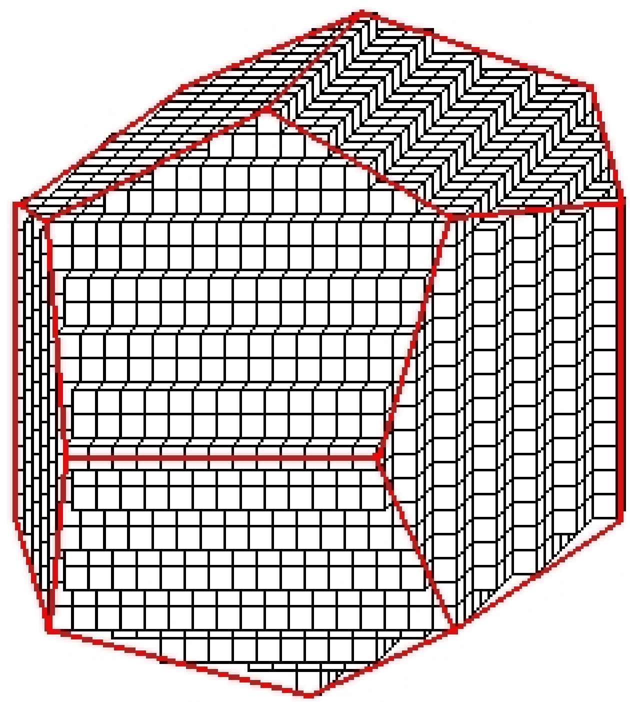
But the oldest *human-made* dodecahedron is,
of course,
Scottish:



It dates back to around 2000 BC!
Nobody knows what these carved stone balls were used for.

The ancient Greeks reinvented the dodecahedron – perhaps inspired by iron pyrite, or ‘fool’s gold’:

But fool's gold cannot have 5-fold symmetry – no crystal can!
What you just saw was a 'pyritohedron':



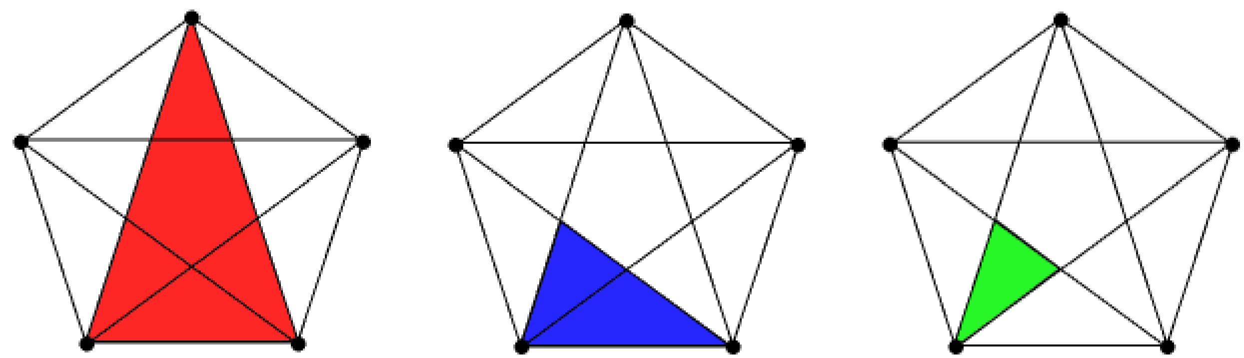
The Greek colonies in Sicily had a lot of pyrite...
and Pythagoreans!

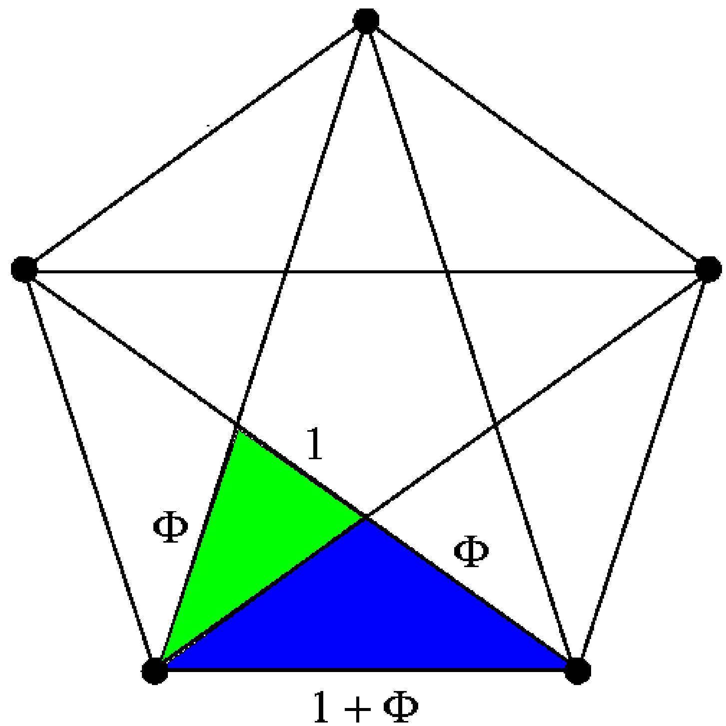
Pythagoras of Samos
(about 582–507 BC)

The Pythagoreans may have invented the regular dodecahedron by 'perfecting' the pyrite crystals they saw.

The Pythagoreans were also fascinated by pentagrams:

The pentagram contains 20 triangles with the same proportions, in three different sizes:





$$\frac{\Phi}{1} = \frac{\Phi + 1}{\Phi}$$

$$\Phi^2 = \Phi + 1$$

$\boldsymbol{\Phi^2 = \Phi + 1}$ **has this solution:**

$$\boldsymbol{\Phi = \frac{1+\sqrt{5}}{2} = 1.6180339\ldots}$$

This is called the *golden ratio*.

$$\boldsymbol{\Phi^2 = \Phi + 1 = 2.6180339\ldots}$$

$$\boldsymbol{\Phi = 1 + \tfrac{1}{\Phi}}$$

$$\boldsymbol{\tfrac{1}{\Phi} = \Phi - 1 = 0.6180339\ldots}$$

It seems the Greeks liked 'continued fractions':

$$\begin{aligned}
\frac{74}{32} &= 2 + \frac{10}{32} \\
&= 2 + \frac{1}{\frac{32}{10}} \\
&= 2 + \frac{1}{3 + \frac{2}{10}} \\
&= 2 + \frac{1}{3 + \frac{1}{\frac{10}{2}}} \\
&= 2 + \frac{1}{3 + \frac{1}{5}}
\end{aligned}$$

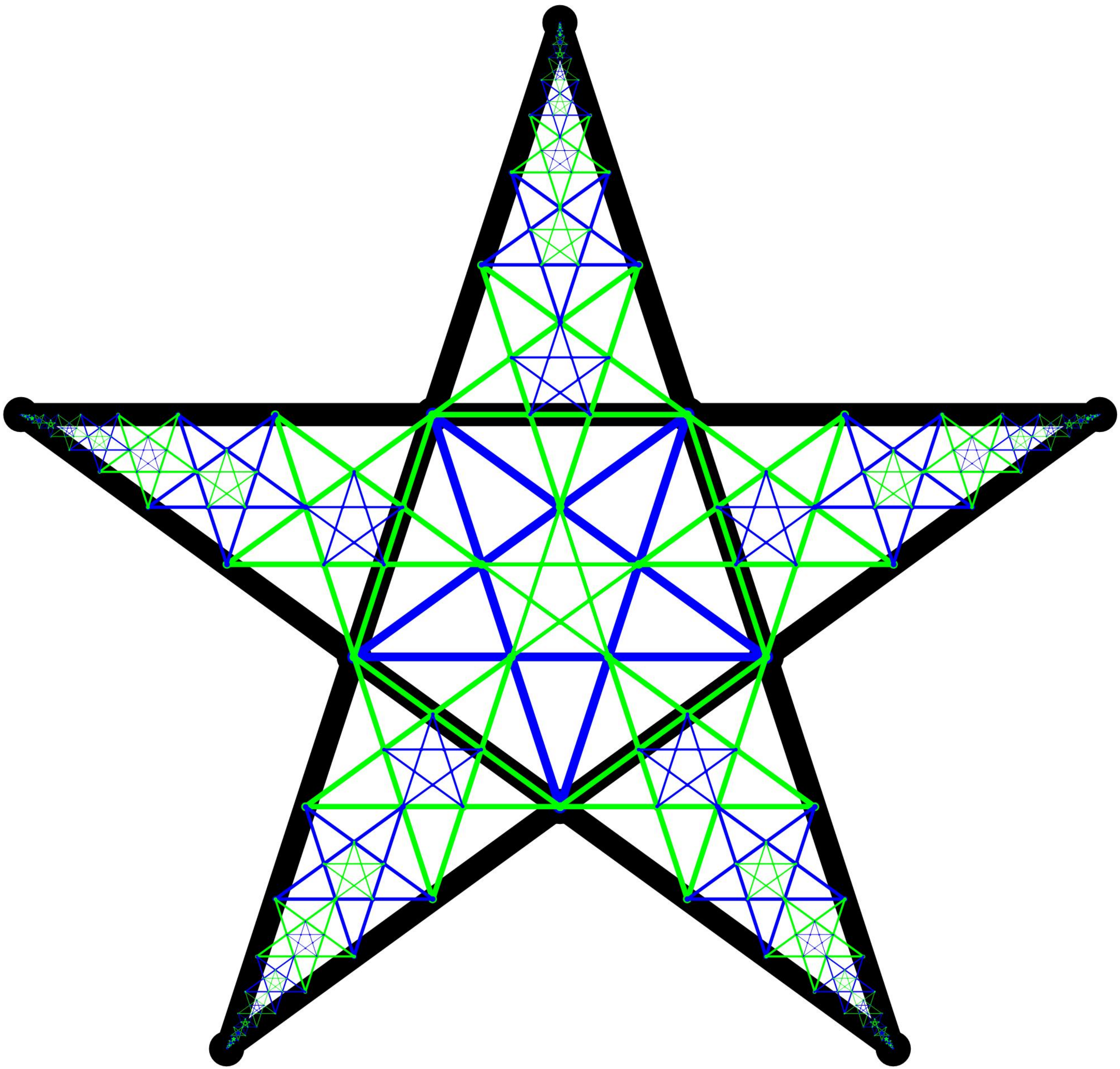
The continued fraction of Φ never ends:

$$\begin{aligned}\Phi &= 1+\tfrac{1}{\Phi}\\ &= 1+\tfrac{1}{1+\frac{1}{\Phi}}\\ &= 1+\tfrac{1}{1+\frac{1}{1+\frac{1}{\Phi}}}\\ &= 1+\tfrac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\Phi}}}}\\ &= \text{etcetera...}\end{aligned}$$

So, Φ is irrational!

We can also use the pentagram to see that the continued fraction expansion for Φ never ends.

Did the Pythagoreans know this?



In fact, any irrational square root gets stuck in a loop:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}$$

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}$$

But the golden ratio is the simplest:

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}$$

Ironically, this makes it the 'most irrational number': the hardest to approximate by simple rational numbers. Why?

We get the best rational approximations of a number by taking its continued fraction expansion and quitting at some point.

This works *really* well when we quit after hitting a *really* big number.

For example, take a look at π:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{\ddots}}}}}$$

15 is pretty big, so this approximation is pretty good:

$$\mathbf{3 + \frac{1}{7} = \frac{22}{7} = 3.1428...}$$

292 is *really* big, so this approximation is *great:*

$$\mathbf{3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113} = 3.1415929...}$$

It matches 6 decimals of

$$\boldsymbol{\pi} \mathbf{= 3.1415926...}$$

This approximation was discovered in 480 AD by the Chinese mathematician Zu Chongzhi!

But now try the golden ratio, Φ:

$$1+\frac{1}{1} = \frac{2}{1} = 2$$

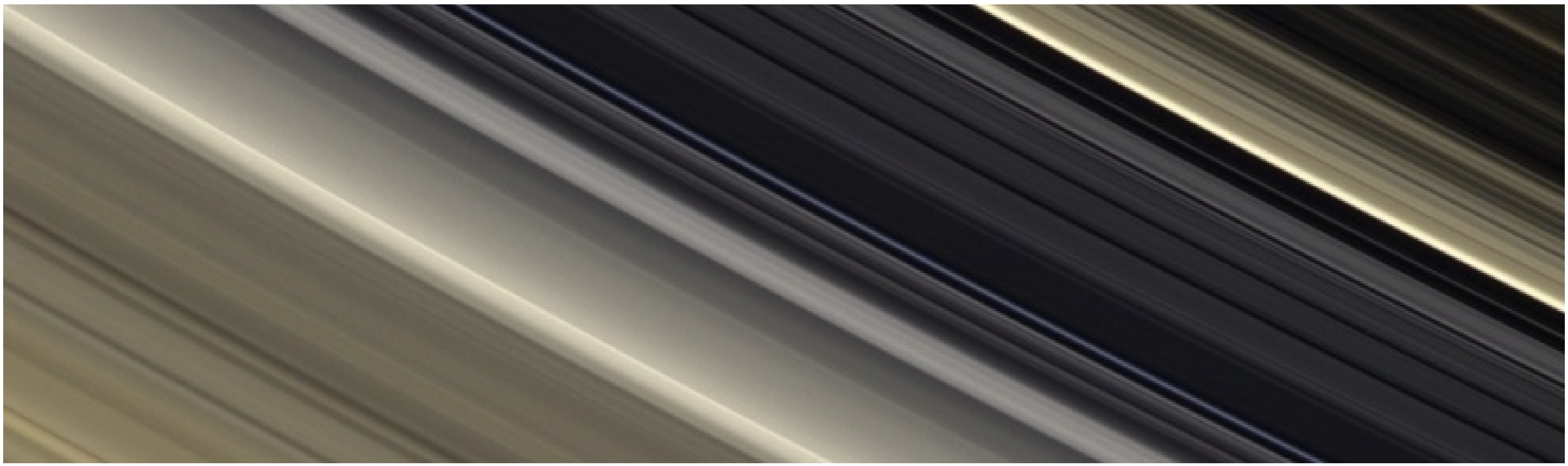
$$1+\frac{1}{1+\frac{1}{1}} = \frac{3}{2} = 1.5$$

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}} = \frac{5}{3} = 1.666...$$

$$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} = \frac{8}{5} = 1.6$$

We're *slowly* creeping our way to $\Phi = 1.6180339\ldots$
To get 6 decimals right, we must go to $\frac{1597}{987}$.

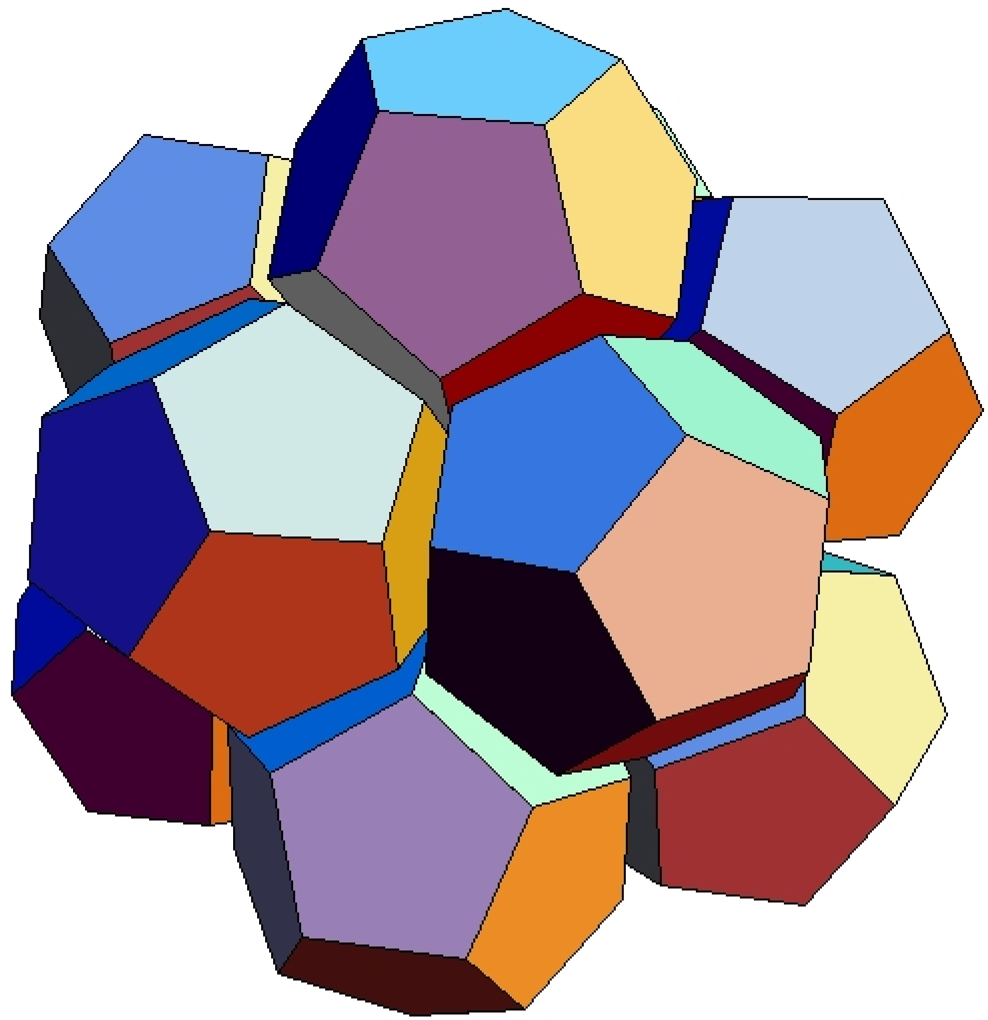
Two piano strings *resonate* when they vibrate at frequencies related by a simple rational number. The same is true for dust in the rings of Saturn...



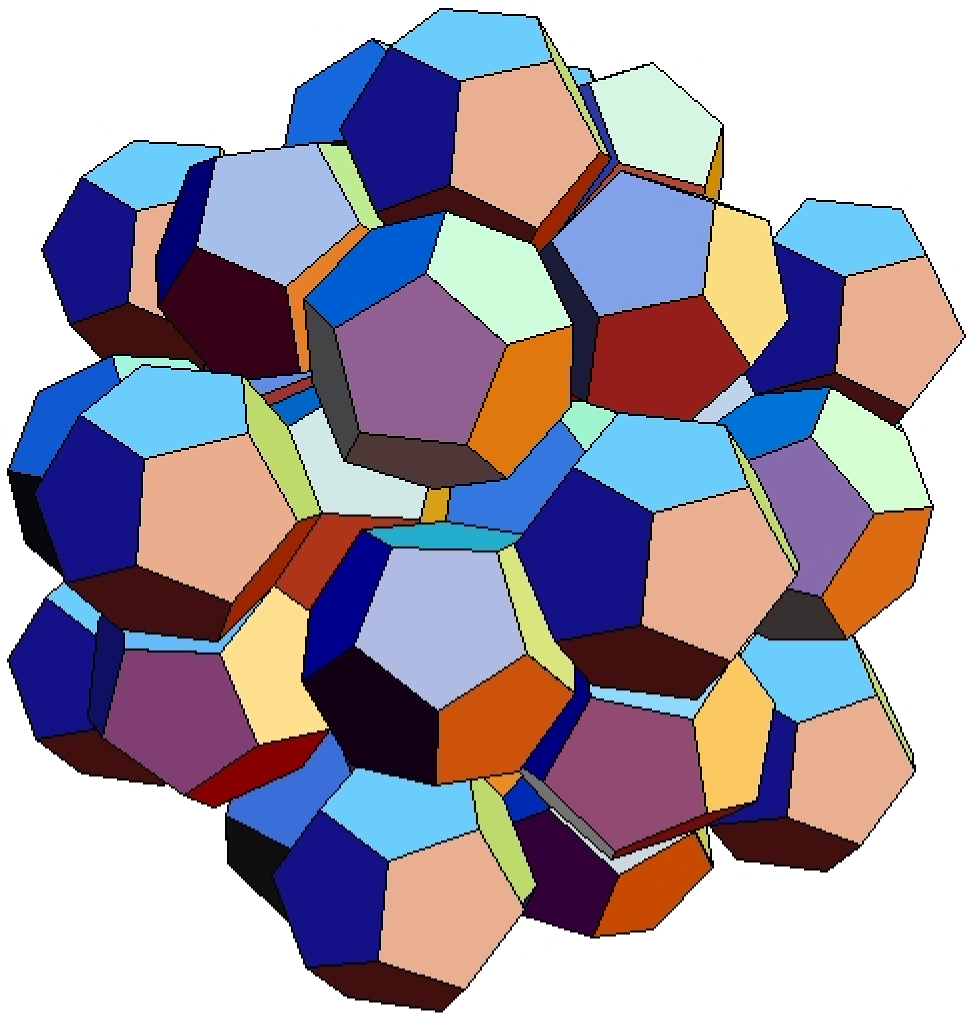
or *anything!*

But for the ratio Φ, there's very little resonance: the 'awkardness' of the number 5 strikes again.

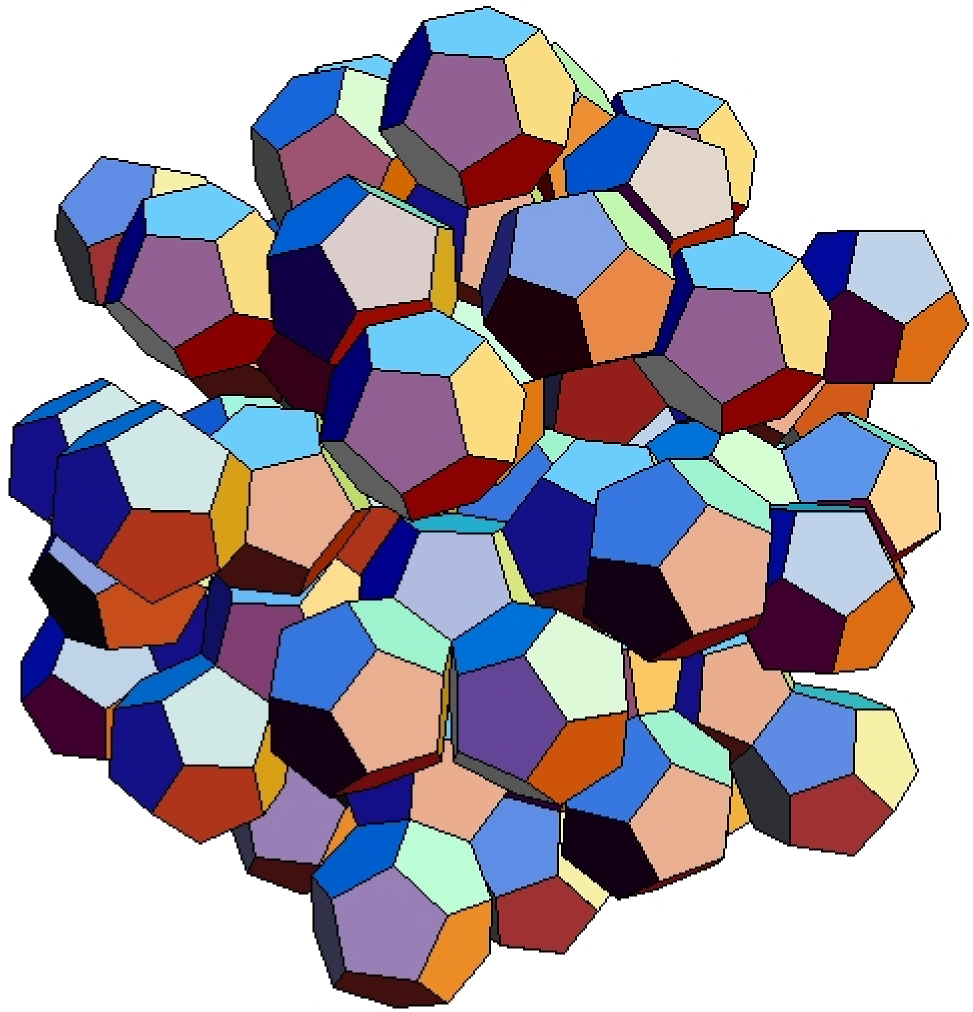
Yet another example:
we can't fill 3d space with dodecahedra.



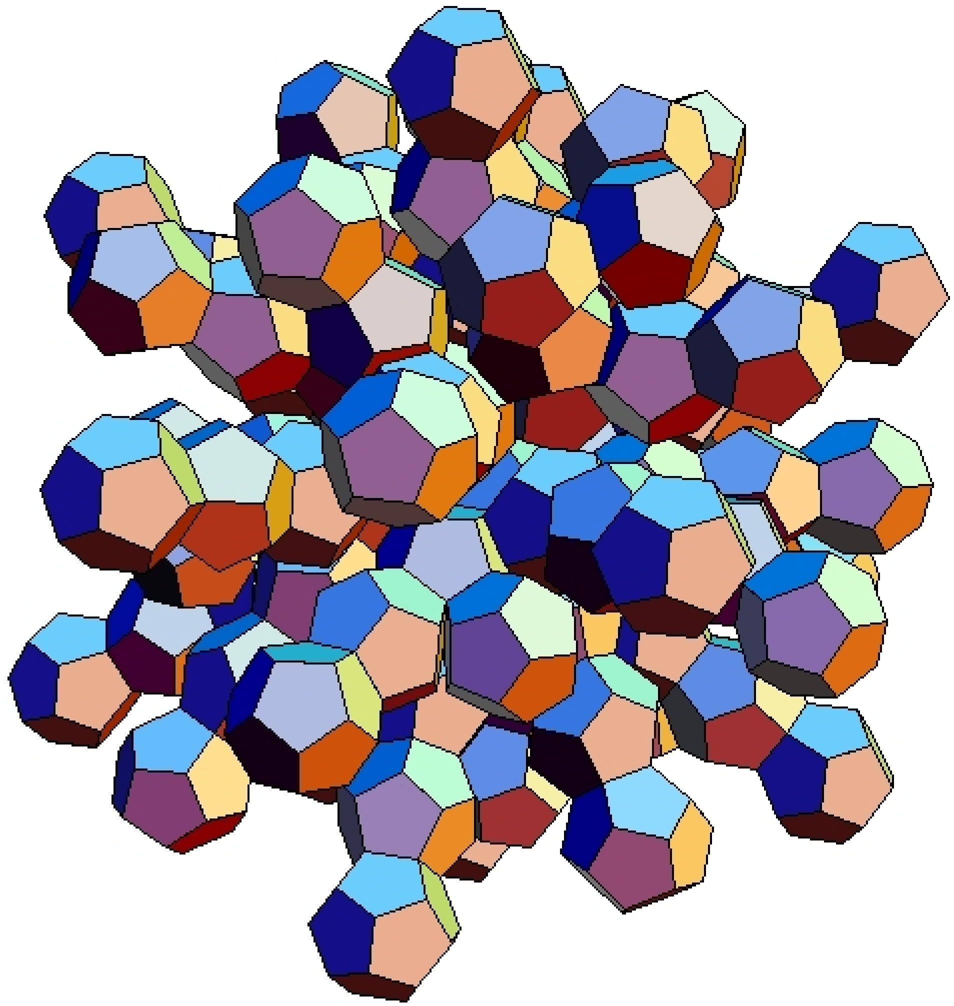
But we can add another layer:



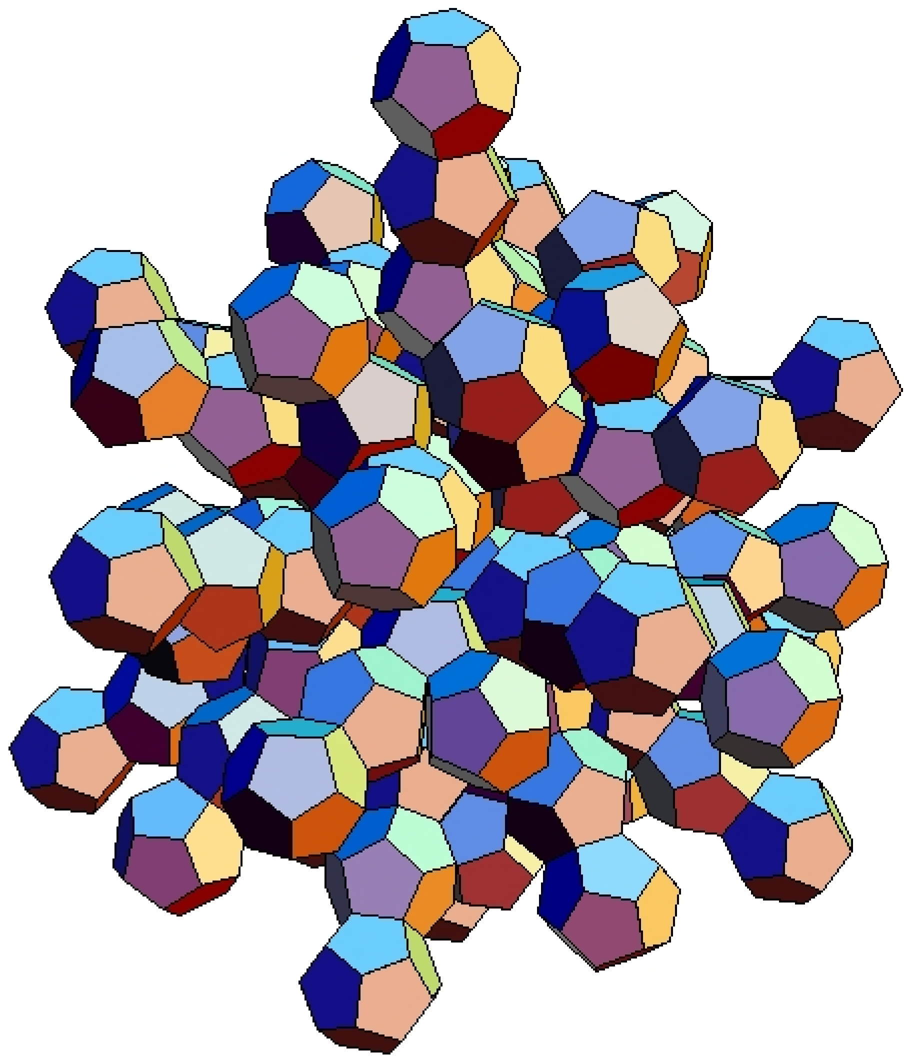
...and another...



...and another...



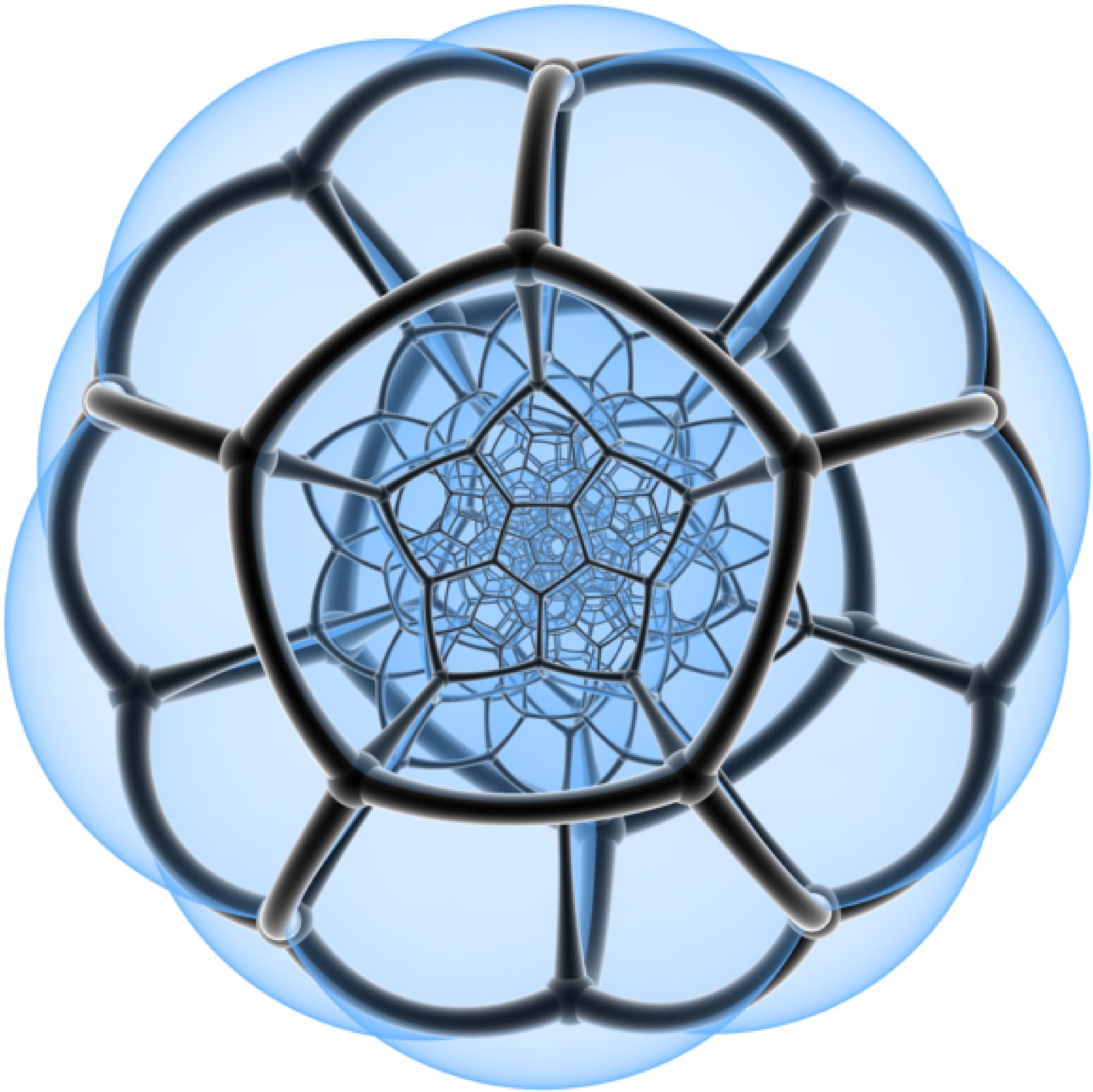
...and one last dodecahedron, for a total of 120...

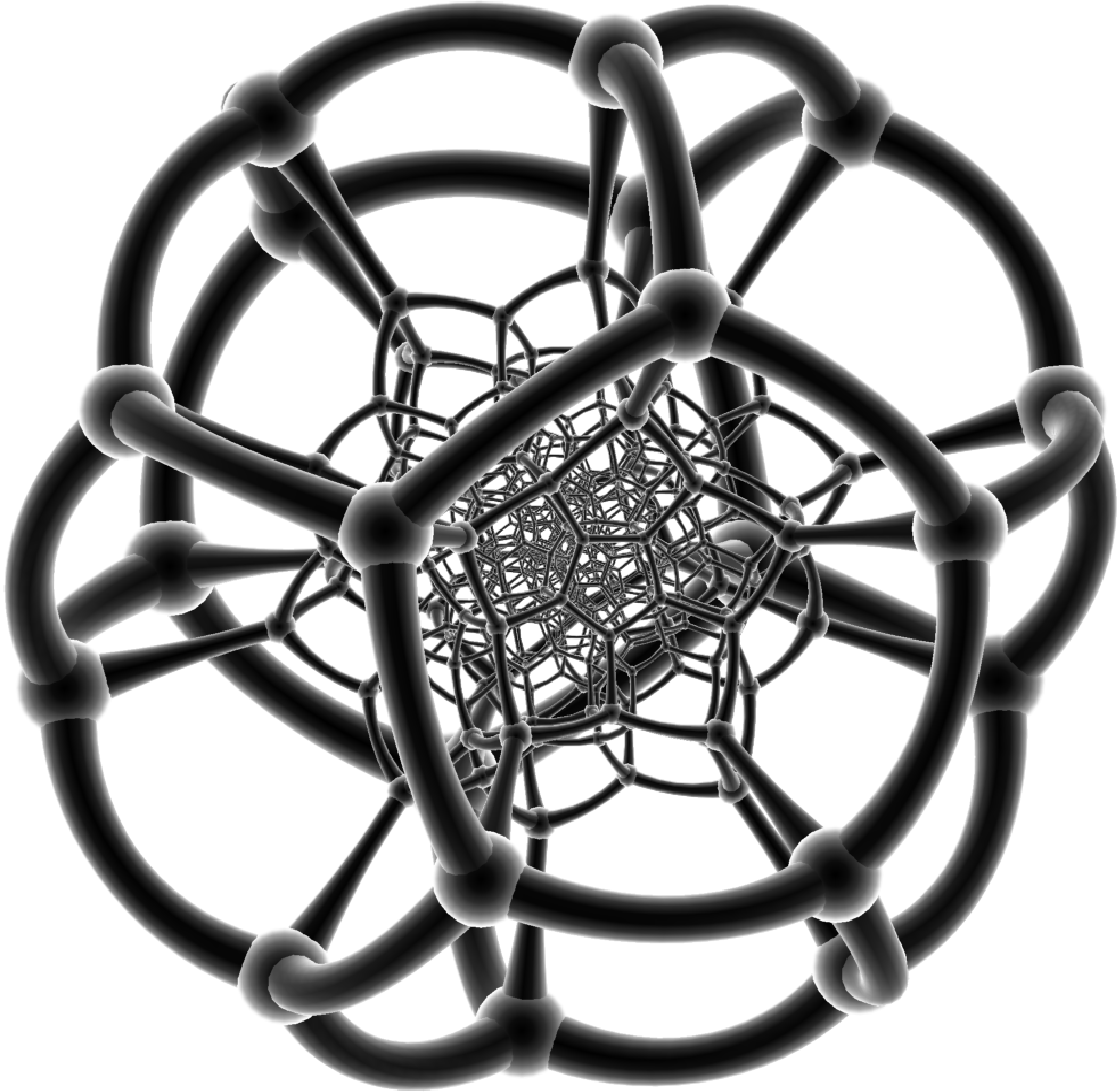


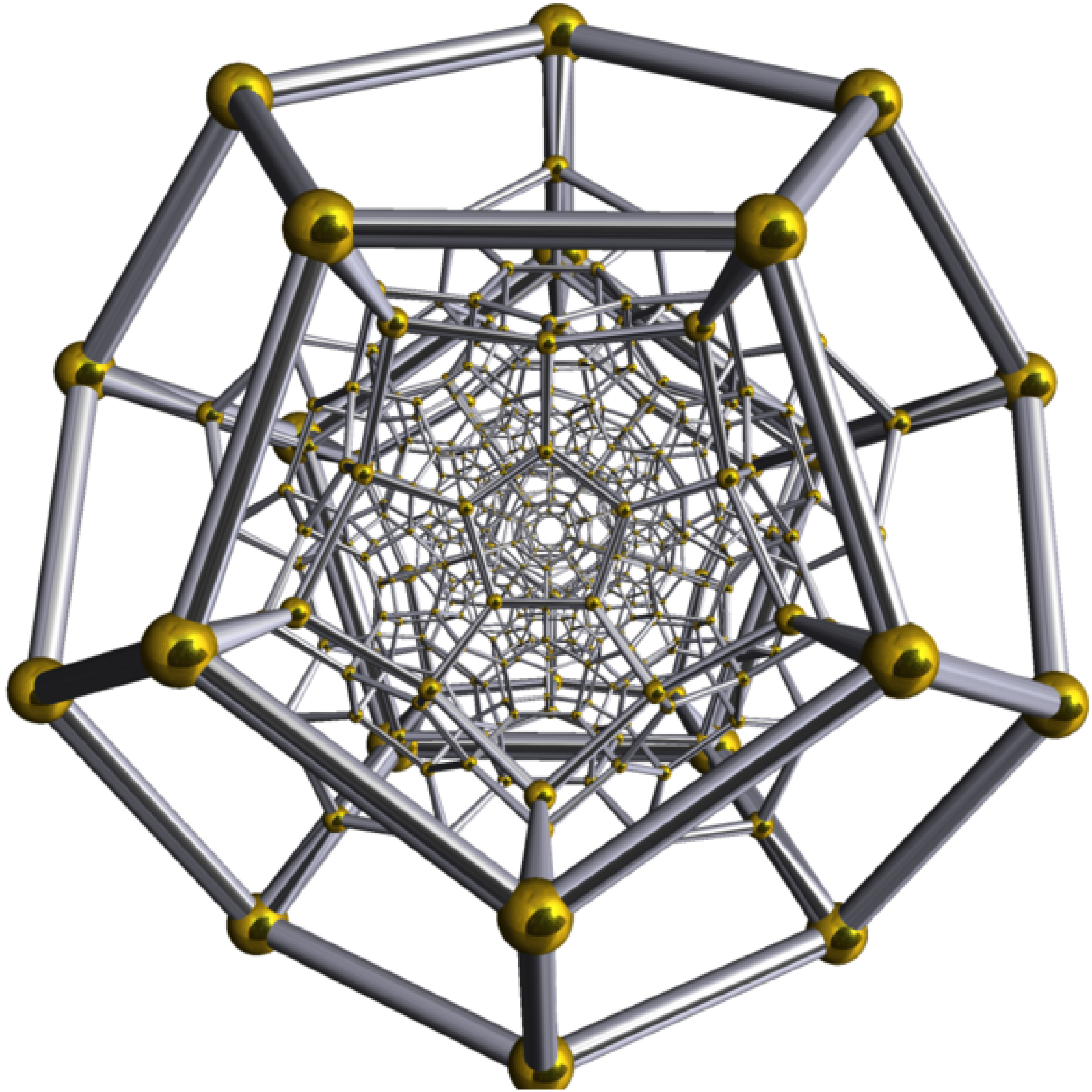
Then we can fold them up into the fourth dimension.
We obtain the '120-cell':
a *4-dimensional* regular solid!

Its surface is 3-dimensional.
We can imagine living in this 3d space and looking around.
It would be a '3-sphere' divided into 120 dodecahedra.

Let's go take a look...





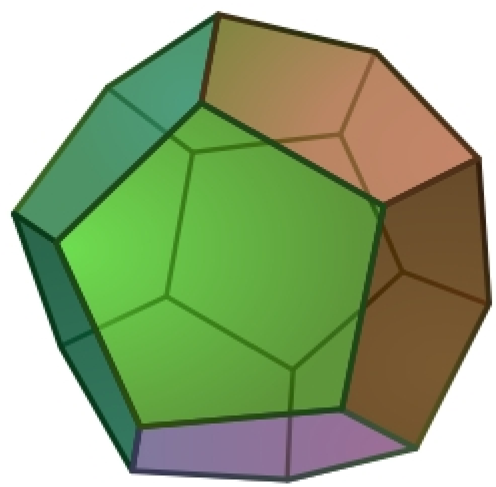


Finally, here's an *different way* to get the 120-cell from the dodecahedron.

The dodecahedron has

$$\mathbf{5 \times 12 = 60}$$

rotational symmetries, since a rotation can carry a face to any of the 12 faces in 5 different ways:



60 is half of 120. Is this a coincidence? No!

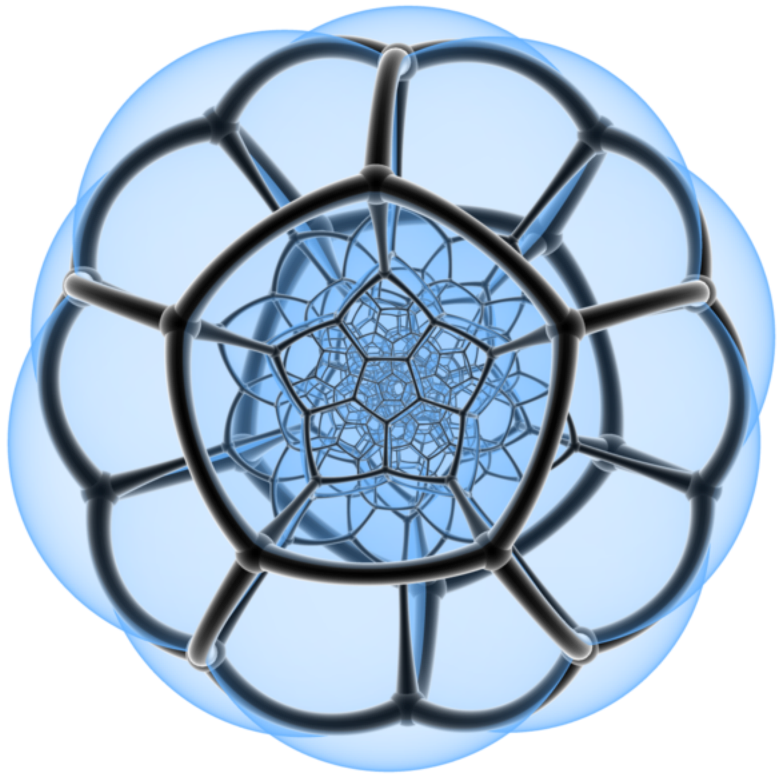
**If you turn a dodecahedron around 360°,
it comes back to the way it was.**

But suppose it were a 'spinor' — a particle like an electron or proton. These don't come back to where they were after turning them around 360° *once*, but they do after turning them around *twice*.

Then the dodecahedron would have $2 \times 60 = 120$ rotational symmetries!

Each of these 120 symmetries has 12 nearest neighbors: the dodecahedron has 6 axes going through opposite faces, and we can rotate these $1/5$ of a turn either clockwise or counterclockwise.

Similarly the 120-cell has 120 dodecahedra, each with 12 nearest neighbors:



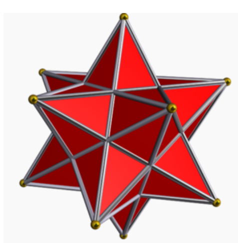
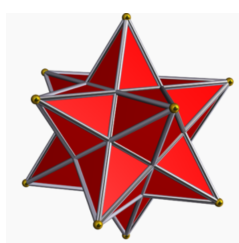
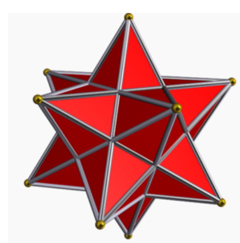
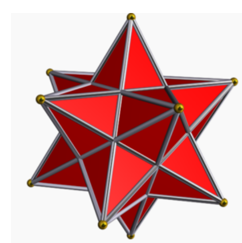
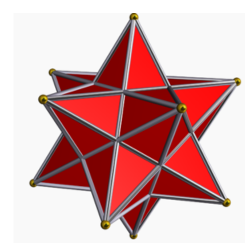
So, the 120-cell is a picture of the rotational symmetries of a 'spinor dodecahedron'!

I've only skimmed the surface in this talk.

Robert Alexander Rankin went deeper, editing papers by Ramanujan with shocking formulas like this:

$$\frac{1}{1+\frac{e^{-2\pi}}{1+\frac{e^{-4\pi}}{1+\frac{e^{-6\pi}}{1+\frac{e^{-8\pi}}{1+\frac{e^{-10\pi}}{\ddots}}}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}\right) e^{2\pi/5}$$

I dedicate this lecture to him!

APPENDIX: THE MOST IRRATIONAL NUMBER

In 1891, Adolf Hurwitz showed every irrational x has infinitely many rational approximations that are 'good' in the sense that:

$$|x - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$$

We can't make the constant $\frac{1}{\sqrt{5}}$ any smaller, thanks to the case $x = \Phi$!

Let's see how it goes...

$$\mathbf{1+\frac{1}{1}=\frac{2}{1}} \qquad \mathbf{\left|\Phi-\frac{2}{1}\right| \simeq 0.8541\cdot\frac{1}{\sqrt{5}\cdot 1^2}}$$

$$\mathbf{1+\frac{1}{1+\frac{1}{1}}=\frac{3}{2}} \qquad \mathbf{\left|\Phi-\frac{3}{2}\right| \simeq 1.056\cdot\frac{1}{\sqrt{5}\cdot 2^2}}$$

$$\mathbf{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}=\frac{5}{3}} \qquad \mathbf{\left|\Phi-\frac{5}{3}\right| \simeq 0.9787\cdot\frac{1}{\sqrt{5}\cdot 3^2}}$$

$$\mathbf{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}=\frac{8}{5}} \qquad \mathbf{\left|\Phi-\frac{8}{5}\right| \simeq 1.008\cdot\frac{1}{\sqrt{5}\cdot 5^2}}$$

$$\mathbf{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}}=\frac{13}{8}} \qquad \mathbf{\left|\Phi-\frac{13}{8}\right| \simeq 0.9969\cdot\frac{1}{\sqrt{5}\cdot 8^2}}$$

CREDITS AND NOTES

1. **SoftCurve 4-inch nickel number 5, available from Home Depot,** `http://www.polyvore.com/cgi/thing?id=737788`**.**
2. **Regular polygons, source unknown.**
3. **Triangular tiling created by 'Fibonacci', from the Wikipedia article Triangular Tiling.**
4. **Square tiling created by 'Fibonacci', from the Wikipedia article Square Tiling.**
5. **Hexagonal tiling created by 'Fibonacci', from Wikipedia article on Hexagonal Tiling.**
6. **Net for the dodecahedron created by 'Cyp', from the Wikipedia article Dodecahedron.**
7. **Penrose tiling created by 'xJaM', from the Wikipedia article Penrose Tiling.**
8. **Penrose tiling created by Clifford Johnson,** `http://asymptotia.com/2006/10/04/irrational-memories/`**.**

9. **Diffraction pattern of icosahedral quasicrystal, Ron Lifschitz,** `http://www.lassp.cornell.edu/lifshitz/quasicrystals.html`**.**

10. **Photo of I'timad al-Daula mausoleum and mathematical reconstruction, from Peter J. Lu and Paul J. Steinhardt, Decagonal and quasi-crystalline tilings in medieval Islamic architecture,** *Science* **315 (2007), 1106–1110,** `http://www.physics.harvard.edu/~plu/publications/`

11. **Photo of Darb-i Imam shrine, also from Peter J. Lu and Paul J. Steinhardt.**

12. **Net for dodecahedron (see above) and picture of dodecahedron created by 'Tomruen' using Robert Webb's *Great Stella* software,** `http://www.software3d.com/Stella.html`**.**

13. **Pariacoto virus, Liang Tang *et al*, The structure of Pariacoto virus reveals a dodecahedral cage of duplex RNA, *Nature Structural Biology* 8 (2001), 77–83,** `http://www.nature.com/nsmb/journal/v8/n1/pdf/nsb0101_77.pdf`**.**

14. **Dodecahedrane created by 'DaMocles-WS0506', from the Wikipedia article Dodecahedrane.**

15. Dodecahedrane synthesis pictures created by 'V8rik', from the Wikipedia article Dodecahedrane.
16. Buckyball created by Michael Ströck, from the Wikipedia article Buckyball. A buckyball is a truncated icosahedron, so it has as many hexagonal faces as an icosahedron has triangles (namely 20), and as many pentagonal faces as a dodecahedron has pentagons (namely 12). It's the pattern used in a soccer ball.
17. Carved stone balls, from Michael Atiyah and Paul Sutcliffe, Polyhedra in physics, chemistry and geometry, available as arxiv:/math-ph/0303071.
18. Pyrite, from Hershel Friedman's webpage Pyrite, `http://www.minerals.net/mineral/sulfides/pyrite/pyrite.htm`.
19. Pyritohedron, from Steven Dutch's webpage Building Isometric Crystals with Unit Cells, `http://www.uwgb.edu/dutchs/symmetry/isometuc.htm`.
20. Pythagoras, from Mactutor webpage Portraits of Pythagoras, `http://www-history.mcs.st-andrews.ac.uk/history/PictDisplay/Pythagoras.html`.
21. Pentagram from Heinrich Cornelius Agrippa's book *Libri Tres de Occulta Philosophia*, scanned by Jörgen Nixdorf, from the Wikipedia article Pentagram.

22. Pentagram with golden triangles, John Baez.

23. Pentagram with golden triangles, John Baez.
24. See the Wikipedia article Golden ratio. For a history of the golden ratio that requires little knowledge of math to enjoy, try Mario Livio, *The Golden Ratio: The Story of Phi, The World's Most Astonishing Number*, Broadway Books, New York, 2002. For a critical review of this book see George Markowsky, The golden ratio, `http://www.ams.org/notices/200503/rev-markowsky.pdf`.
25. For fascinating theories about the Greeks' use of continued fractions see David Fowler, *The Mathematics of Plato's Academy: A New Reconstruction*, Oxford U. Press, 1987.
26. Equations, John Baez.
27. Equations, John Baez.
28. Pentagram with nested pentagrams, James Dolan.
29. Equations, John Baez.
30. Equations, John Baez.
31. See the Wikipedia article Continued fraction, or for much more, Claude Brezinski, *History of Continued Fractions and Padé Approximations*, Springer, New York, 1980.

32. See the Wikipedia articles Pi and Zu Chongzhi.

33. Equations, John Baez.

34. Picture of the Cassini Division in Saturn's rings by NASA, `http://photojournal.jpl.nasa.gov/catalog/PIA08306`. This gap in the rings is formed by a resonance with Saturn's moon Mimas: particles at this distance go around Saturn twice in the time it takes Mimas to go around once. See the Wikipedia article Rings of Saturn.

35. First stage of foldout model of 120-cell from Andrew Weimholt's webpage Regular 4d Polytope Foldouts, `http://www.weimholt.com/andrew/polytope.shtml`.

36. Second stage of 120-cell from Andrew Weimholt's webpage.

37. Third stage of 120-cell from Andrew Weimholt's webpage.

38. Fourth stage of 120-cell from Andrew Weimholt's webpage.

39. Fifth stage of 120-cell from Andrew Weimholt's webpage.

40. Text, John Baez.
41. Bubble picture of 120-cell created by Fritz Obermeyer using Jenn software, from the Wikipedia article 120-Cell.
42. 'Rubber' picture of 120-cell created by Fritz Obermeyer using Jenn software, `http://www.math.cmu.edu/~fho/jenn/polytopes/`.
43. Metallic picture of 120-cell created by 'Tomruen' using Robert Webb's *Great Stella* software, `http://www.software3d.com/Stella.html`.
44. Dodecahedron created by Cyp, from the Wikipedia article Dodecahedron.
45. Text, John Baez.
46. Bubble picture of 120-cell created by Fritz Obermeyer using Jenn software, *op. cit.* I give a proof of this 'spinor dodecahedron' description of the 120-cell in the second appendix of my talk on the number 8.
47. See Eric Weisstein's Mathworld article Ramanujan Continued Fractions. Small stellated dodecahedra created by 'Tomruen' using Robert Webb's *Great Stella* software, `http://www.software3d.com/Stella.html`.

48. See Eric Weisstein's Mathworld article Hurwitz's irrational number theorem, which is interesting even though it attributes this result to Lagrange.