Frobenius monoids, weak bimonoids, and corelations

Brandon Coya

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Circuits

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Engineers often care about pairs of wire:

\[ \phi_1, I_1 \]
\[ \phi_2, I_2 \]

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Engineers often care about pairs of wire:

\[ \phi_1, I_1 \]
[---------------------]
\[ \phi_2, I_2 \]

such that \( I_1 = -I_2 \). They also care about “voltage” \( V \) where \( V = \phi_2 - \phi_1 \).
Meanwhile, there is a category that has morphisms that correspond to circuits made of wire.
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These are also called “corelations” and they determine the category $\text{FinCorel}$. We can draw them as string diagrams:
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```
\begin{ydiagram}
  3 & 4
  \end{ydiagram}
```

Then to study pairs of wires we study the objects \( 2n \in \text{FinCorel} \).
Frobenius monoids

The object 2 can be equipped with two different *Frobenius monoid* structures.
Frobenius monoids

The object $2$ can be equipped with two different *Frobenius monoid* structures. The first Frobenius monoid arises from using the unit and counit pair:

$$i_2: 0 \to 2 \quad e_2: 2 \to 0$$

to build a multiplication and unit:

$$m_2: 4 \to 2 \quad i_2: 0 \to 2$$
Frobenius monoids

The morphisms:

\[ m_2 : 4 \to 2 \quad i_2 : 0 \to 2 \]

make 2 into a monoid:
Frobenius monoids

The morphisms:

\[ m_2 : 4 \to 2 \quad \text{and} \quad i_2 : 0 \to 2 \]

make 2 into a monoid:
Frobenius monoids

The morphisms:

\[ m_2 : 4 \rightarrow 2 \quad i_2 : 0 \rightarrow 2 \]

make 2 into a monoid:
Frobenius monoids

The morphisms:

\begin{align*}
  d_2 &: 4 \to 2 \\
  e_2 &: 0 \to 2
\end{align*}

make 2 into a comonoid:
Frobenius monoids

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\[ d_2 : 4 \rightarrow 2 \]

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make 2 into a comonoid:
Frobenius monoids

Then we get that \((2, m_2, i_2, d_2, e_2)\) is an extraspecial symmetric Frobenius monoid:
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\[
\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} = \text{Diagram 3} \\
\text{Diagram 4} & = \text{Diagram 5} = \text{Diagram 6}
\end{align*}
\]
We can equip $\mathbb{2}$ with a different Frobenius monoid structure via another standard construction:

$$\mu_2 : 4 \to 2$$

$$\nu_2 : 0 \to 2$$
We can equip $2$ with a different Frobenius monoid structure via another standard construction:

$\mu_2 : 4 \to 2$

$\nu_2 : 0 \to 2$

$\delta_2 : 2 \to 4$

$\epsilon_2 : 2 \to 0$
Frobenius monoids

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\[
\begin{align*}
\mu_2 &: 4 \to 2 \\
\nu_2 &: 0 \to 2 \\
\delta_2 &: 2 \to 4 \\
\epsilon_2 &: 2 \to 0
\end{align*}
\]

\((2, \mu_2, \nu_2, \delta_2, \epsilon_2)\) is an extraspecial \textit{commutative} Frobenius monoid.
Frobenius monoids

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Frobenius monoids

\[
\begin{align*}
\text{Motivation} & \quad \text{Frobenius monoids} \\
& \quad \text{Weak bimonoids} \\
& \quad \text{Conclusion}
\end{align*}
\]

Frobenius monoids, weak bimonoids, and corelations
Frobenius monoids
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From Pastro and Street [3] we get the following.

**Theorem**

The following morphisms make 2 into a weak bimonoid:

\[ \mu_2 : 4 \rightarrow 2 \]
\[ \nu_2 : 0 \rightarrow 2 \]
\[ d_2 : 2 \rightarrow 4 \]
\[ e_2 : 2 \rightarrow 0 \]
From Pastro and Street [3] we get the following.

**Theorem**

The following morphisms make $2$ into a weak bimonoid:

- $\mu_2 : 4 \rightarrow 2$
- $\nu_2 : 0 \rightarrow 2$
- $d_2 : 2 \rightarrow 4$
- $e_2 : 2 \rightarrow 0$
Weak bimonoids
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Motivation
Frobenius monoids
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Conclusion
Now let's assign potentials and currents to our morphisms using the "black box" functor $\Box : \text{FinCorel} \to \text{LagRel}_k$ given by Baez and Fong [2].
Black box functor

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the “black box” functor \( \blacksquare : \text{FinCorel} \to \text{LagRel}_k \) given by Baez and Fong [2].

\[
\{(\phi_1, \ldots, I_6) : \phi_1 = \phi_5, I_1 = I_5, \phi_4 = \phi_6, \\
I_4 = I_6, \phi_2 = \phi_3, I_2 + I_3 = 0\}
\]
Now let’s assign potentials and currents to our morphisms using the “black box” functor $\mathbb{H} : \text{FinCorel} \to \text{LagRel}_k$ given by Baez and Fong [2].

$$\{(\phi_1, \ldots, I_6) : \phi_1 = \phi_5, I_1 = I_5, \phi_4 = \phi_6, I_4 = I_6, \phi_2 = \phi_3, I_2 + I_3 = 0\}$$

Then we impose that incoming current is opposite of outgoing current and write difference in potential as voltage.

$$I = I_1 = -I_2, I' = I_3 = -I_4, I'' = I_5 = -I_6$$

$$V = \phi_2 - \phi_1, V' = \phi_4 - \phi_3, V'' = \phi_6 - \phi_5$$
Series and parallel junctions

This results in the space $\{(V, \ldots, I'') : V + V' = V'', I = I' = I''\}$ and we think of the morphism $m_2$ as summing voltages together while equalizing current.
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Doing this with the other multiplication gives us:

$\{(V, \ldots, I'') : I + I' = I'', V = V' = V''\}$

so that $\mu_2$ equalizes voltage and sums voltage. Engineers call this a “parallel” junction.
Now we want to look at the subcategory $\text{FinCorel}^\circ$ of $\text{FinCorel}$ generated by these 8 morphisms.

$m_2: 4 \to 2$

$i_2: 0 \to 2$

$d_2: 2 \to 4$

$e_2: 2 \to 0$

$\mu_2: 4 \to 2$

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$\epsilon_2: 2 \to 0$

and we want to assign voltage and current with a functor $F: \text{FinCorel}^\circ \to \text{LagRel}_k$. 
Then we want the following diagram:

\[
\begin{array}{ccc}
\text{FinCorel}^\circ & \xrightarrow{F} & \text{LagRel}_k \\
\downarrow^{\alpha} & & \downarrow \ \\
\text{FinCorel} & \xleftarrow{i} & \\
\end{array}
\]

where \( \alpha \) comes from the relationships\( V = \phi_2 - \phi_1 \) and \( I = I_1 = -I_2 \).
Then we want the following diagram:

\[
\begin{array}{ccc}
\text{FinCorel}^\circ & \overset{F}{\longrightarrow} & \text{LagRel}_k \\
\downarrow{\alpha} & & \\
\text{FinCorel} & \overset{i}{\longrightarrow} & \text{FinCorel}
\end{array}
\]

where \( \alpha \) comes from the relationships \( V = \phi_2 - \phi_1 \) and \( I = I_1 = -I_2 \). However, this cannot be done.
Instead this led to a lot more work where we define another category which maps into \( \text{FinCorel}^\circ \) and also a subcategory of \( \text{LagRel}_k \). Then we get a nice diagram: [1]
