Reeb Graph Smoothing Via Cosheaves

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Why category theory?

- It is a convenient language for describing persistence modules.
- It gives clues to finding the ‘right’ definitions and concepts.
- It gives immediate access to deeper theorems.
- We are free to drop it when it doesn’t fit.
Preordered sets

Let $P$ be a set with a reflexive transitive relation $\leq$. Then

- objects = \{ elements of $P$ \}
- morphisms = \{ relations $x \leq y$ \}

defines a category $P$.

Directed graphs

A directed graph defines a category:

- $\bullet \to \bullet \to \bullet \to \bullet \to \bullet$

or

- $\bullet \to \bullet \leftrightarrow \bullet \leftrightarrow \bullet \to \bullet$

or

- $\bullet \to \bullet \leftarrow \bullet \leftarrow \bullet \leftrightarrow \bullet \to \bullet$

(Identities and composites are implicit.)
Let $f : X \rightarrow \mathbb{R}$. Consider the category $\mathbf{n}$ defined by

$$0 \rightarrow 1 \rightarrow \cdots \rightarrow n - 1,$$

and select $a_0 \leq a_1 \leq \cdots \leq a_{n-1}$. From

$$X^{a_0} \subseteq X^{a_1} \subseteq \cdots \subseteq X^{a_{n-1}},$$

construct

$$H(X^{a_0}) \rightarrow H(X^{a_1}) \rightarrow \cdots \rightarrow H(X^{a_{n-1}}).$$
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H(X^{a_0}) \to H(X^{a_1}) \to \cdots \to H(X^{a_{n-1}}).
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Definitions

- $X^t := f^{-1}(-\infty, t]$ sublevelset
- $X_s := f^{-1}[s, +\infty)$ superlevelset
- $X^t_s := f^{-1}[s, t]$ interlevelset
The ‘persistence module’

\[ H(X^{a_0}) \rightarrow H(X^{a_1}) \rightarrow \cdots \rightarrow H(X^{a_{n-1}}) \]

can be thought of as a **functor**

\[
\begin{align*}
\mathbf{n} \xrightarrow{F} \textbf{Top} \xrightarrow{H} \textbf{Vect}.
\end{align*}
\]

This means:

- For each object of \( \mathbf{n} \) we have a vector space.
- For each morphism of \( \mathbf{n} \) we have a linear map.
**Sublevelset persistent homology**

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**Generalized persistence modules (Bubenik, Scott 2014)**

A ‘generalized persistence module’ is simply a functor \( \mathbb{V} : \mathcal{C} \rightarrow \mathcal{D} \).

- Usually \( \mathcal{C} \) is a pre-ordered set, such as \( n, \mathbb{N}, \mathbb{Z}, \mathbb{R} \).
- Usually \( \mathcal{D} \) is an abelian category, such as \( \text{Vect}, \text{Ab} \).
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Categories of functors

The collection of functors $\mathcal{C} \to \mathcal{D}$ is itself a category, denoted $\mathcal{D}^\mathcal{C}$. The morphisms are natural transformations $\phi : \mathcal{V} \Rightarrow \mathcal{W}$, defined by the following data:

- For every $c \in \mathcal{C}$ there is a map $\phi_c : \mathcal{V}_c \to \mathcal{W}_c$.
- For every map $f : c \to c'$ in $\mathcal{C}$ the diagram
  
  \[
  \begin{array}{ccc}
  \mathcal{V}_c & \xrightarrow{\phi_c} & \mathcal{W}_c \\
  \downarrow & & \downarrow \\
  \mathcal{V}_{c'} & \xrightarrow{\phi_{c'}} & \mathcal{W}_{c'}
  \end{array}
  \]

  is required to commute.
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### Categories of functors

The collection of functors \( \mathcal{C} \to \mathcal{D} \) is itself a category, denoted \( \mathcal{D}^\mathcal{C} \). The morphisms are **natural transformations** \( \phi : \mathcal{V} \Rightarrow \mathcal{W} \), defined by the following data:

- For every \( c \in \mathcal{C} \) there is a map \( \phi_c : V_c \to W_c \).
- For every map \( f : c \to c' \) in \( \mathcal{C} \) the diagram

\[
\begin{array}{ccc}
V_c & \xrightarrow{\phi_c} & W_c \\
\downarrow{\mathcal{V}[f]} & & \downarrow{\mathcal{W}[f]} \\
V_{c'} & \xrightarrow{\phi_{c'}} & W_{c'}
\end{array}
\]

is required to commute.
Persistent homology takes a filtered space $\mathcal{X} = \{X_t \mid t \in \mathbb{R}\}$ and returns a **barcode** of intervals $[p, q) \subset \mathbb{R}$ or a **persistence diagram** of points $(p, q) \in \mathbb{R}^2$. 
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Algorithmic approach (Edelsbrunner, Letscher, Zomorodian 2000).

- Discretize the $t$-variable.
- Present $X$ as a finite list of cells, attached in sequence.
- Some cells $\sigma$ generate new homology cycles.
- Other cells $\tau$ destroy cycles created by an earlier $\sigma$.
- There is an interval $[t_\sigma, t_\tau)$ for each such pair $(\sigma, \tau)$.
- There is an interval $[t_\sigma, +\infty)$ for each $\sigma$ whose cycle is never destroyed.
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How is this defined?
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Story 1: Persistence diagrams


- Discretize the $t$-variable to integers: $t = 0, 1, 2, \ldots$
- Present $X$ as an increasing sequence:
  \[
  X : \quad X_0 \subset X_1 \subset X_2 \subset \ldots
  \]
- Apply a homology functor $H = H(−; k)$ to the sequence:
  \[
  H(X) : \quad H(X_0) \to H(X_1) \to H(X_2) \to \ldots
  \]
- Observe that $H(X)$ is a graded module over the polynomial ring $k[z]$, where $z$ acts by shifting to the right.
- Decompose this graded module as a direct sum of cyclic submodules.
- Summands $z^s k[z]/(z^{t−s})$ are recorded as intervals $[s, t)$.
- Summands $z^s k[z]$ are recorded as intervals $[s, +\infty)$. 
Using quiver theory (Carlsson, dS 2010).

- Discretize the $t$-variable to integers: $t = 0, 1, \ldots, n - 1$.
- Present $X$ as a sequence of spaces with maps:

$$X : \quad X_0 \to X_1 \to \cdots \to X_{n-1}$$

- Apply a homology functor $H = H(-; k)$ to the sequence:

$$H(X) : \quad H(X_0) \to H(X_1) \to \cdots \to H(X_{n-1})$$

- Observe that $H(X)$ is a representation of the quiver $\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet$.
- Decompose $H(X)$ as a direct sum of indecomposable representations.
- According to Gabriel (1970), the indecomposables are precisely the intervals:

$$0 \to \cdots \to 0 \to k \to \cdots \to k \to 0 \to \cdots \to 0$$

The list of summands of $H(X)$ gives the persistence intervals.
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- Discretize the $t$-variable to integers: $t = 0, 1, \ldots, n - 1$.
- Present $X$ as a sequence of spaces with maps:
  \[ X : X_0 \to X_1 \to \cdots \to X_{n-1} \]
- Apply a homology functor $H = H(-; k)$ to the sequence:
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When the arrows have mixed orientations $\leftarrow$, $\rightarrow$, we get zigzag persistence.
What if we wish to work with a continuous parameter?

Interval decomposition

- Let $\mathcal{V}$ be a persistence module defined over the real numbers $\mathbb{R}$.
- Suppose

$$\mathcal{V} = \bigoplus_{k \in K} \mathbb{I}[a_k, b_k]$$

where $\mathbb{I} = \mathbb{I}[a, b]$ denotes the persistence module with

$$I_t = \begin{cases} k & \text{if } t \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

and all maps $i_s^t$ having full rank. (Open, half-open intervals allowed too.)

- Then we can define the persistence diagram to be

$$\text{Dgm}(\mathcal{V}) = \{(a_k, b_k) \mid k \in K\},$$

a multiset of points in the half-plane above the diagonal.
Problem

Not every $\mathbb{V}$ decomposes into intervals.
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Not every $V$ decomposes into intervals.

**Theorem (Gabriel, Auslander, Ringel–Tachikawa, Webb, Crawley-Boevey)**

Let $V$ be a persistence module over $T \subseteq \mathbb{R}$. In either of the following situations, $V$ decomposes into interval modules:

- $T$ is a finite set; or
- Every $V_t$ is finite-dimensional.

On the other hand, there exists a persistence module over $\mathbb{Z}$ which does not admit an interval decomposition.
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Solution (Chazal, dS, Glisse, Oudot 2016)
Define a measure which counts the number of persistence points in an arbitrary rectangle. Infer the existence of the persistence diagram. This works if the maps $V_s \rightarrow V_t$ are finite-rank whenever $s < t$. 
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Definition 1 (non-functorial)

Let

$$\mu([a, b] \times [c, d]) = r_b^c - r_a^c - r_b^d + r_a^d$$

where $r_s^t = \text{rank}(V_s \to V_t)$. 

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$$\mu([a, b] \times [c, d]) = r^c_b - r^c_a - r^d_b + r^d_a$$

where $r^t_s = \text{rank}(V_s \to V_t)$.

**Definition 2 (functorial)**

Let

$$\mu([a, b] \times [c, d]) = \dim (M_{a,b,c,d} V)$$

where

$$M_{a,b,c,d} V = \left[ \frac{\text{Im}(v^c_b) \cap \text{Ker}(v^d_c)}{\text{Im}(v^c_a) \cap \text{Ker}(v^d_c)} \right].$$
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M_{a,b,c,d} = \left[ \frac{\text{Im}(v_b^c) \cap \text{Ker}(v_c^d)}{\text{Im}(v_a^c) \cap \text{Ker}(v_c^d)} \right].
\]

Note. Each \( M_{a,b,c,d} \) extends to a functor \( \text{Vect}^R \rightarrow \text{Vect} \).
Solution step

It is necessary to show that $\mu$ is additive with respect to splitting a rectangle.

![Diagram showing rectangles R, S, T, U, and V with labels a, b, c, d, p, q. The diagram illustrates the splitting process.]
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Proof 1 (for horizontal split)

$$r_b^c - r_a^c - r_b^d + r_a^d = (r_p^c - r_a^c - r_p^d + r_a^d) + (r_b^c - r_p^c - r_b^d + r_p^d)$$
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Proof 2 (for horizontal split)

There is a short exact sequence

\[
0 \rightarrow \left[ \frac{\text{Im}(v_p^c) \cap \text{Ker}(v_c^d)}{\text{Im}(v_a^c) \cap \text{Ker}(v_c^d)} \right] \rightarrow \left[ \frac{\text{Im}(v_b^c) \cap \text{Ker}(v_d^c)}{\text{Im}(v_a^c) \cap \text{Ker}(v_c^d)} \right] \rightarrow \left[ \frac{\text{Im}(v_c^c) \cap \text{Ker}(v_d^d)}{\text{Im}(v_p^c) \cap \text{Ker}(v_c^d)} \right] \rightarrow 0
\]

or, in other words, a short exact sequence of functors

\[
0 \rightarrow M_{a,p,c,d} \rightarrow M_{a,b,c,d} \rightarrow M_{p,b,c,d} \rightarrow 0
\]
Question (of Morozov)

Is the persistence diagram functorial?

Let $V : \mathbb{R} \to \text{Vect}$ be a persistence module. Select $\ldots < a_2 < a_1 < a_0 < a_1 < \ldots$.

The functorial persistence diagram with respect to $(a_n)$ is the function $(m, n) \mapsto M_{a_m, a_{m+1}, a_n, a_{n+1}} V$ for integers $m < n$.

Pros and cons

• A map $V \to W$ between persistence modules induces a map between f.p.d.
• This method defines a persistence diagram in any abelian category.
• It is not so easy to change the discretization.
• What is the right metric between these diagrams?
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The **functorial persistence diagram** with respect to $(a_n)$ is the function

$$(m, n) \mapsto M_{a_m, a_{m+1}, a_n, a_{n+1}} \mathbb{V}$$

for integers $m < n$. At each point there is a vector space.
Story 1: Persistence diagrams

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Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

The map \{persistence modules\} \rightarrow \{diagrams\} is 1-Lipschitz.
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Relators

The metrics on the two spaces are defined in terms of ‘relators’.

- Two persistence modules may be related by an **interleaving**.
- Two diagrams may be related by a **matching**.

Every relator, of each type, has a size associated with it. The metrics are defined by finding the infimum of the size of relators between a given pair of objects.
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Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

If two persistence modules admit an \(\epsilon\)-interleaving, then their persistence diagrams admit an \(\epsilon\)-matching.
Definition

Let $V, W$ be persistence modules. An $\epsilon$-interleaving between $V, W$ is a pair $(\Phi, \Psi)$ where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

$$\phi_t : V_t \rightarrow W_{t+\epsilon} \quad \text{and} \quad \psi_t : W_t \rightarrow V_{t+\epsilon}$$

such that [various conditions].
Story 2: Interleaving

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**Glisse’s Lemma (Chazal, Cohen-Steiner, Glisse, Guibas, Oudot 2009)**

The proof of the stability theorem relies on the following fact. If $V, W$ are $\epsilon$-interleaved, then there is a 1-parameter family

$$(V_s \mid s \in [0, \epsilon])$$

with $V_0 = V$ and $V_\epsilon = W$, and where $V_r, V_s$ are $|r - s|$-interleaved for all $r, s$. 
Let $V, W$ be persistence modules. An $\epsilon$-interleaving between $V, W$ is a pair $(\Phi, \Psi)$ where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

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The [various conditions] require the diagrams

1. $V_s \xrightarrow{\nu_t^s} V_t \xrightarrow{\nu_{t+2\epsilon}^t} V_{t+2\epsilon}$
2. $W_{s+\epsilon} \xrightarrow{w_t^s} W_t \xrightarrow{w_{t+2\epsilon}^t} W_{t+2\epsilon}$
3. $V_{s+\epsilon} \xrightarrow{\nu_{t+\epsilon}^{s+\epsilon}} V_{t+\epsilon}$
4. $W_{s+\epsilon} \xrightarrow{w_{t+\epsilon}^s} W_t \xrightarrow{w_{t+2\epsilon}^t} W_{t+2\epsilon}$

[Diagrams]

The [various conditions] require the diagrams to commute for all $s < t$. 

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such that [various conditions].

The [various conditions] amount to the assertion that there is a unique way to get from any of the $V_t, W_t$ to any other. All compositions of the $\nu_s^t, w_s^t, \phi_t, \psi_t$ with the same start and end point must agree.
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Interleavor categories (Chazal, dS, Glisse, Oudot 2016)

An $\epsilon$-interleaved pair of modules $(V, W, \Phi, \Psi)$ is ‘the same thing’ as a persistence module defined over the set $I = \mathbb{R} \times \{0, \epsilon\}$ (two copies of the real line) with the partial order

$$(s, a) \leq (t, b) \iff \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}$$
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R \times \{0, \epsilon\}:
Interleavings for classical persistence modules

Two classical persistence modules $\mathcal{V}$, $\mathcal{W}$ are $\epsilon$-interleaved iff the following functor extension problem has a solution:

$$
\begin{align*}
\text{Vect} & \quad \mathcal{V} \quad \mathcal{W} \\
R & \quad R \times \{0, \epsilon\} \quad R
\end{align*}
$$

Here $R \times \{0, \epsilon\}$ has the partial order

$$(s, a) \leq (t, b) \iff \begin{cases} 
    s \leq t & \text{if } a = b \\
    s + \epsilon \leq t & \text{if } a \neq b
\end{cases}$$
Proof of Glisse’s Lemma

Consider the set $J = \mathbb{R} \times [0, \varepsilon]$ with the partial order

$$(s, a) \leq (t, b) \iff s + |a - b| \leq t$$

This contains the interleavor category $I$ as a sub-poset. An $\varepsilon$-interleaving between two persistence modules corresponds to a functor $I \rightarrow \text{Vect}$ which restricts to $V, W$ on the two respective copies of the real line.

An interpolation $(V_t)$ is found constructing an extension of the functor to $J$:

Since $I$ is a full subcategory of $J$, and $\text{Vect}$ contains all limits and colimits, the problem is solved by taking a left or right Kan extension.
Story 2: Interleaving

Proof of Glisse’s Lemma

Consider the set \( J = \mathbb{R} \times [0, \epsilon] \) with the partial order

\[(s, a) \leq (t, b) \iff s + |a - b| \leq t\]

This contains the interleavor category \( I \) as a sub-poset. An \( \epsilon \)-interleaving between two persistence modules corresponds to a functor \( I \to \text{Vect} \) which restricts to \( \mathbb{V}, \mathbb{W} \) on the two respective copies of the real line.

An interpolation \( (\mathbb{V}_t) \) is found constructing an extension of the functor to \( J \):

\[
\begin{array}{c}
\text{Vect} \\
\uparrow \\
\mathbb{R} \times \{0, \epsilon\} \\
\downarrow \\
\mathbb{R} \times [0, \epsilon]
\end{array}
\]

Since \( I \) is a full subcategory of \( J \), and \( \text{Vect} \) contains all limits and colimits, the problem is solved by taking a left or right Kan extension.
Question (of Morozov)

Is the persistence diagram functorial?

The persistence diagram is a map \{\text{persistence modules}\} \rightarrow \{\text{diagrams in the upper half-plane}\}.

What are the morphisms that make these into categories?

• A morphism \(V_1 \rightarrow V_2\) could be an interleaving pair \((\phi, \psi)\).

• A morphism \(\text{Dgm}_1 \rightarrow \text{Dgm}_2\) could be a matching between points.

For both notions there is an associative composition law with identities.

Question (of Morozov, reworded)

Does an \(\epsilon\)-interleaving between two persistence modules specify an \(\epsilon\)-matching between their diagrams, in a way that respects composition?

Answer

\((\text{Bauer, Lesnick 2015})\)

Almost. See recent work of Ulrich Bauer and Michael Lesnick.
Question (of Morozov)

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\[ \{ \text{persistence modules} \} \rightarrow \{ \text{diagrams in the upper half-plane} \} \]

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For both notions there is an associative composition law with identities.

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Does an \( \epsilon \)-interleaving between two persistence modules specify a \( \epsilon \)-matching between their diagrams, in a way that respects composition?

Answer 2\(^+ \) (Bauer, Lesnick 2015)
Almost. See recent work of Ulrich Bauer and Michael Lesnick.
Two classical persistence modules $V, W$ are $\epsilon$-interleaved iff the following functor extension problem has a solution:

\[
\begin{array}{ccc}
\text{Vect} & \rightarrow & \rightarrow \\
\uparrow & & \uparrow \\
V & \rightarrow & \rightarrow \\
R & \rightarrow & R \times \{0, \epsilon\} & \leftarrow & R \\
W & \leftarrow & \leftarrow \\
\end{array}
\]

Here $R \times \{0, \epsilon\}$ has the partial order

\[(s, a) \leq (t, b) \iff \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}\]
Interleaving for generalized persistence modules over a poset

Two persistence modules $V, W : P \to C$ are $\Omega$-interleaved iff the following functor extension problem has a solution:

Here $P \cup_\Omega P$ has the partial order

$$(s, a) \leq (t, b) \iff \begin{cases} 
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where $\Omega : P \to P$ is a translation.
Translations (Bubenik, dS, Scott 2015)

$\text{Trans}_P$ is the poset of functions $\Omega : P \to P$ that are order-preserving and satisfy $x \leq \Omega x$ for all $x \in P$. 
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Superlinear Families

A superlinear family is a 1-parameter family of translations of $P$

$$(\Omega_\epsilon \mid \epsilon \in [0, \infty))$$

such that

$$\Omega_{\epsilon_1} \Omega_{\epsilon_2} \leq \Omega_{\epsilon_1+\epsilon_2}$$

for all $\epsilon_1, \epsilon_2 \in [0, \infty)$. 
Transitions (Bubenik, dS, Scott 2015)

Trans\textsubscript{P} is the poset of functions \(\Omega : P \rightarrow P\) that are order-preserving and satisfy \(x \leq \Omega x\) for all \(x \in P\).

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Sublinear Projections

A sublinear projection is a map \(\pi : \text{Trans}_P \rightarrow [0, \infty]\) such that

\[
\pi(\Omega_1 \Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2)
\]

for all \(\Omega_1, \Omega_2 \in \text{Trans}_P\).
A **superlinear family** is a 1-parameter family of translations of $P$

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Reeb Graph Smoothing Via Cosheaves
Superlinear Families

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such that

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for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Examples of superlinear families

- $P = \mathbb{R}$,
  $\Omega_{\epsilon}(t) = t + \epsilon$.

- $P = \{\text{compact intervals in the real line}\}$,
  $\Omega_{\epsilon}([a, b]) = [a - \epsilon, b + \epsilon]$.

- $P = \{\text{closed subsets of a metric space } X\}$,
  $\Omega_{\epsilon}(V) = V^\epsilon = \{x \in X \text{ such that } d(x, V) \leq \epsilon\}$. 

A superlinear family is a 1-parameter family of translations of $P$

$\left( \Omega_{\epsilon} \mid \epsilon \in [0, \infty) \right)$

such that

$\Omega_{\epsilon_1} \Omega_{\epsilon_2} \leq \Omega_{\epsilon_1 + \epsilon_2}$

for all $\epsilon_1, \epsilon_2 \in [0, \infty)$. 

Interleaving distance (Bubenik, dS, Scott 2015)

Given a superlinear family $(\Omega_{\epsilon})$ of translations of $P$, we define the interleaving distance $d_i(V, W)$ as

$\inf \left( \epsilon \mid V, W \text{ are } \Omega_{\epsilon}-interleaved \right)$

between generalized persistence modules $V, W : P \rightarrow C$. 

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Reeb Graph Smoothing Via Cosheaves
A superlinear family is a 1-parameter family of translations of \( P \)

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for all \( \epsilon_1, \epsilon_2 \in [0, \infty) \).

Interleaving distance (Bubenik, dS, Scott 2015)

Given a superlinear family \( (\Omega_\epsilon) \) of translations of \( P \), we define the interleaving distance

\[ d_i(V, W) = \inf (\epsilon \mid V, W \text{ are } \Omega_\epsilon\text{-interleaved}) \]

between generalized persistence modules \( V, W : P \to C \).
### Sublinear Projections

A **sublinear projection** is a map $\pi : \text{Trans}_P \rightarrow [0, \infty]$ such that

$$\pi(\Omega_1 \Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2)$$

for all $\Omega_1, \Omega_2 \in \text{Trans}_P$.

### Interleaving distance (Bubenik, dS, Scott 2015)

Given a sublinear projection family $\pi : \text{Trans}_P \rightarrow [0, \infty]$, we define the interleaving distance

$$d_i(\mathcal{V}, \mathcal{W}) = \inf (\pi(\Omega) \mid \mathcal{V}, \mathcal{W} \text{ are } \Omega\text{-interleaved})$$

between generalized persistence modules $\mathcal{V}, \mathcal{W} : \mathbf{P} \rightarrow \mathbf{C}$.
Functoriality

Suppose \( V, W : P \to C \) and \( H : C \to D \) are functors. Then

\[
d_i(HV, HW) \leq d_i(V, W)
\]

for any superlinear family or sublinear projection.

Proof.

An \( \Omega \)-interleaving of \( V, W \) gives an \( \Omega \)-interleaving of \( HV, HW \):
Functoriality

Suppose $\mathcal{V}, \mathcal{W}: P \to C$ and $H: C \to D$ are functors. Then

$$d_i(H\mathcal{V}, H\mathcal{W}) \leq d_i(\mathcal{V}, \mathcal{W})$$

for any superlinear family or sublinear projection.
Functoriality
Suppose $V, W: P \to C$ and $H: C \to D$ are functors. Then
\[ d_i(HV, HW) \leq d_i(V, W) \]
for any superlinear family or sublinear projection.

Example: sublevelset persistence
Let $X$ be a topological space and $f, g: X \to \mathbb{R}$ be functions with $\|f - g\|_\infty \leq \epsilon$.
- The persistence modules $V, W: \mathbb{R} \to \text{Top}$ defined
  \[ V(t) = f^{-1}(-\infty, t], \quad W(t) = g^{-1}(-\infty, t], \]
  are $\epsilon$-interleaved.
  (There are natural inclusions $V(t) \subseteq W(t + \epsilon)$ and $W(t) \subseteq V(t + \epsilon)$.)
**Functoriality**

Suppose $V, W : P \to C$ and $H : C \to D$ are functors. Then

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**Example: sublevelset persistence**

Let $X$ be a topological space and $f, g : X \to \mathbb{R}$ be functions with $\|f - g\|_\infty \leq \epsilon$.

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  are $\epsilon$-interleaved.

  (There are natural inclusions $V(t) \subseteq W(t + \epsilon)$ and $W(t) \subseteq V(t + \epsilon)$.)

- For any homology functor $H : \text{Top} \to \text{Vect}$, the persistence modules $HV, HW : \mathbb{R} \to \text{Vect}$ are $\epsilon$-interleaved.
Story 3: Interleaving Metrics
Interleavings for generalized persistence modules over a poset

Two persistence modules $V, W : P \rightarrow C$ are $\Omega$-interleaved iff the following functor extension problem has a solution:

Here $P \cup_\Omega P$ has the partial order

$$(s, a) \leq (t, b) \iff \begin{cases} s \leq t & \text{if } a = b \\ \Omega s \leq t & \text{if } a \neq b \end{cases}$$

where $\Omega : P \rightarrow P$ is a translation.
Interleavings for generalized persistence modules over an arbitrary category

Two persistence modules $V, W : D \to C$ are $\Delta$-interleaved iff the following functor extension problem has a solution:

Here $\Delta$ is a cospan. The two functors $l_1, l_2$ are full-and-faithful. Every object of $\Delta$ is of the form $l_1(d)$ or $l_2(d)$. 
Example: dynamical system interleavings

Let $D$ be the category defined by the directed graph

Thus $D$ has one object and morphisms $\{0, 1, 2, 3, \ldots\}$. 
Example: dynamical system interleavings

Let $\mathbf{D}$ be the category defined by the directed graph

Thus $\mathbf{D}$ has one object and morphisms $\{0, 1, 2, 3, \ldots \}$.  

- Functors $\mathbf{D} \rightarrow \mathbf{Top}$ are \textit{discrete dynamical systems}.  

\[ \begin{array}{c}
\bullet \\
\nearrow
\end{array} \]
Example: dynamical system interleavings

Let $D$ be the category defined by the directed graph

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- Functors $D \rightarrow \text{Top}$ are discrete dynamical systems.

Let $\Delta_n$ be the category with two objects $\bullet_1$ and $\bullet_2$ and morphisms

$$\text{Mor}(\bullet_1, \bullet_1) = \text{Mor}(\bullet_1, \bullet_1) = \{0, 1, 2, 3, \ldots\}$$

$$\text{Mor}(\bullet_1, \bullet_2) = \text{Mor}(\bullet_2, \bullet_1) = \{n, n + 1, n + 2, n + 3, \ldots\}$$

with addition as composition.
Example: dynamical system interleavings

Let $D$ be the category defined by the directed graph

Thus $D$ has one object and morphisms $\{0, 1, 2, 3, \ldots \}$.

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$\text{Mor}(\bullet_1, \bullet_2) = \text{Mor}(\bullet_2, \bullet_1) = \{n, n+1, n+2, n+3, \ldots \}$

with addition as composition.

- $\Delta_n$-interleavings are shift-equivalences.
A functor $T : \mathbb{R} \to \text{Set}$ can be thought of as a **merge tree**.

Let $X$ be a topological space and $f : X \to \mathbb{R}$ a function. Then

$$T(t) = \pi_0 f^{-1}(-\infty, t]$$

$$T[s \leq t] = \pi_0 \left[ f^{-1}(-\infty, t] \subseteq f^{-1}(-\infty, t] \right]$$

defines the **sublevelset merge tree** of $(X, f)$.
Merge trees (Cagliari, Ferri, Pozzi 2001, & Morozov, Beketayev, Weber 2013)

- A functor $T : \mathbb{R} \rightarrow \textbf{Set}$ can be thought of as a merge tree.
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  defines the sublevelset merge tree of $(X, f)$.

- If $f, g : X \rightarrow \mathbb{R}$ with $\|f - g\|_{\infty} \leq \epsilon$ then $d_i(T, U) \leq \epsilon$. 

**Interleaving Distance between Merge Trees**

Given a function $f : X \rightarrow \mathbb{R}$, we can track the evolution of homology groups of its sublevel sets, $F_a$. We get a sequence of groups, $H_0(F_a)$, connected by a shift map $\bar{f}^{-1}(a)$, where

$$\bar{f}^{-1}(a)$$

is the component of the level set

$$\bar{f}(H_0(F_a)) = \pi_0\left[\bar{f}^{-1}(a)\right]$$

in the tree $T$. To find the image of $X$ under $f$, we follow the path from the tree $X$ sublevel set $x$ to the range of $f$. Therefore, there is a well-defined map $\hat{T}$, a merge tree extend to infinity. This formulation and grows until it merges with another component at a saddle. We note that $f$ is defined by

$$f(x) = \max_{y \in \hat{T}} (x, y)$$

so induces a map between homology groups, $H_0(F_a)$, and $H_0(F_b)$, where $a \leq b$. The $\bar{f}$ function works as a shift map in the definition of

$$\hat{T} = \left\{ (x, y) \in X \times X \mid x \leq y \right\}$$

and $d_i(T, U)$.

Let $g : X \rightarrow \mathbb{R}$ be a topological space and $f : X \rightarrow \mathbb{R}$ a function. Then

$$T(t) = \pi_0 f^{-1}(-\infty, t]$$

defines the sublevelset merge tree of $(X, f)$.
A functor \( F : \text{Int} \to \text{Set} \) can be thought of as a graph over the real line. (Technically we require \( F \) to satisfy a cosheaf condition.)

Let \( X \) be a topological space and \( f : X \to \mathbb{R} \) a function. Then

\[
F_f(I) = \pi_0 f^{-1}(I)
\]

\[
F_f[I \subseteq J] = \pi_0 \left[ f^{-1}(I) \subseteq f^{-1}(J) \right]
\]

defines the Reeb graph of \((X, f)\).
Reeb Graph Smoothing Via Cosheaves

Reeb graphs (dS, Munch, Patel 2016)

- A functor $F : \text{Int} \to \text{Set}$ can be thought of as a graph over the real line. (Technically we require $F$ to satisfy a cosheaf condition.)
- Let $X$ be a topological space and $f : X \to \mathbb{R}$ a function. Then
  
  $$F_f(I) = \pi_0 f^{-1}(I)$$
  $$F_f[I \subseteq J] = \pi_0 \left[ f^{-1}(I) \subseteq f^{-1}(J) \right]$$

  defines the Reeb graph of $(X, f)$.

- If $f, g : X \to \mathbb{R}$ with $\|f - g\|_{\infty} \leq \epsilon$ then $d_i(F, G) \leq \epsilon$. 

![Reeb graph diagram](image.png)
Story 5: Reeb Graphs & Reeb Cosheaves
Story 5: Reeb Graphs & Reeb Cosheaves

Reeb graphs

- An **R-space** \((X, f)\) is a topological space \(X\) with a map \(f : X \to \mathbb{R}\).
- An **R-space** is a **Reeb graph** if \(X\) is a graph and each \(f^{-1}(t)\) is finite.
Reeb graphs

- An **R-space** \((X, f)\) is a topological space \(X\) with a map \(f : X \to \mathbb{R}\).
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Reeb graphs

- An **R-space** \((X, f)\) is a topological space \(X\) with a map \(f : X \to \mathbb{R}\).
- An **R-space** is a **Reeb graph** if \(X\) is a graph and each \(f^{-1}(t)\) is finite.

Reeb functor

- The **Reeb functor** converts a (constructible) **R-space** into a Reeb graph:

  \[
  (X, f) \mapsto ((X/\sim), \overline{f})
  \]

  where \(x \sim y\) iff \(x, y\) are in the same component of the same levelset of \(f\).
Story 5: Reeb Graphs & Reeb Cosheaves

Reeb Graph Smoothing Via Cosheaves
Reeb cosheaves (dS, Munch, Patel 2016)

- Let \( \text{Int} \) denote the poset of open intervals, with respect to inclusion.
- A Reeb graph gives rise to a functor \( F : \text{Int} \rightarrow \text{Set} \) that is \textit{constructible} and satisfies the \textit{cosheaf condition} for unions of intervals.

\[
F(I = \bigcup I_{\alpha}) = \colim \big( \bigcup_{\alpha, \beta} F(I_{\alpha} \cap I_{\beta}) \big) \Rightarrow \bigcup_{\alpha} F(I_{\alpha})
\]
Reeb cosheaves (dS, Munch, Patel 2016)

- Let $\text{Int}$ denote the poset of open intervals, with respect to inclusion.
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Story 5: Reeb Graphs & Reeb Cosheaves

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Reeb cosheaves (dS, Munch, Patel 2016)

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Reeb cosheaves (dS, Munch, Patel 2016)

- Let $\text{Int}$ denote the poset of open intervals, with respect to inclusion.
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Reeb cosheaves (dS, Munch, Patel 2016)

- Let \( \text{Int} \) denote the poset of open intervals, with respect to inclusion.
- A Reeb graph corresponds to a functor \( F : \text{Int} \rightarrow \text{Set} \) that is constructible and satisfies the cosheaf condition for unions of intervals.

![Diagram]

- \( \mathbb{R}\text{-spaces} \)
- \( \mathbb{R}\text{-Top} \)
- \( \mathbb{R}\text{-Top}^c \)
- Reeb graphs
- Pre-cosheaves
- Cosheaves
- Constructible cosheaves
1.2 Reeb graphs and Reeb cosheaves

Our starting point is a topological space $X$ equipped with a continuous real-valued function $f : X \to \mathbb{R}$. We call the pair $(X, f)$ a space fibered over $\mathbb{R}$ or, more succinctly, an $\mathbb{R}$-space. For reasons of convenience we will often abbreviate $(X, f)$ simply to $f$. The context will indicate whether we are thinking of $f$ as a function or as an $\mathbb{R}$-space.

We can think of an $\mathbb{R}$-space as a 1-parameter family of topological spaces $f_a$. The topology on $X$ gives information on how these spaces relate to each other. For instance, each levelset can be partitioned into connected components. How can we track these components as the parameter $a$ varies? An answer is provided by the Reeb graph.

The geometric Reeb graph of an $\mathbb{R}$-space $f$ is an $\mathbb{R}$-space $\tilde{f}$ defined as follows. First, we define an equivalence relation on the domain of $f$ by saying two points $x, x_0 \in X$ are equivalent if they lie on the same levelset $f_a$ and do not the same component of that levelset. Let $X_{\tilde{f}}$ be the quotient space defined by this equivalence relation, and let $\bar{f} : X_{\tilde{f}} \to \mathbb{R}$ be the function inherited from $f$. This is the Reeb graph. See, for example, Figure 1.

If $f$ is a Morse function on a compact manifold, or a piecewise linear function on a compact polyhedron, then its Reeb graph is topologically a finite graph with vertices at each critical value of $f$. This situation is well studied. These examples are included in a larger class, the constructible $\mathbb{R}$-spaces, which have similar good behavior. We will say more about this in Section 2. If we work in greater generality, the quotient $X_{\tilde{f}}$ can be badly behaved. Among other things, we would need to pay attention to the distinction between connected components and path components. This is not an issue for constructible $\mathbb{R}$-spaces, where the two concepts lead to the same outcome.

We now indicate an alternate way of recording the information stored in the geometric Reeb graph. The abstract Reeb graph or Reeb cosheaf of an $\mathbb{R}$-space $f$ is defined to be the following collection of data (see Figure 2):

- For each open interval $I \subseteq \mathbb{R}$, let $F(I)$ be the set of path-components of $f_I$;
- For $I \subseteq J$, let $F[I \subseteq J] = \pi_0[f_I \subseteq f_J]$.

Let $F$ denote the entirety of this data. It is easily confirmed that $F$ is a functor (see Section 1.3) from the category of open intervals to the category of sets. As such, $F$ is sometimes called a pre-cosheaf on the real line in the category of sets. The important point is that this information, in the constructible case, is enough to recover the geometric Reeb graph; see Figure 3. The other important point is that it is sometimes easier to work with the pre-cosheaf than with the geometric Reeb graph.

Reeb functor (two versions)

- The **Reeb functor** converts a (constructible) $\mathbb{R}$-space into a Reeb graph:

$$ (X, f) \mapsto ((X/\sim), \bar{f}) $$

where $x \sim y$ iff $x, y$ are in the same component of the same levelset of $f$.

or

- The **Reeb functor** converts a constructible $\mathbb{R}$-space into a Reeb cosheaf:

$$ F(I) = \pi_0 f^{-1}(I) $$

$$ G[I \subseteq J] = \pi_0 \left[ f^{-1}(I) \subseteq f^{-1}(J) \right] $$

Vin de Silva Pomona College

Reeb Graph Smoothing Via Cosheaves
Translation operators on $\textbf{Int}$

We define a 1-parameter semigroup $(\Omega_\epsilon)$ of functors $\textbf{Int} \to \textbf{Int}$ by setting

$$\Omega_\epsilon(I) = I^\epsilon = "\epsilon\text{-neighbourhood of } I"$$
Translation operators on \textbf{Int}

We define a 1-parameter semigroup \((\Omega_\epsilon)\) of functors \textbf{Int} \(\rightarrow\) \textbf{Int} by setting

\[\Omega_\epsilon(I) = I^\epsilon = \text{"\(\epsilon\)-neighbourhood of } I\text{"}\]

Reeb interleaving

An \textbf{\(\epsilon\)-interleaving} between \(F, G\) is given by two families of maps

\[\phi_I : F(I) \rightarrow G(I^\epsilon), \quad \psi_I : G(I) \rightarrow F(I^\epsilon)\]

which are natural with respect to inclusions \(I \subseteq J\) and such that

\[\psi_I^\epsilon \circ \phi_I = F[I \subseteq I^{2\epsilon}], \quad \phi_I^\epsilon \circ \psi_I = G[I \subseteq I^{2\epsilon}]\]

for all \(I\).
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for all $I$.

Stability Theorem

If $f, g : X \to \mathbb{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $d_i(F, G) \leq \epsilon$. 
Translation operators on \textbf{Int}

We define a 1-parameter semigroup \((\Omega_\epsilon)\) of functors \textbf{Int} \to \textbf{Int} by setting

\[
\Omega_\epsilon(I) = I^\epsilon = \text{“\(\epsilon\)-neighbourhood of \(I\)”}
\]
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### Cosheaf Smoothing Theorem

If \(F : \text{Int} \to \text{Set}\) is a (constructible) cosheaf, then so is \(F\Omega_\epsilon : \text{Int} \to \text{Set}\).
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Corollary: Reeb Smoothing

There is a 1-parameter semigroup of ‘smoothing’ operations on Reeb graphs.
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Progressive smoothing algorithm by Dmitriy Smirnov & Song Yu:

- $\epsilon = 0.00$
- $\epsilon = 0.14$
- $\epsilon = 0.27$
- $\epsilon = 0.41$
- $\epsilon = 0.55$
- $\epsilon = 0.68$
- $\epsilon = 0.82$
- $\epsilon = 0.95$
- $\epsilon = 1.09$
- $\epsilon = 1.23$
- $\epsilon = 1.36$
- $\epsilon = 1.50$
Progressive smoothing algorithm by Dmitriy Smirnov & Song Yu:

Discretized Reeb Graphs

- A discrete Reeb graph is a diagram

\[ E \xrightarrow{\ell} V \xrightarrow{\phi} R \]

where \( E, V \) are finite sets and \( \phi_\ell(e) < \phi_r(e) \) for each \( e \in E \).

- Each \( v \in V \) has a **left-** and **right-degree**:

\[
\deg_l(v) = \# r^{-1}(v), \quad \deg_r(v) = \# \ell^{-1}(v), \quad \deg(v) = (\deg_l(v), \deg_r(v)).
\]

- The discrete Reeb graph is **reduced** if \( \deg(v) \neq (1, 1) \) for all \( v \).

The **critical radius** of a reduced graph is

\[
\epsilon_{crit} = \frac{1}{2} \min \{ \phi_r(e) - \phi_\ell(v) \mid e \in E, \deg_r(\ell(e)) > 1, \deg_l(r(e)) > 1 \}
\]
Progressive smoothing algorithm by Dmitriy Smirnov & Song Yu:

**Algorithm: smooth by \( \epsilon \)**

- If \( \text{deg}(v) = (1, ?) \) then \( v \) moves by \( +\epsilon \).
- If \( \text{deg}(v) = (?, 1) \) then \( v \) moves by \( -\epsilon \).
- If \( \text{deg}(v) = (?, ?) \) then split \( v \) into two and move by \( \pm\epsilon \).

Valid up to the critical radius. Recompute at critical radius and recurse.
Progressive smoothing algorithm by Dmitriy Smirnov & Song Yu:

- **epsilon = 0.00**
- **epsilon = 0.14**
- **epsilon = 0.27**
- **epsilon = 0.41**
- **epsilon = 0.55**
- **epsilon = 0.68**
- **epsilon = 0.82**
- **epsilon = 0.95**
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- **epsilon = 1.23**
- **epsilon = 1.36**
- **epsilon = 1.50**
Let $V, W : P \to \textbf{Vect}$ be persistence modules and let $\Phi : V \Rightarrow W$. Then we can define a persistence module $\text{Im}(\Phi)$ with

- $[\text{Im}(\Phi)](t) = \text{Im}(V_t \xrightarrow{\phi_t} W_t)$ for all $t$.
- $[\text{Im}(\Phi)](s \leq t) = \text{the map induced by the horizontal maps in:}$

$$
\begin{array}{ccc}
V_s & \to & V_t \\
\downarrow{\phi_s} & & \downarrow{\phi_t} \\
W_s & \to & W_t
\end{array}
$$

We can similarly define $\text{Ker}(\Phi)$ and $\text{Coker}(\Phi)$. 
Let $V, W : P \to \text{Vect}$ be persistence modules and let $\Phi : V \Rightarrow W$. Then we can define a persistence module $\text{Im}(\Phi)$ with

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  \phi_t &\downarrow
  \end{align*}$

We can similarly define $\text{Ker}(\Phi)$ and $\text{Coker}(\Phi)$.

**Example**

Suppose $p : X \to Y$ is a map of spaces, $f : X \to \mathbb{R}$, and $g : Y \to \mathbb{R}$. If $f \leq gp$, then $p$ carries the $t$-sublevelset of $f$ into the $t$-sublevelset of $g$, for all $t$, and the persistence module $\text{Im}(H(p))$ is defined.
Three ways of thinking of a map between persistence modules (over $\mathbb{N}$, say)

A functor $\mathbb{2} \to \text{Vect}^{\mathbb{N}}$:

\[
\begin{array}{ccccccccc}
F_0 & \rightarrow & F_1 & \rightarrow & F_2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
G_0 & \rightarrow & G_1 & \rightarrow & G_2 & \rightarrow & \cdots 
\end{array}
\]

A functor $\mathbb{N} \times \mathbb{2} \to \text{Vect}$:

\[
\begin{array}{ccccccccc}
F_0 & \rightarrow & F_1 & \rightarrow & F_2 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
G_0 & \rightarrow & G_1 & \rightarrow & G_2 & \rightarrow & \cdots 
\end{array}
\]

A functor $\mathbb{N} \to \text{Vect}^2$:

\[
\begin{array}{ccccccccc}
F_0 & \Rightarrow & F_1 & \Rightarrow & F_2 & \Rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
G_0 & \Rightarrow & G_1 & \Rightarrow & G_2 & \Rightarrow & \cdots 
\end{array}
\]
The exponential law

The following categories of functors

$$(D^P)^W = D^P \times^W = (D^W)^P$$

are equal for any three categories $D, P, W$. 
Story 6: Generalised Factors

The exponential law

The following categories of functors

$$(D^P)^W = D^P \times W = (D^W)^P$$

are equal for any three categories $D$, $P$, $W$.

Image, Kernel, Cokernel functors

The operations Im, Ker and Coker can be thought of as functors $\text{Vect}^2 \to \text{Vect}$.

- Each operation converts any $(V \xrightarrow{\alpha} W)$ into a vector space.
- Given a commutative square, there are induced maps between images, kernels, cokernels.
Story 6: Generalised Factors

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The following categories of functors

\[(D^P)^W = D^P \times W = (D^W)^P\]

are equal for any three categories \(D, P, W\).

Image, Kernel, Cokernel functors

The operations \(\text{Im}, \text{Ker}\) and \(\text{Coker}\) can be thought of as functors \(\text{Vect}^2 \rightarrow \text{Vect}\).

- Each operation converts any \((V \xrightarrow{\alpha} W)\) into a vector space.
- Given a commutative square, there are induced maps between images, kernels, cokernels.

Proposition (Bubenik, dS, Scott)

The image persistence of \(\Phi : V \Rightarrow W\) is equal to the composite

\[P \xrightarrow{\hat{\Phi}} \text{Vect}^2 \xrightarrow{\text{Im}} \text{Vect}\]

where \(\hat{\Phi}\) is the interpretation of \(\Phi\) as a functor \(P \rightarrow \text{Vect}^2\).
Generalized factor persistence (Bubenik, dS, Scott)

Given

- a category of persistence modules $\mathcal{D}^P$;
- a category $\mathcal{W}$, which we call the auxiliary category;
- a functor $\mathcal{D}^W \xrightarrow{N} \mathcal{E}$, which we call the generalized factor.
Generalized factor persistence (Bubenik, dS, Scott)

Given

- a category of persistence modules $\mathbf{D}^P$;
- a category $\mathbf{W}$, which we call the *auxiliary category*;
- a functor $\mathbf{D}^W \xrightarrow{N} \mathbf{E}$, which we call the *generalized factor*.

Then any functor $F : \mathbf{W} \to \mathbf{D}^P$ determines a persistence module in $\mathbf{E}^P$, by

$$
(D^P)^W = \mathbf{D}^{W \times P} = (D^W)^P \xrightarrow{f} \mathbf{E}^P
$$

$$
F \xrightarrow{\hat{F}} \mathbf{N} \xrightarrow{\hat{N}} \mathbf{F}
$$
Reductions of 2-dimensional persistence

Let $\mathbb{V} = (V(s, t)) \in \text{Vect}^{\mathbb{R} \times \mathbb{R}}$ be a two-dimensional persistence module. Think of this as a family $(\mathbb{W}_t)$ of 1-dimensional persistence modules. We will define various generalized factors $N : \text{Vect}^{\mathbb{R}} \to \text{Vect}$.

- Fix $a$ and define $N(\mathbb{W}) = \mathbb{W}(a)$.
- Fix $a < b$ and define $N(\mathbb{W}) = \text{Im}(\mathbb{W}(a) \to \mathbb{W}(b))$.
- Fix $a < b \leq c < d$ and define

$$N(\mathbb{W}) = \left[ \frac{\text{Im} (\mathbb{W}(b) \to \mathbb{W}(c)) \cap \text{Ker} (\mathbb{W}(c) \to \mathbb{W}(d))}{\text{Im} (\mathbb{W}(a) \to \mathbb{W}(c)) \cap \text{Ker} (\mathbb{W}(c) \to \mathbb{W}(d))} \right]$$

Then there is a 1-parameter persistence module associated to each of these functors.
Suppose $Z$ is the category defined by:

$$
\bullet \longrightarrow \bullet \leftarrow \bullet \longrightarrow \bullet
$$

An element of $\text{Vect}^Z$ is a diagram

$$W : \quad W_1 \xrightarrow{f} W_2 \xleftarrow{g} W_3 \xrightarrow{h} W_4$$

Then, for example, the functor $\text{Vect}^Z \rightarrow \text{Vect}$ defined by

$$N(W) = \left[ \frac{g(h^{-1}(0))}{f(W_1)} \right]$$

does not pick out the part of $W$ supported over $W_2, W_3$. 

Therefore, given a zigzag of persistence modules $V_1 V_2 V_3 V_4 \rightarrow \leftarrow \rightarrow \leftarrow$ we can construct a single persistence module which extracts the $[2, 3]$ part.
Zigzag factors

Suppose $Z$ is the category defined by:

\[ \bullet \longrightarrow \bullet \leftarrow \bullet \longrightarrow \bullet \]

An element of $\text{Vect}^Z$ is a diagram

\[ \mathbb{W} : W_1 \xrightarrow{f} W_2 \xleftarrow{g} W_3 \xrightarrow{h} W_4 \]

Then, for example, the functor $\text{Vect}^Z \to \text{Vect}$ defined by

\[ N(\mathbb{W}) = \left[ \frac{g(h^{-1}(0))}{f(W_1)} \right] \]

picks out the part of $\mathbb{W}$ supported over $W_2, W_3$.

Therefore, given a zigzag of persistence modules

\[ \mathbb{V}_1 \xrightarrow{f} \mathbb{V}_2 \xleftarrow{g} \mathbb{V}_3 \xrightarrow{h} \mathbb{V}_4 \]

we can constrict a single persistence module which extracts the $[2, 3]$ part.
Tame persistence modules

Let $\mathbb{V} : \mathbb{R} \to \text{Vect}$ be a persistence module. If the maps $V_s \to V_t$ have finite rank whenever $s < t$, then $\mathbb{V}$ has a persistence diagram. If $\mathbb{V}$ has an interval decomposition, then the summands are identified exactly by the points in the diagram. However, it is not guaranteed that $\mathbb{V}$ has an interval decomposition.
Story 7: The observable category

**Tame persistence modules**

Let \( \mathbb{V} : \mathbb{R} \rightarrow \text{Vect} \) be a persistence module. If the maps \( V_s \rightarrow V_t \) have finite rank whenever \( s < t \), then \( \mathbb{V} \) has a persistence diagram. If \( \mathbb{V} \) has an interval decomposition, then the summands are identified exactly by the points in the diagram. However, it is not guaranteed that \( \mathbb{V} \) has an interval decomposition.

**Ephemeral modules (Chazal, Crawley-Boevey, dS 2016)**

A persistence module \( \mathbb{V} \) is *ephemeral* if \( v^t_s = 0 \) whenever \( s < t \). Then:

- The ephemeral modules comprise a **Serre subcategory** of the category of persistence modules.
- We can form the Serre quotient category by formally inverting all maps whose kernels and cokernels are ephemeral.
- In this category, every q-tame persistence module admits an interval decomposition.

Perhaps this is the ‘correct’ category for real-parameter persistence?
A **Serre subcategory** is a full subcategory $\mathbf{C}$ of an Abelian category such that for any short exact sequence

$$0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$$

we have

$$V \in \mathbf{C} \iff U \in \mathbf{C} \text{ and } W \in \mathbf{C}.$$ 

Equivalently, the subcategory $\mathbf{C}$ is closed under subobjects, quotient objects, and extensions.
A **Serre subcategory** is a full subcategory \( \mathcal{C} \) of an Abelian category such that for any short exact sequence
\[
0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0
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we have
\[
V \in \mathcal{C} \iff U \in \mathcal{C} \text{ and } W \in \mathcal{C}.
\]
Equivalently, the subcategory \( \mathcal{C} \) is closed under subobjects, quotient objects, and extensions.

Noise systems (Scolamiero et al., 2016)
Noise in topological data analysis can be studied by considering a nested family \( (\mathcal{C}_\epsilon \mid \epsilon \in [0, \infty)) \) satisfying an enriched version of the Serre conditions:
\[
V \in \mathcal{C}_\epsilon \Rightarrow U \in \mathcal{C}_\epsilon \text{ and } W \in \mathcal{C}_\epsilon
\]
\[
V \in \mathcal{C}_{\epsilon_1 + \epsilon_2} \iff U \in \mathcal{C}_{\epsilon_1} \text{ and } W \in \mathcal{C}_{\epsilon_2}.
\]
for any short exact sequence.
Collaborators

Peter Bubenik, Gunnar Carlsson, Fred Chazal, William Crawley-Boevey, Marc Glisse, Dmitriy Morozov, Vidit Nanda, Steve Oudot, Elizabeth Munch, Amit Patel, Jonathan Scott, Dmitriy Smirnov, Anastasios Stefanou, Song Yu
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