Outline

- Examples and motivation
- Creating new diagrams from old
- Controllability and observability
Motivation (Why diagrams?)

In physics, the informal use of graphical calculi for symmetric monoidal categories goes back to Penrose, as early as 1971. But it wasn’t until 1991 that Joyal and Street proved the soundness of working with diagrams in a symmetric monoidal category. Selinger went further in 2007. Selinger extended their results to graphical calculi for dagger compact closed categories.

This is important because the categories that appear in control systems that I will talk about are dagger compact closed categories.
Motivation (Conceptual simplification)

The following equation is one of those used in the definition of a compact closed category:

\[ \lambda_A^{-1} \circ (\epsilon_A \otimes A) \circ \alpha_{A,A^*,A}^{-1} \circ (A \otimes \eta_A) \circ \rho_A = id_A \]

Using semicolon notation for composition:

\[ \rho_A; (A \otimes \eta_A); \alpha_{A,A^*,A}^{-1}; (\epsilon_A \otimes A); \lambda_A^{-1} = id_A \]

The same equation, written in 2d:

\[ A \quad A^* \quad A \]

\[ = \]

\[ A \]
Motivation (Communication)

How would you describe this electric circuit if you had no 2d language?
Examples

- Circuit diagrams
- Bond graphs
- Block diagrams
- Petri nets
- ZX calculus / ZW calculus
- Feynman diagrams
- Graphical linear algebra
- Natural language
Bond graphs

From en.wikipedia.org/wiki/Bond_graph
Block diagrams
Petri nets

From the PGF manual, tutorial chapter
ZX calculus

From A Simplified Stabilizer ZX-calculus (arXiv:1602.04744)
Feynman diagrams

From math.ucr.edu/home/baez/control/control_talk_erlangen.pdf
Graphical linear algebra

From GraphicalLinearAlgebra.net
John walks in the park with a dog

From Compositional Distributional Semantics with Compact Closed Categories and Frobenius Algebras (arXiv:1505.00138)
2d reasoning is not just for nerd sniping \textit{à la} xkcd/356:

\begin{itemize}
  \item communication
  \item intuition
  \item calculation
  \item understanding dualities
\end{itemize}
In 1d everything has to be done in series. By adding a second dimension, things can be done in parallel. Series compositions are usually denoted $\circ$, while parallel compositions are usually denoted $\otimes$. Note, though, that parallel composition of circuits is not the same as parallel circuits:

If we know all of the fundamental building block diagrams, any diagram can be built from series and parallel compositions of the building blocks.
Building diagrams

For linear passive electric circuits, those building blocks are resistors, capacitors, inductors, junctions, and a symmetric braiding:

\[ R \quad C \quad L \]

where \( R \), \( L \), and \( C \) are positive real numbers. The symmetric braiding can be taken as implicit when the category of diagrams is a PROP.
Simplifying diagrams

Often we want to be able to say two diagrams are “the same”, possibly preferring one over the other if it is particularly nice for calculations. Using rewrite rules, complicated-appearing diagrams can be reduced to simpler-appearing diagrams, and vice versa.

\[ R_1 \rightarrow R_2 \rightarrow R_1 + R_2 \]

Having a complete set of rewrite rules means any two equivalent diagrams can be connected by a chain of local applications of those rules. The rewrite rules are confluent if they are one-way, and local applications of those rules to any two equivalent diagrams will always lead to the same final diagram.
Control systems building blocks

where \( c \in k \).

- Multiplication by a scalar
- Integration
- Addition
- Duplication
- Zero
- Delete
- Cup
- Cap
Two contravariant endofunctors give two different dualities on this equational system, denoted here with † and *. Roughly, † can be thought of as ‘inverse’ and * can be thought of as ‘transpose’.
Rewrite rules
Controllability and Observability

For a system of matrix equations,

- **Differential:**
  \[ \dot{x} = Ax + Bu \]

- **Linear:**
  \[ y = Cx + Du \]

we can find a signal flow diagram that ‘contains’ these equations.

\( u \) is the input vector \( (\in \mathbb{R}^m) \)
\( x \) is the ‘state’ vector \( (\in \mathbb{R}^n) \)
\( y \) is the output vector \( (\in \mathbb{R}^p) \).
Controllability and Observability

\[ n - 1 = \text{an epimorphism for a controllable system} \]

\[ n - 1 = \text{a monomorphism for an observable system} \]
Controllability and Observability

\[
\begin{align*}
\{n-1\} &= \text{is an epimorphism for a controllable system} \\
\{n-1\} &= \text{is a monomorphism for an observable system}
\end{align*}
\]
Controllability and Observability

\[
\begin{align*}
A & \overset{f}{\longrightarrow} D \\
B & \quad \\
C & \quad \\
\end{align*}
\]

\[\longleftrightarrow \star \]

\[
\begin{align*}
A^* & \overset{f}{\longrightarrow} D^* \\
C^* & \\
B^* & \\
\end{align*}
\]
Future work

Much is still missing in this story:
- Stability
- Nonlinearities
- Time dependence
- Continuous / discrete hybrid systems
- Relational signal flow diagrams ($A, B, C,$ and/or $D$ as (linear) relations)
References


Paweł Sobociński’s Graphical Linear Algebra blog: http://graphcallinearalgebra.net