Aim: Diagrams as Language

For example, in the language of circuits:

\[ i, \phi \rightarrow \text{current} \rightarrow \text{voltage} \obey Ohm's law \]

Interpreting syntax as semantics should be compositional: the meaning of an expression should be derivable from the meaning of its parts.

We want to cast this in the language of categories.
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Syntax → Semantics

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\text{current } i \\
\text{and voltage } \phi
\end{array}\]

Interpreting syntax as semantics should be compositional: the meaning of an expression should be derivable from the meaning of its parts.

We want to cast this in the language of categories.
The syntax of circuits

Let’s think about interconnecting circuits.
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  I. take pushouts (additionally, coproducts)
  II. transfer decorations
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I. take pushouts (additionally, coproducts)
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What structures allow us to do these?

I. a category with finite colimits
II. a lax symmetric monoidal functor
Let $\mathcal{C}$ be a category with finite colimits, and let

$$F: (\mathcal{C}, +) \longrightarrow (\text{Set}, \times)$$

be a lax symmetric monoidal functor.

We call $d$ the decoration on the cospan. Actually, these decorated cospans are the morphisms of a bicategory, and a morphism in $F \text{Cospan}$ is an isomorphism class of decorated cospans. Kenny will say more about this shortly.
Theorem

Let $\mathcal{C}$ be a category with finite colimits, and let

$$F : (\mathcal{C}, +) \longrightarrow (\text{Set}, \times)$$

be a lax symmetric monoidal functor. Then there is a symmetric monoidal category, $F\text{Cospan}$, where

- an object is an object of $\mathcal{C}$
- a morphism from $X$ to $Y$ is a cospan $X \to N \leftarrow Y$ in $\mathcal{C}$ together with an element $d \in FN$. 

Actually, these decorated cospans are the morphisms of a bicategory, and a morphism in $F\text{Cospan}$ is an isomorphism class of decorated cospans. Kenny will say more about this shortly.
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We call $d$ the *decoration* on the cospan.

*Actually, these decorated cospans are the morphisms of a bicategory, and a morphism in $F\text{Cospan}$ is an isomorphism class of decorated cospans. Kenny will say more about this shortly.*
We compose decorated cospans by taking the pushout, then transferring the decoration.

$$\begin{pmatrix}
N & FN \\
\rightarrow & , \\
X & \leftarrow & Y \\
& d & 1
\end{pmatrix}$$
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\[
\left( \begin{array}{c}
N \\
X
\end{array} \right) \rightarrow \left( \begin{array}{c}
FN \\
Y
\end{array} \right) \leftarrow \left( \begin{array}{c}
1 \\
d
\end{array} \right) \right); \left( \begin{array}{c}
M \\
Y
\end{array} \right) \rightarrow \left( \begin{array}{c}
FM \\
Z
\end{array} \right) \leftarrow \left( \begin{array}{c}
1 \\
e
\end{array} \right) \right)
\]
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\[
\begin{pmatrix}
\xrightarrow{N} & \xleftarrow{Y}, & \uparrow_{d} & FN \\
X & Y & 1
\end{pmatrix};
\begin{pmatrix}
\xrightarrow{M} & \xleftarrow{Z}, & \uparrow_{e} & FM \\
Y & Z & 1
\end{pmatrix}
\]

= 

\[
\begin{pmatrix}
\xrightarrow{N} & \xleftarrow{M}, & \xrightarrow{Z} & \uparrow_{(d,e)} & FN \times FM \\
X & Y & M & Z & 1
\end{pmatrix}
\]
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\[
\left( \begin{array}{ccc}
N & FN \\
\downarrow X & \downarrow Y & \downarrow 1
\end{array} \right); \left( \begin{array}{ccc}
M & FM \\
\downarrow Y & \downarrow Z & \downarrow 1
\end{array} \right)
\]

= \[
\left( \begin{array}{ccc}
N + Y & M \\
\downarrow j_N & \downarrow j_M \\
\downarrow X & \downarrow Y & \downarrow Z
\end{array} \right)
\]
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\[
\begin{pmatrix}
N \\
X
\end{pmatrix}
\leftarrow
\begin{pmatrix}
FN \\
Y
\end{pmatrix},
\begin{pmatrix}
M \\
Y
\end{pmatrix}
\leftarrow
\begin{pmatrix}
FM \\
Z
\end{pmatrix},
\begin{pmatrix}
d \\
1
\end{pmatrix},
\begin{pmatrix}
e \\
1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
N +_Y M \\
X
\end{pmatrix}
\leftarrow
\begin{pmatrix}
j_N \\
N
\end{pmatrix}
\leftarrow
\begin{pmatrix}
j_M \\
M
\end{pmatrix},
\begin{pmatrix}
F(N +_Y M) \\
F(N + M)
\end{pmatrix}
\leftarrow
\begin{pmatrix}
\varphi_N,M \\
FN \times FM
\end{pmatrix},
\begin{pmatrix}
\uparrow F[j_N,j_M] \\
\uparrow \varphi_N,M \\
\uparrow (d,e)
\end{pmatrix}
\]
We compose decorated cospans by taking the pushout, then transferring the decoration.

\[
\begin{pmatrix}
N & FN \\
X & Y & 1
\end{pmatrix}
; 
\begin{pmatrix}
M & FM \\
Y & Z & 1
\end{pmatrix}
= 
\begin{pmatrix}
N +_Y M \\
N & M \phantom{+} \\
X & Y & Z
\end{pmatrix}
; 
\begin{pmatrix}
F(N + M) \\
F(N + Y M) \phantom{+} \\
X & Z
\end{pmatrix}
; 
\begin{pmatrix}
\phi_{N,M} \\
\phi_{j_N,j_M} \phantom{+} \\
Y & Z
\end{pmatrix}
; 
\begin{pmatrix}
(d,e) \\
(d,e) \phantom{+} \\
X & Z
\end{pmatrix}
\]
Let $\mathcal{C}$ be a category with finite colimits, and let

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be a lax symmetric monoidal functor. Then there is a symmetric monoidal category, $\mathcal{F}\text{Cospan}$, where

- an object is an object of $\mathcal{C}$
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Examples

Let $1 : (\mathcal{C}, +) \longrightarrow (\text{Set}, \times)$ be the constant map on a one element set. Then $1\text{Cospan}$ is just the category of cospans in $\mathcal{C}$.

Let $\mathcal{M} : (1, +) \longrightarrow (\text{Set}, \times)$ be a commutative monoid. Then $\mathcal{M}\text{Cospan}$ is just the monoid $\mathcal{M}$ considered as a one object category.
Let $\mathcal{C}$ be a category with finite colimits, and let

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**Examples**

Let $1: (\mathcal{C},+) \longrightarrow (\text{Set}, \times)$ be the constant map on a one element set. Then $1\text{Cospan}$ is just the category of cospans in $\mathcal{C}$.

Let $M: (1,+) \longrightarrow (\text{Set}, \times)$ be a commutative monoid. Then $M\text{Cospan}$ is just the monoid $M$ considered as a one object category.
Example: circuits

Define \( \text{Circ}: (\text{FinSet}, +) \rightarrow (\text{Set}, \times) \) on objects by

\[
\text{Circ}(N) = \left\{ \left. \text{circuits with nodes } N \right| E \xrightarrow{s} N \right\},
\]

on morphisms \( f: N \rightarrow M \) by

\[
\left( E \xrightarrow{s} N \right) \mapsto \left( E \xrightarrow{f \circ s} M \right),
\]

and with the lax structure maps \( \text{Circ}(N) \times \text{Circ}(M) \rightarrow \text{Circ}(N + M) \) defined by

\[
\left( E \xrightarrow{s} N, \ E' \xrightarrow{s'} M \right) \mapsto \left( E + E' \xrightarrow{s + s'} N + M \right).
\]

This is a lax symmetric monoidal functor.
Example: circuits

Define \( \text{Circ}: (\text{FinSet}, +) \longrightarrow (\text{Set}, \times) \) on objects by

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\text{Circ}(N) = \left\{ \text{circuits with nodes } N \right\} = \left\{ E \xrightarrow{s} N \right\},
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on morphisms \( f: N \rightarrow M \) by

\[
\left( E \xrightarrow{s} \xleftarrow{t} N \right) \longmapsto \left( E \xrightarrow{f \circ s} \xleftarrow{f \circ t} M \right),
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\left( E \xrightarrow{s} \xleftarrow{t} N , \ E' \xrightarrow{s'} \xleftarrow{t'} M \right) \longmapsto \left( E + E' \xrightarrow{s+s'} \xleftarrow{t+t'} N + M \right).
\]

This is a lax symmetric monoidal functor.
Note: \( F \) maps \( N \) to the set of decorations on \( N \).
Then we have the composite of decorated cospans
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Theorem: functors

Suppose we have a monoidal natural transformation

\[(C, +) \xrightarrow{(A, F, G)} (\text{Set}, \times)\]

between lax symmetric monoidal functors \(A, F, G\), where \(A\) preserves finite colimits.

This functor sends objects \(X\) to \(AX\), and morphisms \(N X Y\) to \(AN X Y\).
Theorem: functors

Suppose we have a monoidal natural transformation

\[(C, +) \xrightarrow{\theta} (\text{Set}, \times)\]

between lax symmetric monoidal functors \(A, F, G\), where \(A\) preserves finite colimits. Then we can define a symmetric monoidal functor

\[T : FCosp \longrightarrow GCosp.\]

This functor sends objects \(X\) to \(AX\), and morphisms

\[
\begin{pmatrix}
X \\
N
\end{pmatrix}
\begin{pmatrix}
Y
\end{pmatrix}
\xrightarrow{\begin{pmatrix}
FN \\
FN 1
\end{pmatrix}}
\begin{pmatrix}
AN \\
GAN \theta_N
\end{pmatrix}
\begin{pmatrix}
AX \\
AY
\end{pmatrix}
\]
Example: counting components
Consider the monoidal natural transformation

\[(\text{FinSet}, +) \xrightarrow{(1, \cdot)} (\text{Set}, \times)\]

\[\text{Circ}\]

\[\#\]

\[\mathbb{N}\]

defined by \(\#_{\mathbb{N}}(E \Rightarrow N) = |E|\).
Example: counting components
Consider the monoidal natural transformation

\[(\text{FinSet}, +) \overset{\text{Circ}}{\longrightarrow} (\text{Set}, \times)\]

defined by \(\#_N(E \Rightarrow N) = |E|\).

This defines a symmetric monoidal functor \(R: \text{CircCospan} \to \mathbb{N}\) that sends an open circuit to the number of resistors it contains.

For example,

\[
R\left(\begin{array}{c}
\bullet \\
\end{array}\right) = 2
\]
Summary

We want functorial semantics for diagram languages. Decorated cospans allows construction of
- symmetric monoidal categories from lax symmetric monoidal functors
- symmetric monoidal functors from monoidal natural transformations
In fact, decorated cospan categories are hypergraph categories: categories where we can interpret network-style diagrams.

A limitation, however, is that decorated cospan categories have a very free notion of composition: they completely separate compositional structure from semantic structure.

To handle more interaction between composition and semantics, we must use decorated *corelations*. This can handle all hypergraph categories.

I’ll talk about this on Tuesday.
Thanks for listening.

For more
The paper: arXiv:1502.00872
My website: http://www.brendanfong.com/
John Baez’s website: http://math.ucr.edu/baez/networks/