Decorated Cospans

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AMS Fall Western Sectional
UC Riverside
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Aim: Diagrams as Language

Syntax $\longrightarrow$ Semantics

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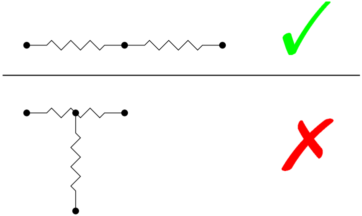
Syntax $\longrightarrow$ Semantics

For example, in the language of circuits:



$$\rightsquigarrow \left\{ (i, \phi) \;\middle|\; \begin{array}{c} \text{current } i \text{ and} \\ \text{voltage } \phi \\ \text{obey Ohm's law} \end{array} \right\}$$

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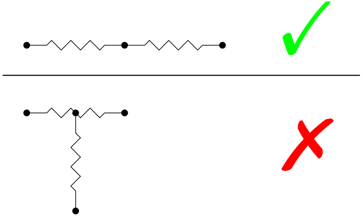

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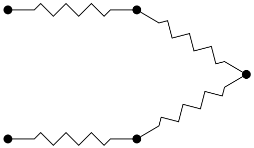
We want to cast this in the language of categories.

The syntax of circuits

Let's think about interconnecting circuits.



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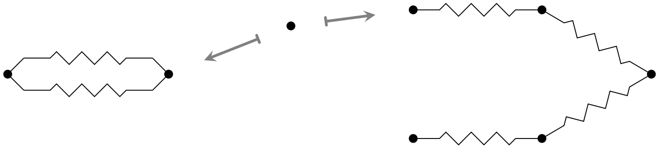

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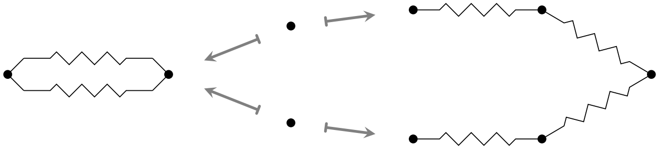

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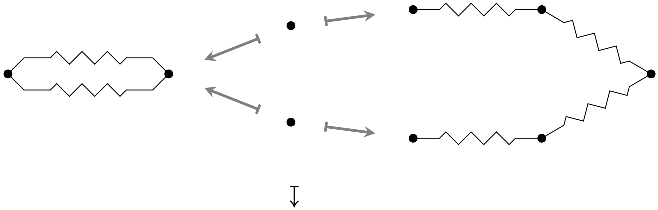

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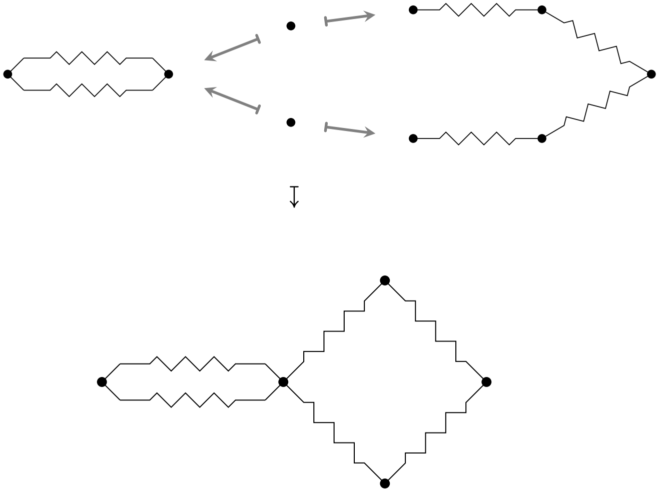

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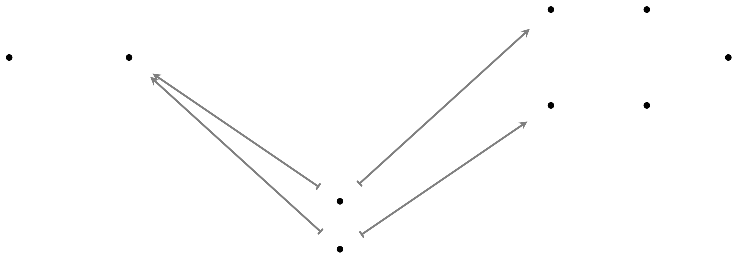

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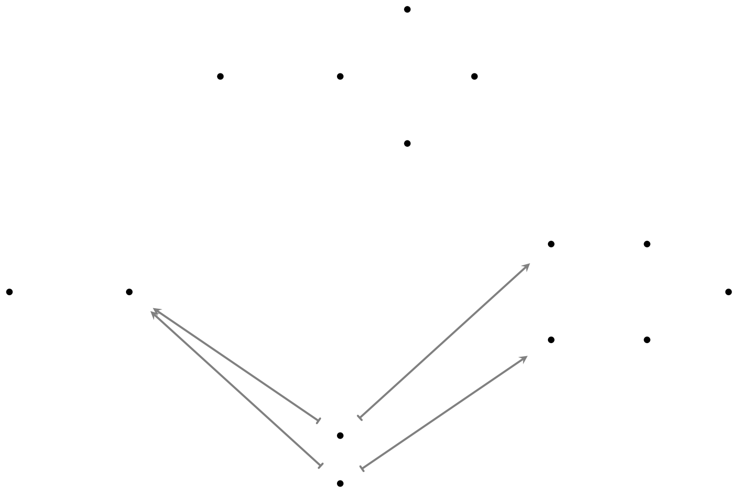

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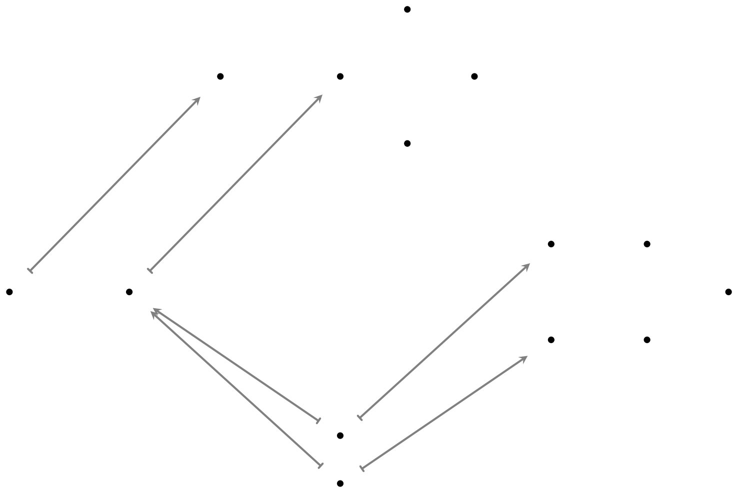

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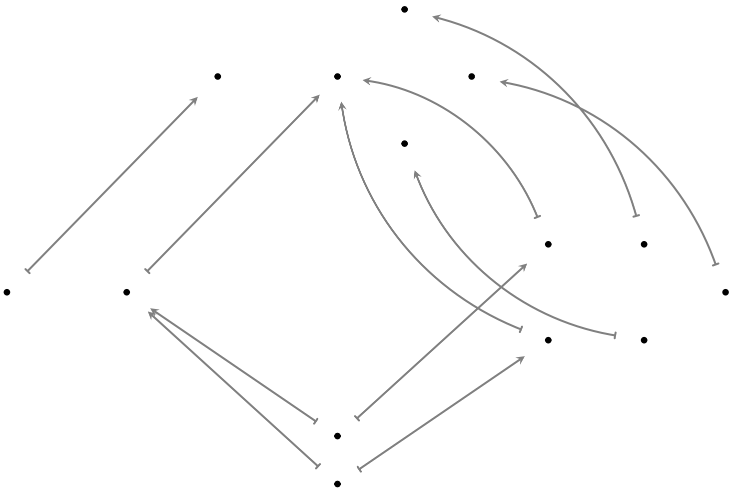

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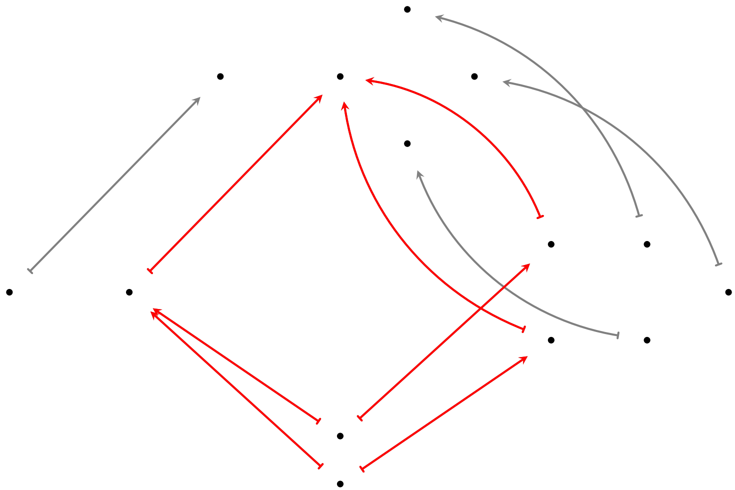

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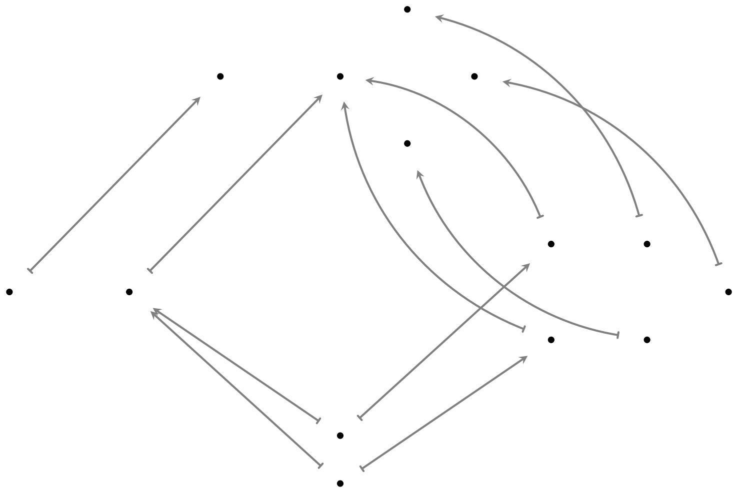

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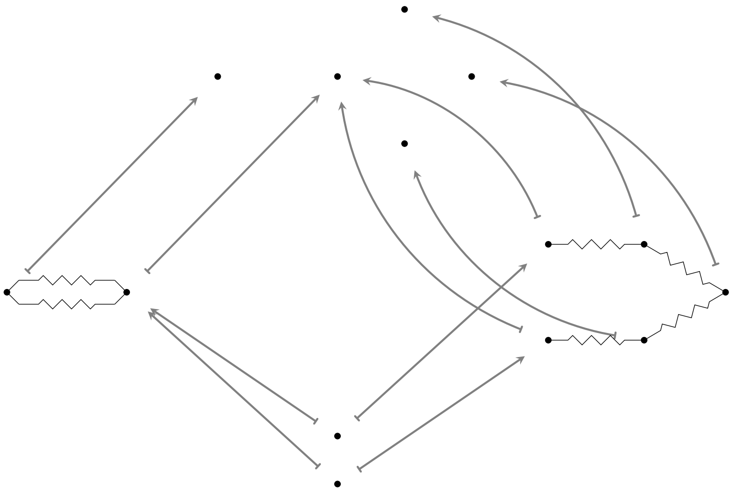

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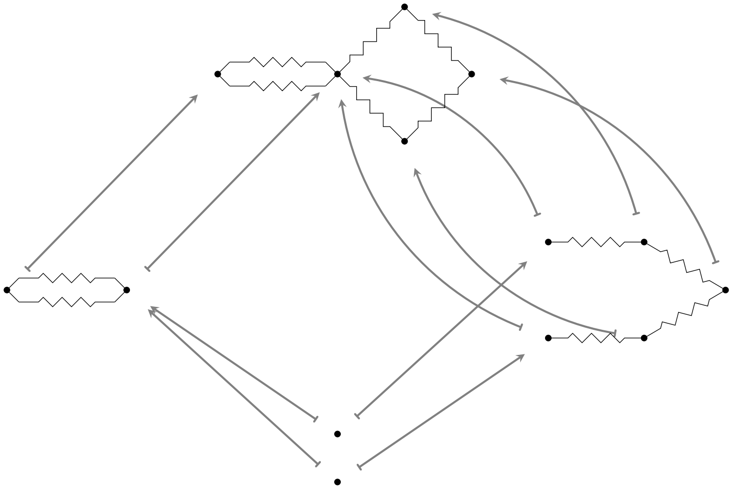

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I. take pushouts (additionally, coproducts)
II. transfer decorations

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I. a category with finite colimits
II. a lax symmetric monoidal functor

Theorem

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be a lax symmetric monoidal functor. Then there is a symmetric monoidal category, $F\mathsf{Cospan}$, where

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*Actually, these decorated cospans are the morphisms of a bicategory, and a morphism in $F\mathsf{Cospan}$ is an isomorphism class of decorated cospans. Kenny will say more about this shortly.

We compose decorated cospans by taking the pushout, then transferring the decoration.

$$\left(\begin{array}{ccccc} & & N & & FN \\ & \nearrow & & \nwarrow & \uparrow d \\ X & & & Y, & 1 \end{array} \right)$$

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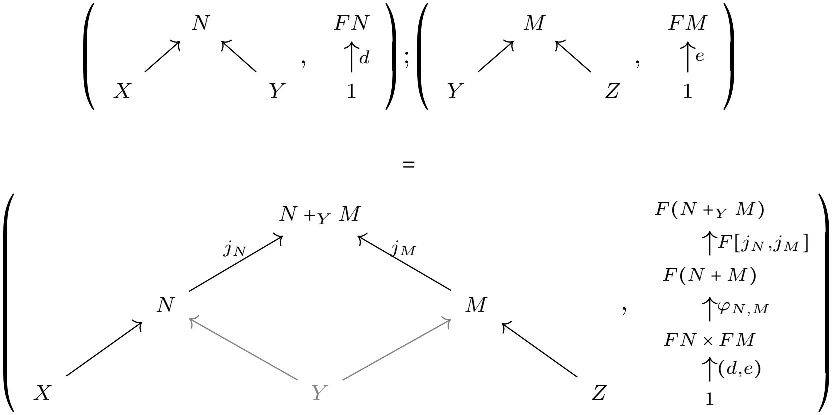

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Examples

Let $1\colon (\mathcal{C}, +) \longrightarrow (\mathsf{Set}, \times)$ be the constant map on a one element set. Then $1\mathsf{Cospan}$ is just the category of cospans in $\mathcal{C}$.

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Let $M\colon (1, +) \longrightarrow (\mathsf{Set}, \times)$ be a commutative monoid. Then $M\mathsf{Cospan}$ is just the monoid M considered as a one object category.

Example: circuits

Define $\mathsf{Circ}\colon (\mathsf{FinSet}, +) \longrightarrow (\mathsf{Set}, \times)$ on objects by

$$\mathsf{Circ}(N) = \left\{ \begin{array}{c} \text{circuits with} \\ \text{nodes } N \end{array} \right\} = \left\{ E \underset{t}{\overset{s}{\rightrightarrows}} N \right\},$$

on morphisms $f\colon N \to M$ by

$$\left(E \underset{t}{\overset{s}{\rightrightarrows}} N \right) \longmapsto \left(E \underset{f\circ t}{\overset{f\circ s}{\rightrightarrows}} M \right),$$

and with the lax structure maps $\mathsf{Circ}(N) \times \mathsf{Circ}(M) \to \mathsf{Circ}(N+M)$ defined by

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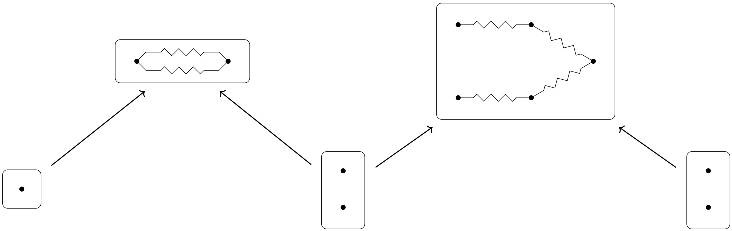
Note: F maps N to the set of decorations on N.

Then we have the composite of decorated cospans



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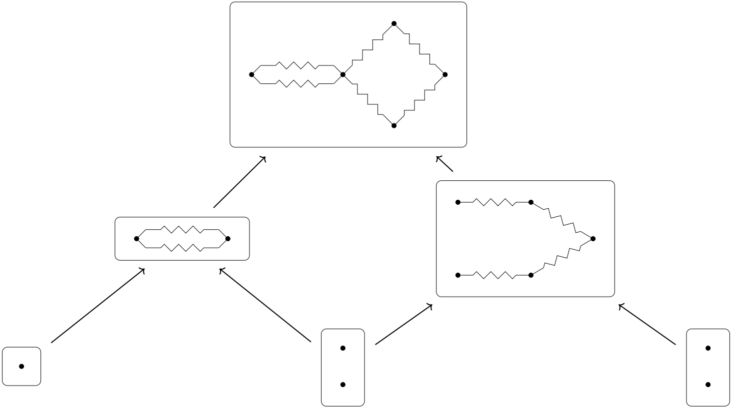

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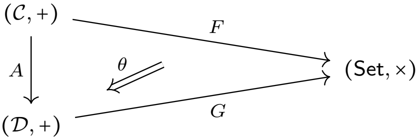

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$$\begin{array}{ccc} (\mathcal{C},+) & \xrightarrow{\quad F \quad} & \\ {\scriptstyle A}\downarrow & \overset{\theta}{\Swarrow} & (\mathsf{Set},\times) \\ (\mathcal{D},+) & \xrightarrow[\quad G \quad]{} & \end{array}$$

between lax symmetric monoidal functors A, F, G, where A preserves finite colimits. Then we can define a symmetric monoidal functor

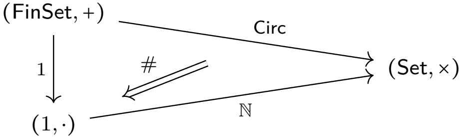
$$T\colon F\mathsf{Cospan} \longrightarrow G\mathsf{Cospan}.$$

This functor sends objects X to AX, and morphisms

$$\left(\begin{array}{ccc} & N & \\ \nearrow & & \nwarrow \\ X & & Y \end{array} \;,\; \begin{array}{c} FN \\ \uparrow d \\ 1 \end{array} \right) \text{ to } \left(\begin{array}{ccc} & AN & \\ \nearrow & & \nwarrow \\ AX & & AY \end{array} \;,\; \begin{array}{c} GAN \\ \uparrow \theta_N \\ FN \\ \uparrow d \\ 1 \end{array} \right).$$

Example: counting components

Consider the monoidal natural transformation



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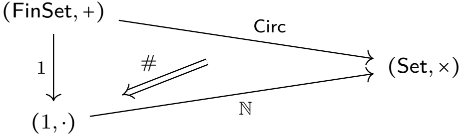

defined by $\#_N(E \rightrightarrows N) = |E|$.

This defines a symmetric monoidal functor R: CircCospan $\longrightarrow \mathbb{N}$ that sends an open circuit to the number of resistors it contains.

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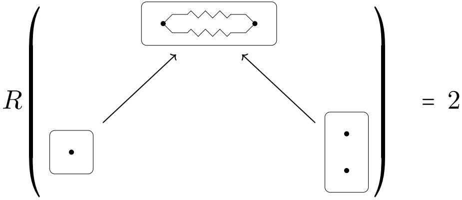

Summary

We want functorial semantics for diagram languages.

Decorated cospans allows construction of

- symmetric monoidal categories from lax symmetric monoidal functors
- symmetric monoidal functors from monoidal natural transformations

In fact, decorated cospan categories are hypergraph categories: categories where we can interpret network-style diagrams.

A limitation, however, is that decorated cospan categories have a very free notion of composition: they completely separate compositional structure from semantic structure.

To handle more interaction between composition and semantics, we must use decorated *corelations*. This can handle all hypergraph categories.

I'll talk about this on Tuesday.

Thanks for listening.

For more
The paper: arXiv:1502.00872
My website: http://www.brendanfong.com/
John Baez's website: http://math.ucr.edu/baez/networks/