Algebras of Open Systems on the Operad of Wiring Diagrams

Dmitry Vagner

Joint work with David I. Spivak and Eugene Lerman

Remember these?

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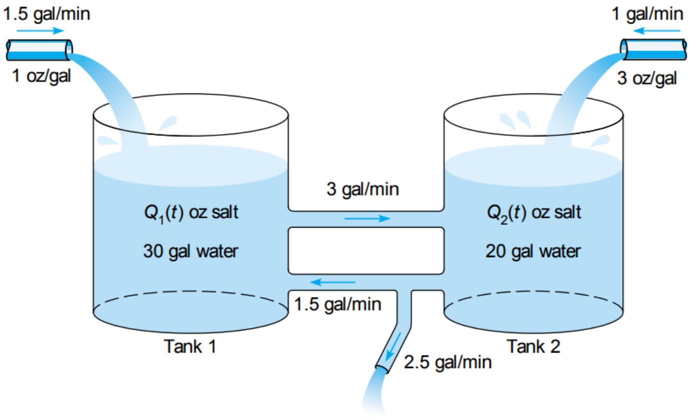

source: Figure 7.1.6 in Boyce & DiPrima

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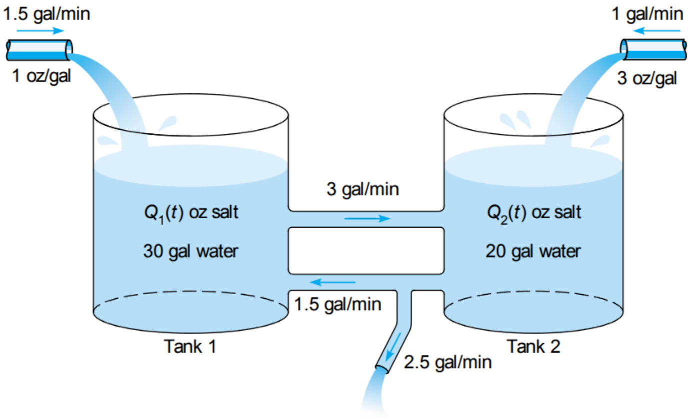

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$$\begin{cases} \dot{Q}_1 = -.1Q_1 + .075Q_2 + 1.5 \\ \dot{Q}_2 = .1Q_1 - .2Q_2 + 3 \end{cases} \tag{1}$$

Distill Into Underlying Network

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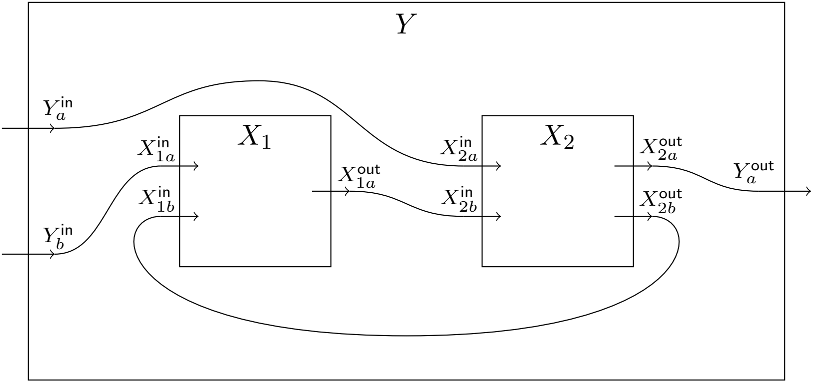
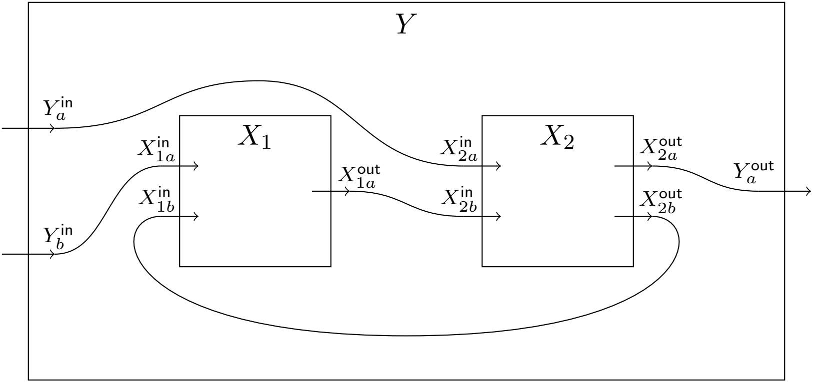


Distill Into Underlying Network (the operad)



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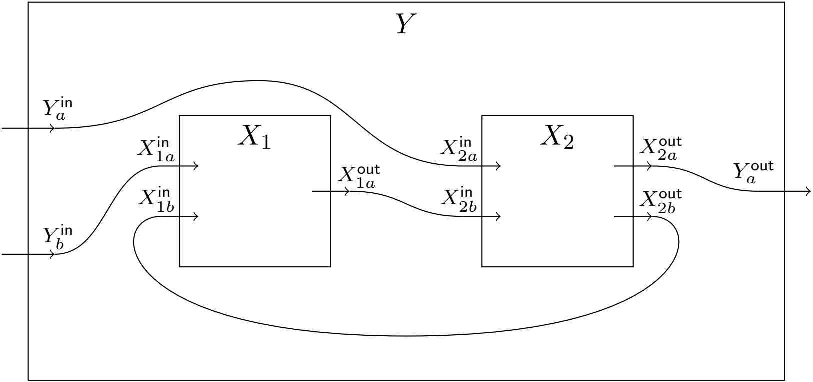
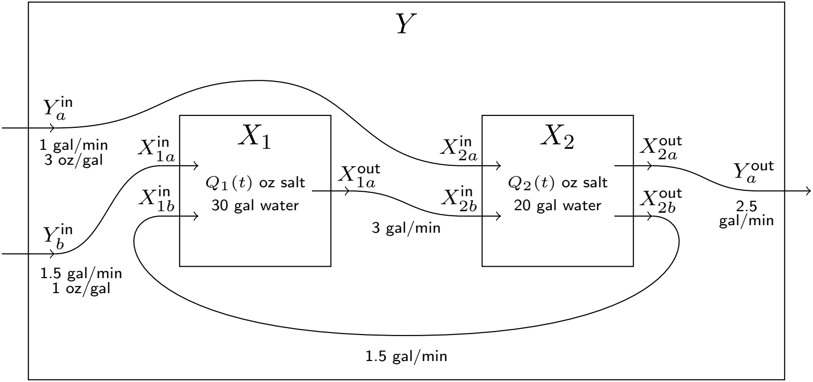


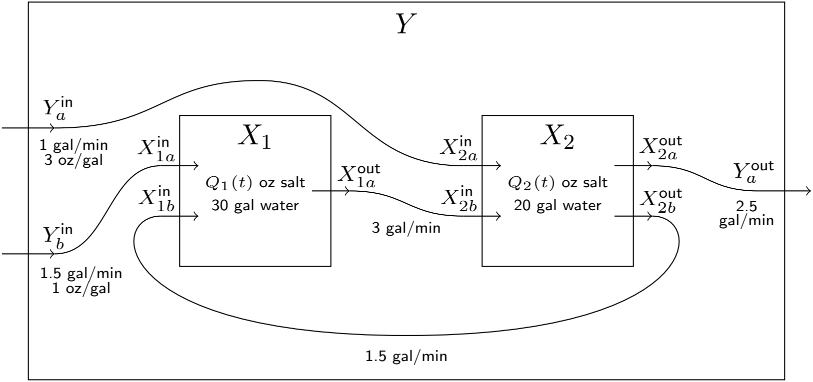
wiring diagram $\Phi : X_1, X_2 \to Y$

Add Content



Y
X_1
X_2
Y_a^{in}
1 gal/min
3 oz/gal
Y_b^{in}
1.5 gal/min
1 oz/gal
X_{1a}^{in}
X_{1b}^{in}
$Q_1(t)$ oz salt
30 gal water
X_{1a}^{out}
3 gal/min
X_{2a}^{in}
X_{2b}^{in}
$Q_2(t)$ oz salt
20 gal water
X_{2a}^{out}
X_{2b}^{out}
Y_a^{out}
2.5
gal/min
1.5 gal/min

Add Content (the algebra)



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$$\hom_{\mathcal{OM}}(X_1, \ldots, X_n; Y) \coloneqq \hom_{\mathcal{M}}(X_1 \otimes \cdots \otimes X_n, Y)$$

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This allows us to avoid distracting subscripts.

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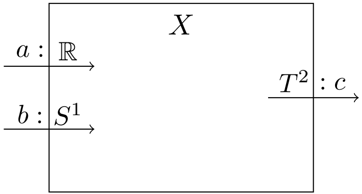

Figure: A box $X = (\{a, b\}, \{c\})$ with two input ports, $a : \mathbb{R}$ and b:S^1, and one output port, $c : T^2$.

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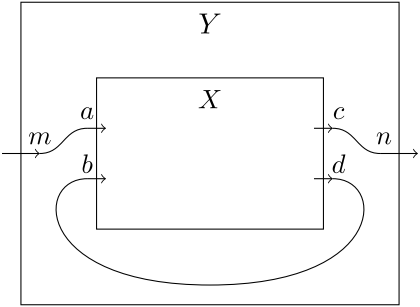

Figure: A Wiring Diagram $\Phi : X \to Y$.

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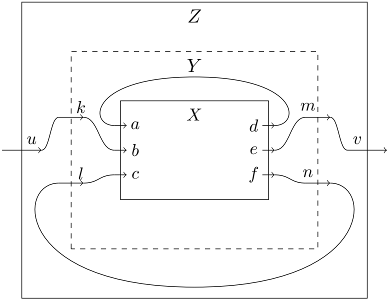

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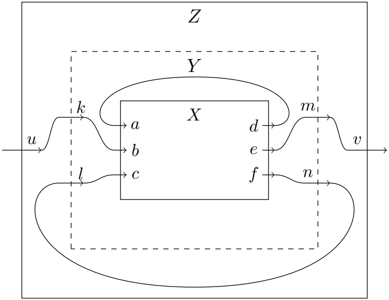

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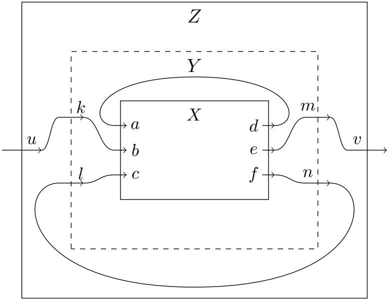

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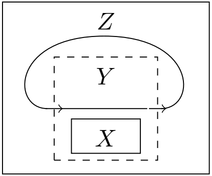

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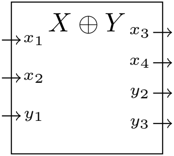

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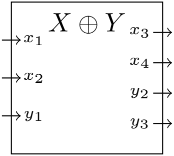

Monoidal unit, denoted '0', is the closed (without wires) box.

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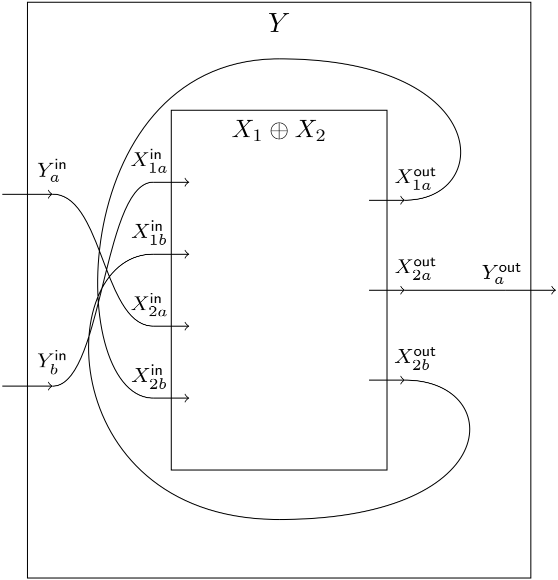

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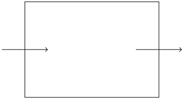

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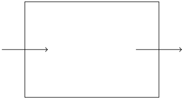

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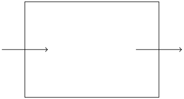

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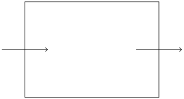

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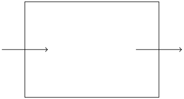

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$$\begin{cases} f^{\mathsf{in}} \colon M \times U^{\mathsf{in}} \to TM \\ f^{\mathsf{out}} \colon M \to U^{\mathsf{out}} \end{cases}$$

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$$\begin{cases} f^{\mathsf{in}} \colon \overline{S} \times \overline{X^{\mathsf{in}}} \to T\overline{S} \\ f^{\mathsf{out}} \colon \overline{S} \to \overline{X^{\mathsf{out}}} \end{cases}$$

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$$\overline{X} = \prod_{a \in A} X(a)$$

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$$\begin{bmatrix} \dot{m} \\ u^{\mathsf{out}} \end{bmatrix} = \begin{bmatrix} f^{M,M} & f^{M,U} \\ f^{U,M} & 0 \end{bmatrix} \begin{bmatrix} m \\ u^{\mathsf{in}} \end{bmatrix}$$

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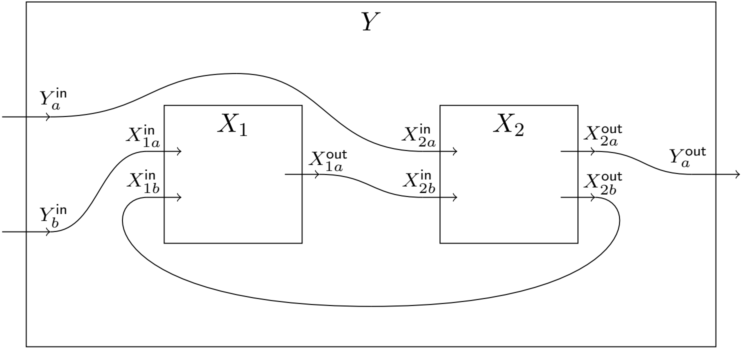
$$g = \begin{bmatrix} g^{S,S} & g^{S,X} \\ g^{X,S} & g^{X,X} \end{bmatrix} = \begin{bmatrix} f^{S,X} & 0 \\ 0 & I \end{bmatrix} \overline{\varphi} \begin{bmatrix} f^{X,S} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} f^{S,S} & 0 \\ 0 & 0 \end{bmatrix}$$

back to our example



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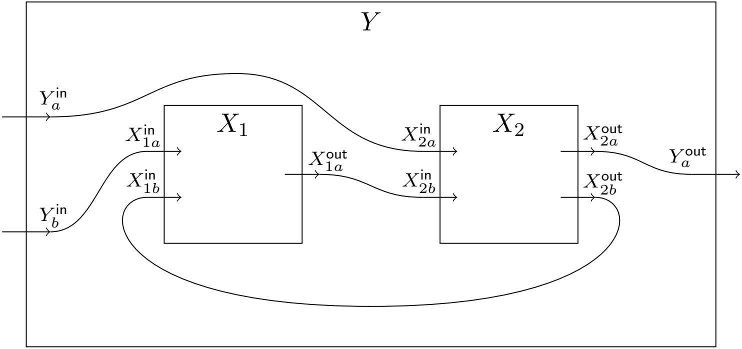

$\overline{\varphi}$ given by
$$\begin{bmatrix} \overline{X_{1a}^{\mathrm{out}}} \\ \overline{X_{2a}^{\mathrm{out}}} \\ \overline{X_{2b}^{\mathrm{out}}} \\ \overline{Y_a^{\mathrm{in}}} \\ \overline{Y_b^{\mathrm{in}}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \overline{X_{1a}^{\mathrm{in}}} \\ \overline{X_{1b}^{\mathrm{in}}} \\ \overline{X_{2a}^{\mathrm{in}}} \\ \overline{X_{2b}^{\mathrm{in}}} \\ \overline{Y_a^{\mathrm{out}}} \end{bmatrix}$$

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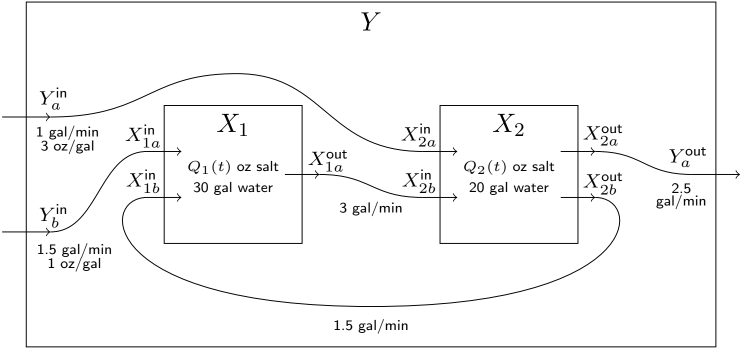

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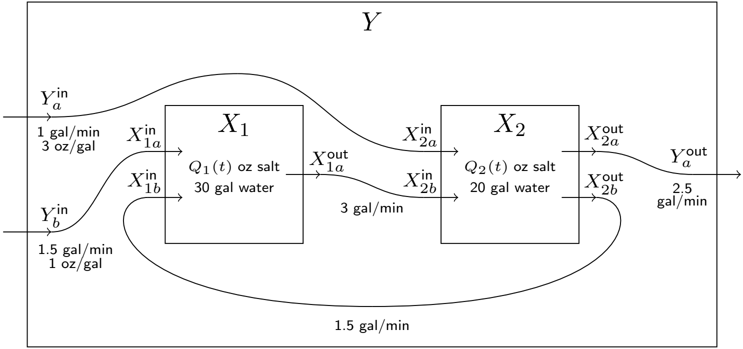

ODS for $X_1 \oplus X_2$ is
$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ X_{1a}^{\text{out}} \\ X_{2a}^{\text{out}} \\ X_{2b}^{\text{out}} \end{bmatrix} = \begin{bmatrix} -.1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -.2 & 0 & 0 & 1 & 1 \\ .1 & 0 & 0 & 0 & 0 & 0 \\ 0 & .125 & 0 & 0 & 0 & 0 \\ 0 & .075 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ X_{1a}^{\text{in}} \\ X_{1b}^{\text{in}} \\ X_{2a}^{\text{in}} \\ X_{2ba}^{\text{in}} \end{bmatrix}$$

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this is what we wanted!