Abstract Dynamical Systems

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Goal: categorical framework for modeling and analysis of systems

- Systems as boxes
- Inhabitants
- Channels of info flow as wires

Analyse the behavior of the composite system using analysis of the particular systems components and their wired interconnection.

- Coherent zoom in/out subsystems, due to compositionality [operad algebras]
- Appropriate notions of time for abstract systems [sheaves]
Outline

1. The operad of wiring diagrams
2. Interval sheaves
3. Continuous and discrete machines
4. Total and deterministic variations
Monoidal category of wiring diagrams

* A $\mathcal{C}$-typed finite set is $X$ together with typing function $X \xrightarrow{\tau} \operatorname{ob}\mathcal{C}$; these form a comma category $\mathbf{TFS}_\mathcal{C}$, cocartesian monoidal.

The monoidal category $\mathcal{W}_\mathcal{C}$ has

- objects labeled boxes, i.e. $X = (X^{\text{in}}, X^{\text{out}}) \in \mathbf{TFS}_\mathcal{C}^2$

- morphisms

\[
(X^{\text{in}}, \phi^{\text{in}} \xrightarrow{\phi^{\text{in}}} X^{\text{out}} + Y^{\text{in}}, Y^{\text{out}} \xrightarrow{\phi^{\text{out}}} X^{\text{out}}) \in \mathbf{TFS}_\mathcal{C}^2
\]

- tensor product $X_1 \oplus X_2 = (X_1^{\text{in}} + X_2^{\text{in}}, X_1^{\text{out}} + X_2^{\text{out}})$
If \( C \) finitely complete, dependent product \( \hat{X} = \prod_x \tau(x) \) gives strong monoidal \( (-) : \text{TFS}_C^{\text{op}} \to C \) passage to \( C \)-context.

Model systems as algebras for \( \mathcal{W}_C \Leftrightarrow \) the underlying operad \( \mathcal{OW}_C \); monoidal world for formal language, operadic world for visual.

A lax monoidal functor \( F : \mathcal{W}_C \to \text{Cat} \) gives semantics to boxes, composite formula to wiring diagrams

\[
F(X_1) \times \ldots \times F(X_n) \xrightarrow{F_{X_1 \ldots X_n}} F(X_1 + \ldots + X_n) \xrightarrow{F_\phi} FY.
\]

Dynamical Systems as Algebras

- Continuous (open) dynamical systems (previous talk)
- Discrete dynamical systems, or (finite case) Moore machines
**Modeling Time: Categories of intervals**

\[ \mathbb{R}_{\geq 0} \] positive reals, \( \text{Tr}_p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) translation-by-\( p \).

- **Category \( \text{Int} \) of continuous intervals** has objects \( \mathbb{R}_{\geq 0} \), morphisms \( \text{Int}(\ell, \ell') = \{ \text{Tr}_p \mid p \in \mathbb{R}_{\geq 0} \text{ and } p \leq \ell' - \ell \} \); equivalently via image
  \[ [0, \ell] \overset{p}{\subseteq} [0, \ell'] \]

- **Category \( \text{Int}_N \) of discrete intervals**, \( \text{ob} = \mathbb{N} \), \( n \xrightarrow{\text{Tr}_p} n' \) by \( p \in \mathbb{N} \).

If \( A : \text{Int}^{\text{op}} \to \text{Set} \), view section \( x \in A(\ell') \) & restriction \( A(\text{Tr}_p)(x) \)
Sheaves on intervals

For $\ell \in \text{Int}$ and $0 \leq p \leq \ell$, the pairs $p \xrightarrow{[0,p]} \ell$, $(\ell-p) \xrightarrow{[p,\ell]} \ell$ form a cover for $\ell$. These generate a coverage for $\text{Int}$; similarly for $\text{Int}_N$.

$\tilde{\text{Int}}$ and $\tilde{\text{Int}}_N$ are the toposes of continuous and discrete interval sheaves, i.e. $\text{Int}_{(N)}$-presheaves whose compatible sections glue.

Examples

- $\tilde{\text{Int}}_N \simeq \text{Grph}$, so every graph gives a discrete interval sheaf
- $F: \text{Set} \rightarrow \tilde{\text{Int}}$ by $F(X)(\ell) = \{ f: [0,\ell] \rightarrow X \}$, sheaf of functions
- $\text{Ext}_\epsilon: \tilde{\text{Int}} \rightarrow \tilde{\text{Int}}$ by $\text{Ext}_\epsilon(A)(\ell) = A(\ell + \epsilon)$, $\epsilon$-extension sheaf

Idea: $\tilde{\text{Int}}_{(N)}$-labeled boxes have ports carrying very general time-based signals, expressed as sheaves of ‘all possible behaviors’.
Abstract machines

- **A continuous machine** with input & output $A \& B \in \Int$ is

$$p^i \quad \text{S} \quad p^o$$

$A$ $B$

$S$ - state sheaf
$p^i$ - input sheaf map
$p^o$ - output sheaf map

$\text{Mch}(A, B) = \Int /_{A \times B}$ the topos of continuous $(A, B)$-machines.

- **For** $A, B \in \Int_N$, **discrete machines** $\text{Mch}_N(A, B) = \Int_N /_{A \times B}$. 
Continuous machines form a $\mathcal{W}_{\text{Int}}$-algebra

Functor $\text{Mch}: \mathcal{W}_{\text{Int}} \rightarrow \text{Cat}$ by $(X^{\text{in}}, X^{\text{out}}) \mapsto \text{Mch}(\hat{X}^{\text{in}}, \hat{X}^{\text{out}})$ and

Finally, lax monoidal structure by taking products of spans:

$$(S \xymatrix{(p^i,p^o) \ar[r] & \hat{X}^{\text{in}} \times \hat{X}^{\text{out}}, \quad T \xymatrix{(q^i,q^o) \ar[r] & \hat{Z}^{\text{in}} \times \hat{Z}^{\text{out}}}) \mapsto (p^i \times q^i, p^o \times q^o)$$
Total and deterministic machines

Characteristics of interest: for initial state and input, the machine

- uniquely evolves or ‘stays idle’ \(\xrightarrow{\sim}\) determinism
- always evolves \(\xrightarrow{\sim}\) totality

Continuous machines \(A \xrightarrow{S} B\) are neither in general:

\[\begin{aligned}
&\text{Starting in state germ } s_0, \text{ for input } a \text{ over } \ell\text{-interval, there may or may not be } s_0\text{-extension.}
\end{aligned}\]

\(\star\) A total machine would have at least one extension, whereas a deterministic machine would have maximum one extension.
There exist subalgebras of $\text{Mch}_N$: $\mathcal{W}_{\text{Int}} \rightarrow \text{Cat}$ of total and deterministic machines, by imposing conditions on $p^i$ and $q^i$.

There are algebra maps from discrete dynamical systems

$\mathcal{W}_{\text{Set}} \xrightarrow{\alpha} \mathcal{W}_{\text{Int}_N} \xrightarrow{\text{DDS}} \text{Cat}$

and from continuous dynamical systems

$\mathcal{W}_{\text{Euc}} \xrightarrow{\beta} \mathcal{W}_{\text{Int}} \xrightarrow{\text{CDS}} \text{Cat}$

Algebra maps ‘translate’ between various processes; can then interconnect arbitrary systems & study them on common ground.
Thank you for your attention!