Metrics on Functor Categories & Reeb Graph Operations

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AMS Sectional Meeting, UC Riverside
9–10 November 2019
Edelsbrunner, Letscher, Zomorodian 2000

Persistent homology takes a filtered space $\mathbb{X} = \{X_t \mid t \in \mathbb{R}\}$ and returns a barcode of intervals $[p, q) \subset \mathbb{R}$ or a persistence diagram of points $(p, q) \in \mathbb{R}^2$. 

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Persistence diagrams


- Discretize the $t$-variable to integers: $t = 0, 1, 2, \ldots$
- Present $X$ as an increasing sequence:
  \[ X : X_0 \subset X_1 \subset X_2 \subset \ldots \]
- Apply a homology functor $H = H(-; k)$ to obtain a persistence module:
  \[ H(X) : H(X_0) \to H(X_1) \to H(X_2) \to \ldots \]
- Observe that $H(X)$ is a graded module over the polynomial ring $k[z]$, where $z$ acts by shifting to the right.
- Decompose this graded module as a direct sum of cyclic submodules.
- Summands $z^s k[z]/(z^{t-s})$ are recorded as intervals $[s, t)$.
- Summands $z^s k[z]$ are recorded as intervals $[s, +\infty)$.
The map \{\text{persistence modules}\} \rightarrow \{\text{diagrams}\} is 1-Lipschitz.
Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

The map \{persistence modules\} → \{diagrams\} is 1-Lipschitz.

Relators

The metrics on the two spaces are defined in terms of ‘relators’.

- Two persistence modules may be related by an **interleaving**.
- Two diagrams may be related by a **matching**.

Every relator, of each type, has a size associated with it. The metrics are defined by finding the infimum of the size of relators between a given pair of objects. (Compare the geodesic distance in a Riemannian manifold.)

Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

If two persistence modules admit an \(\epsilon\)-interleaving, then their persistence diagrams admit an \(\epsilon\)-matching.
Interleaving of Persistence Modules

**Definition**

Let $V, W$ be persistence modules. An $\epsilon$-interleaving between $V, W$ is a pair $(\Phi, \Psi)$ where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

$$\phi_t : V_t \rightarrow W_{t+\epsilon} \quad \psi_t : W_t \rightarrow V_{t+\epsilon}$$

such that [various conditions].

The [various conditions] require the diagrams

![Diagram](diagram.png)

to commute for all $s < t$. 
**Definition**

Let $V, W$ be persistence modules. An $\epsilon$-interleaving between $V, W$ is a pair $(\Phi, \Psi)$ where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

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such that [various conditions].

**Interleavor categories (Chazal, dS, Glisse, Oudot 2016)**

An $\epsilon$-interleaved pair of modules $(V, W, \Phi, \Psi)$ is ‘the same thing’ as a persistence module defined over the set $I = \mathbb{R} \times \{0, \epsilon\}$ (two copies of the real line) with the partial order

$$(s, a) \leq (t, b) \iff \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}$$

$\mathbb{R} \times \{0, \epsilon\}$:
Two classical persistence modules \( V, W \) are \( \epsilon \)-interleaved iff the following functor extension problem has a solution:

\[
\begin{array}{ccc}
\text{Vect} & \xrightarrow{V} & R \\
R & \xrightarrow{\cdot} & R \times \{0, \epsilon\}
\end{array}
\]

Here \( R \times \{0, \epsilon\} \) has the partial order:

\[
(s, a) \leq (t, b) \iff \begin{cases} 
    s \leq t & \text{if } a = b \\
    s + \epsilon \leq t & \text{if } a \neq b
\end{cases}
\]

Interleaving of Persistence Modules
Two persistence modules $\mathbb{V}, \mathbb{W} : P \to C$ are $\Omega$-interleaved iff the following functor extension problem has a solution:

Here $P \cup_\Omega P$ has the partial order

$$(s, a) \leq (t, b) \iff \begin{cases} s \leq t & \text{if } a = b \\ \Omega s \leq t & \text{if } a \neq b \end{cases}$$

where $\Omega : P \to P$ is a translation.
Interleaving Metrics on Functor Categories

Translations (Bubenik, dS, Scott 2015)

\( \text{Trans}_P \) is the poset of functions \( \Omega : P \to P \) that are order-preserving and satisfy \( x \leq \Omega x \) for all \( x \in P \).

Superlinear Families

A **superlinear family** is a 1-parameter family of translations of \( P \)

\[ (\Omega_\epsilon \mid \epsilon \in [0, \infty)) \]

such that

\[ \Omega_{\epsilon_1} \Omega_{\epsilon_2} \leq \Omega_{\epsilon_1 + \epsilon_2} \]

for all \( \epsilon_1, \epsilon_2 \in [0, \infty) \).

Sublinear Projections

A **sublinear projection** is a map \( \pi : \text{Trans}_P \to [0, \infty] \) such that

\[ \pi(\Omega_1 \Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2) \]

for all \( \Omega_1, \Omega_2 \in \text{Trans}_P \).
Superlinear Families

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for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Examples of superlinear families

- $P = \mathbb{R}$,
  $$\Omega_\epsilon(t) = t + \epsilon.$$

- $P = \{\text{compact intervals in the real line}\}$,
  $$\Omega_\epsilon([a, b]) = [a - \epsilon, b + \epsilon].$$

- $P = \{\text{closed subsets of a metric space } X\}$,
  $$\Omega_\epsilon(V) = V^\epsilon = \{x \in X \text{ such that } d(x, V) \leq \epsilon\}.$$
Superlinear Families

A **superlinear family** is a 1-parameter family of translations of $\mathbf{P}$

$$\{\Omega_\epsilon \mid \epsilon \in [0, \infty)\}$$

such that

$$\Omega_{\epsilon_1} \Omega_{\epsilon_2} \leq \Omega_{\epsilon_1 + \epsilon_2}$$

for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Interleaving distance (Bubenik, dS, Scott 2015)

Given a superlinear family $\{\Omega_\epsilon\}$ of translations of $\mathbf{P}$, we define the interleaving distance

$$d_i(\mathbf{V}, \mathbf{W}) = \inf (\epsilon \mid \mathbf{V}, \mathbf{W} \text{ are } \Omega_\epsilon\text{-interleaved})$$

between generalized persistence modules $\mathbf{V}, \mathbf{W} : \mathbf{P} \to \mathbf{C}$. 
A **sublinear projection** is a map \( \pi : \text{Trans}_P \rightarrow [0, \infty] \) such that

\[
\pi(\Omega_1 \Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2)
\]

for all \( \Omega_1, \Omega_2 \in \text{Trans}_P \).

**Interleaving distance** (Bubenik, dS, Scott 2015)

Given a sublinear projection family \( \pi : \text{Trans}_P \rightarrow [0, \infty] \), we define the interleaving distance

\[
d_i(\mathbb{V}, \mathbb{W}) = \inf (\pi(\Omega) \mid \mathbb{V}, \mathbb{W} \text{ are } \Omega\text{-interleaved})
\]

between generalized persistence modules \( \mathbb{V}, \mathbb{W} : P \rightarrow C \).
Functoriality

Suppose $\mathcal{V}, \mathcal{W} : \mathbf{P} \to \mathbf{C}$ and $H : \mathbf{C} \to \mathbf{D}$ are functors. Then

$$d_i(H\mathcal{V}, H\mathcal{W}) \leq d_i(\mathcal{V}, \mathcal{W})$$

for any superlinear family or sublinear projection.

Proof.

An $\Omega$-interleaving of $\mathcal{V}, \mathcal{W}$ gives an $\Omega$-interleaving of $H\mathcal{V}, H\mathcal{W}$:
Two persistence modules $\mathbb{V}, \mathbb{W} : P \rightarrow C$ are $\Omega$-interleaved iff the following functor extension problem has a solution:

Here $P \cup_\Omega P$ has the partial order

$$(s, a) \leq (t, b) \iff \begin{cases} s \leq t & \text{if } a = b \\ \Omega s \leq t & \text{if } a \neq b \end{cases}$$

where $\Omega : P \rightarrow P$ is a translation.
Interleavings for generalized persistence modules over an arbitrary category

Two persistence modules $V, W : D \to C$ are $\Delta$-interleaved iff the following functor extension problem has a solution:

Here $\Delta$ is a cospan. The two functors $l_1, l_2$ are full-and-faithful. Every object of $\Delta$ is of the form $l_1(d)$ or $l_2(d)$.

Bubenik, dS, Scott

Example: dynamical system interleavings

Let $D$ be the category defined by the directed graph

\[ \bullet \rightarrow \bullet \]

Thus $D$ has one object and morphisms $\{0, 1, 2, 3, \ldots \}$.

- Functors $D \rightarrow \text{Top}$ are **discrete dynamical systems**.

Let $\Delta_n$ be the category with two objects $\bullet_1$ and $\bullet_2$ and morphisms

\[
\begin{align*}
\text{Mor}(\bullet_1, \bullet_1) &= \text{Mor}(\bullet_1, \bullet_1) = \{0, 1, 2, 3, \ldots \} \\
\text{Mor}(\bullet_1, \bullet_2) &= \text{Mor}(\bullet_2, \bullet_1) = \{n, n + 1, n + 2, n + 3, \ldots \}
\end{align*}
\]

with addition as composition.

- $\Delta_n$-interleavings are **shift-equivalences**.
Interleaving Metrics on Functor Categories

Categories with a flow (dS, Munch, Stefanou 2018)

Interleaving distance defined on categories with a coherent \([0, \infty)\)-action.

Examples

- Functor categories \(C^P\), equipped with a superlinear family \((\Omega_\epsilon)\) on \(P\).
- Poset \(S\) of subsets of a metric space \(X\); ‘thickening’ action on \(S\):

  \[ A \mapsto A^\epsilon = \{ x \in X \mid d(x, A) \leq \epsilon \} \]

Interleaving distance = Hausdorff distance.
Reeb graphs

- An R-space \((X, f)\) is a topological space \(X\) with a map \(f : X \to \mathbb{R}\).
- An R-space is a Reeb graph if each \(f^{-1}(t)\) is finite.

Reeb functor

- The Reeb functor converts a (constructible) R-space into a Reeb graph:

\[
(X, f) \mapsto ((X/\sim), \bar{f})
\]

where \(x \sim y\) iff \(x, y\) are in the same component of the same levelset of \(f\).
Reeb Graphs & Reeb Cosheaves

\( \mathbb{E}_0 \times [a_0, a_1] \quad \mathbb{E}_1 \times [a_1, a_2] \quad \mathbb{E}_2 \times [a_2, a_3] \quad \mathbb{E}_3 \times [a_3, a_4] \quad \mathbb{E}_4 \times [a_4, a_5] \)

\( V_0 \quad V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5 \)
Reeb Graphs & Reeb Cosheaves

\[ E_0 \times [a_0, a_1] \quad E_1 \times [a_1, a_2] \quad E_2 \times [a_2, a_3] \quad E_3 \times [a_3, a_4] \quad E_4 \times [a_4, a_5] \]

\[ V_0 \quad V_1 \quad V_2 \quad V_3 \quad V_4 \quad V_5 \]
Reeb cosheaves (dS, Munch, Patel 2016)

- Let $\text{Int}$ denote the poset of open intervals, $\subseteq$.
- A Reeb graph gives rise to a functor $F = \pi_0 f^{-1} : \text{Int} \to \text{Set}$ that is constructible and satisfies the cosheaf condition for unions of intervals.

$$F(I = \bigcup I_{\alpha}) = \text{colim} \left[ \bigsqcup_{\alpha, \beta} F(I_{\alpha} \cap I_{\beta}) \Rightarrow \bigsqcup_{\alpha} F(I_{\alpha}) \right]$$
Reeb Graphs & Reeb Cosheaves

Reeb cosheaves (dS, Munch, Patel 2016)

- Let $\text{Int}$ denote the poset of open intervals, $\subseteq$.
- A Reeb graph is the same thing as a functor $F = \pi_0 f^{-1} : \text{Int} \to \text{Set}$ that is constructible and satisfies the cosheaf condition for unions of intervals.

$$F(I = \bigcup I_\alpha) = \text{colim} \left[ \coprod_{\alpha, \beta} F(I_\alpha \cap I_\beta) \Rightarrow \coprod_\alpha F(I_\alpha) \right]$$
Reeb graphs & Reeb cosheaves

Reeb cosheaves (dS, Munch, Patel 2016)

- Let $\text{Int}$ denote the poset of open intervals, $\subseteq$.
- A Reeb graph is the same thing as a functor $F = \pi_0 f^{-1} : \text{Int} \to \text{Set}$ that is constructible and satisfies the cosheaf condition for unions of intervals.
Reeb functor (two versions)

1. The **Reeb functor** converts a (constructible) $\mathbb{R}$-space into a Reeb graph:

$$(X, f) \mapsto ((X/\sim), \bar{f})$$

where $x \sim y$ iff $x, y$ are in the same component of the same levelset of $f$.

Or

2. The **Reeb functor** converts a constructible $\mathbb{R}$-space into a Reeb cosheaf:

$$F(I) = \pi_0 f^{-1}(I)$$

$$G[I \subseteq J] = \pi_0 \left[ f^{-1}(I) \subseteq f^{-1}(J) \right]$$
Reeb Graphs & Reeb Cosheaves

Translation operators on \textbf{Int}

We define a 1-parameter semigroup \((\Omega_\epsilon)\) of functors \(\text{Int} \to \text{Int}\) by setting

\[
\Omega_\epsilon(I) = I^\epsilon = \text{“}\epsilon\text{-neighbourhood of } I\text{”}
\]

Reeb interleaving distance (dS, Munch, Patel 2016)

An \(\epsilon\text{-interleaving}\) between \(F, G\) is given by two families of maps

\[
\phi_I : F(I) \to G(I^\epsilon), \quad \psi_I : G(I) \to F(I^\epsilon)
\]

which are natural with respect to inclusions \(I \subseteq J\), and such that for all \(I\)

\[
\psi_{I^\epsilon} \circ \phi_I = F[I \subseteq I^{2\epsilon}], \quad \phi_{I^\epsilon} \circ \psi_I = G[I \subseteq I^{2\epsilon}].
\]
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An \(\epsilon\)-interleaving between \(F, G\) is given by two families of maps

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which are natural with respect to inclusions \(I \subseteq J\), and such that for all \(I\)

\[ \psi_I \epsilon \circ \phi_I = F[I \subseteq I^{2\epsilon}], \quad \phi_I \epsilon \circ \psi_I = G[I \subseteq I^{2\epsilon}] \]

Stability Theorem

If \(f, g : X \rightarrow \mathbb{R}\) with \(\|f - g\|_\infty \leq \epsilon\) then \(d_i(F, G) \leq \epsilon\).

Universal ReebMetric (Bauer, Landi, Mémoli 2018)

The \textit{universal metric} \(d_u(F, G)\) is the largest that satisfies the stability theorem.
Translation operators on $\mathbf{Int}$

We define a 1-parameter semigroup $(\Omega_\epsilon)$ of functors $\mathbf{Int} \to \mathbf{Int}$ by setting

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Translation operators on $\mathbf{Int}$

We define a 1-parameter semigroup $(\Omega_\epsilon)$ of functors $\mathbf{Int} \to \mathbf{Int}$ by setting

$$\Omega_\epsilon(I) = I^\epsilon = "\epsilon\text{-neighbourhood of } I"$$

Cosheaf Smoothing Theorem

If $F : \mathbf{Int} \to \mathbf{Set}$ is a (constructible) cosheaf, then so is $F\Omega_\epsilon : \mathbf{Int} \to \mathbf{Set}$. 
Translation operators on \textbf{Int}

We define a 1-parameter semigroup \((\Omega_\epsilon)\) of functors \textbf{Int} \to \textbf{Int} by setting

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Cosheaf Smoothing Theorem

If \(F : \textbf{Int} \to \textbf{Set}\) is a (constructible) cosheaf, then so is \(F\Omega_\epsilon : \textbf{Int} \to \textbf{Set}\).

Corollary: Reeb Smoothing

There is a 1-parameter semigroup of ‘smoothing’ operations on Reeb graphs.
Translation operators on **Int**

We define a 1-parameter semigroup \( (\Omega_\epsilon) \) of functors \( \text{Int} \to \text{Int} \) by setting

\[
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**Cosheaf Smoothing Theorem**

If \( F : \text{Int} \to \text{Set} \) is a (constructible) cosheaf, then so is \( F\Omega_\epsilon : \text{Int} \to \text{Set} \).

**Corollary: Reeb Smoothing**

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Cosheaf Smoothing Theorem

If \(F : \textbf{Int} \to \textbf{Set}\) is a (constructible) cosheaf, then so is \(F\Omega_\epsilon : \textbf{Int} \to \textbf{Set}\).

Corollary: Reeb Smoothing

There is a 1-parameter semigroup of ‘smoothing’ operations on Reeb graphs.
Reeb space operations

Reeb graphs

- An **R-space** $(X, f)$ is a topological space $X$ with a map $f : X \to \mathbb{R}$.
- An **R-space** is a **Reeb graph** if $X$ is a graph and each $f^{-1}(t)$ is finite.

Reeb functor

- The **Reeb functor** converts a (constructible) **R-space** into a Reeb graph:

$$ (X, f) \mapsto ((X/\sim), \bar{f}) $$

where $x \sim y$ iff $x, y$ are in the same component of the same levelset of $f$. 
### Reeb spaces

- A **B-space** \((X, f)\) is a topological space \(X\) with a map \(f : X \to B\).
- A **B-space** is a **Reeb B-space** if each \(f^{-1}(t)\) is finite.

### Reeb functor

- The **Reeb functor** converts a (constructible) **B-space** into a Reeb **B-space**:

  \[(X, f) \mapsto ((X/\sim), \overline{f})\]

  where \(x \sim y\) iff \(x, y\) are in the same component of the same **fiber** of \(f\).
Example: Universal Cover

Let $B$ be a (locally well-behaved) topological space. Then

$$\text{Path}(B, b_0) = \{ \text{paths } \gamma : [0, 1] \to B \text{ with } \gamma(0) = b_0 \}$$

is a $B$-space with respect to the evaluation map

$$e : \text{Path}(B, b_0) \to B; \gamma \mapsto \gamma(1).$$

Then

$$\text{Univ}(B) = \text{Reeb} \left[ \text{Path}(B, b_0), e \right]$$

is the universal cover of $B$. 
Let $X = (X, f)$ and $Y = (Y, g)$ be Reeb graphs.

- A **relator** for $X, Y$ is an $(\mathbb{R} \times \mathbb{R})$-space
  $$W \xrightarrow{F} \mathbb{R} \times \mathbb{R}$$
  such that
  $$\text{Reeb}[W, p_1 \circ F] \cong (X, f),$$
  $$\text{Reeb}[W, p_2 \circ F] \cong (Y, g).$$

- The **deviation** of a relator
  $$\text{dev}(W) = \sup_{w \in W} |p_1(F(w)) - p_2(F(w))|$$
  measures how far $F(W)$ deviates from the diagonal.

- The **universal distance** between $X, Y$ is defined
  $$d_u(X, Y) = \inf \{ \text{dev}(W) \mid W \text{ is a relator for } X, Y \}$$
Let $B$ be a topological semigroup with operation $\circ$.

**Reeb $B$-space convolutions**

The $\pi_0$-convolution of Reeb spaces

$$X \xrightarrow{f} B, \quad Y \xrightarrow{g} B$$

is defined to be

$$(X, f) \ast (Y, g) = \text{Reeb} \left[ X \times Y, f \circ g \right]$$

**Reeb graph convolutions**

$$(X, f) \ast (Y, g) = \text{Reeb} \left[ X \times Y, f + g \right]$$
Reeb space operations

Reeb graph convolutions

$$(X, f) \ast (Y, g) = \text{Reeb} \left[ X \times Y, f + g \right]$$

Examples

- The $\sigma$-smoothing of a Reeb graph $X = (X, f)$ is given by the formula
  $$X^\sigma = X \ast [-\sigma, \sigma].$$

- The intervals $[-\sigma, \sigma]$, for $\sigma \geq 0$, form a semigroup under $\ast$.

- More generally, the convolution of intervals is their Minkowski sum:
  $$[m_1 - \sigma_1, m_1 + \sigma_1] \ast [m_2 - \sigma_2, m_2 + \sigma_2] = [m - \sigma, m + \sigma]$$
  where $m = m_1 + m_2$ and $\sigma = \sigma_1 + \sigma_2$.

- The **merge-tree** and **split-tree** of $X$ are given by the formulas
  $$\text{Merge}(X) = X \ast [0, +\infty), \quad \text{Split}(X) = X \ast (-\infty, 0].$$

- Thus $X \ast [-R, R]$, when $R \gg 0$, combines the merge and split trees of $X$. 
Let $X, Y$ be Reeb graphs. Then

$$d_i(X^\sigma, Y^\sigma) \leq d_i(X, Y), \quad d_u(X^\sigma, Y^\sigma) \leq d_u(X, Y)$$

and

$$d_i(X, X^\sigma) \leq d_u(X, X^\sigma) \leq \sigma$$

for all $\sigma \geq 0$. 

Analogy: Gaussian kernel smoothing

Is there a theory of $\pi_0$ signal processing?
Metric properties of Reeb graph smoothing

Let $X, Y$ be Reeb graphs. Then

$$d_i(X^\sigma, Y^\sigma) \leq d_i(X, Y), \quad d_u(X^\sigma, Y^\sigma) \leq d_u(X, Y)$$

and

$$d_i(X, X^\sigma) \leq d_u(X, X^\sigma) \leq \sigma$$

for all $\sigma \geq 0$.

Analogy: Gaussian kernel smoothing

Is there a theory of $\pi_0$ signal processing?
## Acknowledgements

### Collaborators

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