Supplying bells and whistles in symmetric monoidal categories

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Every object is equipped with an algebraic structure, compatible with $\otimes$. 

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Table 13: Summary of mixed-integer and their graphical languages.
Example: **Set** supplies comonoids

In \((\textbf{Set}, \times)\), we have commutative comonoids:

- **terminal:** $\varepsilon$

- **diagonal:** $\delta$

Moreover, morphisms are comonoid homomorphisms.

\[
\begin{align*}
\varepsilon_X \otimes Y & = \delta_X \\
X \otimes \varepsilon_Y & = \delta_Y
\end{align*}
\]

(In fact, we can recover products from this perspective.)
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- **terminal:**
  \[ \epsilon \]

- **diagonal:**
  \[ \delta \]

There are compatible with ×:

\[
\epsilon_{X \otimes Y} = \epsilon_X \oplus \epsilon_Y \\
\delta_{X \otimes Y} = \delta_X \oplus \delta_Y
\]

Moreover, morphisms are comonoid homomorphisms.

\[
f(\epsilon_X) = \epsilon_Y \\
f(\delta_X) = \delta_Y
\]

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There are compatible with \(\times\):

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\epsilon_X \otimes Y = \epsilon_Y \quad \epsilon_X \epsilon_Y = \delta_X \delta_Y
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Moreover, morphisms are comonoid homomorphisms.

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f \epsilon_Y = \epsilon_X \quad f \delta_Y = \delta_X
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Moreover, morphisms are comonoid homomorphisms.

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\begin{align*}
\delta_X \otimes \epsilon_Y & = \epsilon_X \\
\epsilon_X \otimes \delta_Y & = \delta_X
\end{align*}
\]

(In fact, we can recover products from this perspective.)
Outline

I. Props and theories
II. Definition
III. Examples
IV. An equivalent definition
V. Some fun facts
A **prop** is a symmetric strict monoidal category where the monoid of objects is \((\mathbb{N}, +)\).

**Examples:**

- **Bij**: the prop of bijections
- **Cob**: the prop of unoriented 1-cobordisms
- **FinSet**: the prop of functions
- **Cospan**: the prop of cospans of functions
**Key idea:** Props describe algebraic (monoidal) theories

For example, \( \text{FinSet} \) is the theory of commutative monoids.

\[
\text{SymMonCat}_{\text{strong}}(\text{FinSet}, \mathcal{C}) \cong \text{CommMon}(\mathcal{C})
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Props

Key idea: Props describe algebraic (monoidal) theories

For example, FinSet is the theory of commutative monoids.

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\text{SymMonCat}_{\text{strong}}(\text{FinSet}, \mathcal{C}) \cong \text{CommMon}(\mathcal{C})
\]

Bij: the prop for objects
Cob: the prop for self-duality
FinSet: the prop for commutative monoids
Cospan: the prop for special commutative frobenius monoids
Definition of Supply

Let $\mathbb{P}$ be a prop, and $C$ be a SMC.

A supply $s$ of $\mathbb{P}$ in $C$ is

- for each $c \in C$, a strong SMF $s_c : \mathbb{P} \to C$
Definition of Supply

Let $\mathbb{P}$ be a prop, and $\mathcal{C}$ be a SMC.

A **supply** $s$ of $\mathbb{P}$ in $\mathcal{C}$ is

- for each $c \in \mathcal{C}$, a strong SMF $s_c: \mathbb{P} \to \mathcal{C}$

such that for all $m, n \in \mathbb{N}$, $c, d \in \mathcal{C}$, $\mu: m \to n \in \mathbb{P}$

(i) $s_c(m) = c^\otimes m$

(ii) the strongators are the unique coherence maps

\[
c^\otimes m \otimes c^\otimes n \to c^\otimes (m+n)
\]

(iii)

\[
\begin{array}{ccc}
c^\otimes m \otimes d^\otimes m & \xrightarrow{s_c(\mu) \otimes s_d(\mu)} & c^\otimes n \otimes d^\otimes n \\
\sigma & & \sigma \\
(c \otimes d)^\otimes m & \xrightarrow{s_{c \otimes d}(\mu)} & (c \otimes d)^\otimes n
\end{array}
\]

\[
\begin{array}{ccc}
I & = & I \\
\sigma & & \sigma \\
I^\otimes m & \xrightarrow{s_I(\mu)} & I^\otimes n
\end{array}
\]
A morphism \( f: c \to d \) is an \( s \)-homomorphism if for all \( \mu : m \to n \) in \( \mathbb{P} \):

\[
\begin{align*}
  c \otimes m & \xrightarrow{s_c(\mu)} c \otimes n \\
  f \otimes m & \xrightarrow{s_d(\mu)} f \otimes n \\
  d \otimes m & \xrightarrow{s_d(\mu)} d \otimes n
\end{align*}
\]
Examples

- Every symmetric monoidal category uniquely supplies $\text{Bij}$. Moreover, this unique supply is homomorphic.

- A category $C$ supplies $\text{Cob}$ iff it is self-dual compact closed. For example, $(\text{Mat}, \otimes)$ supplies $\text{Cob}$.

- Supply homomorphisms are orthogonal matrices.

- A category supplies $\text{Cospan}$ iff it is hypergraph. For example, $(\text{Rel}, \times)$ supplies $\text{Cospan}$.

- Supply homomorphisms are bijections.

  (The homomorphisms of $\text{FinSet}^{\text{op}}$ are functions.)

- A category homomorphically supplies $\text{FinSet}^{\text{op}}$ iff the monoidal product is a categorical product.
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A key theorem

**Theorem**
Let $s$ be a supply of $\mathbb{P}$ in $\mathcal{C}$. Then all coherence maps of $\mathcal{C}$ are $s$-homomorphisms.

For example,

\[
\begin{align*}
((a \otimes b) \otimes c)^{\otimes m} & \xrightarrow{s_{(a \otimes b) \otimes c}(\mu)} (a \otimes b) \otimes c)^{\otimes n} \\
(a \otimes (b \otimes c))^{\otimes m} & \xrightarrow{s_{a \otimes (b \otimes c)}(\mu)} (a \otimes (b \otimes c))^{\otimes n}
\end{align*}
\]
Define $\text{inc}: C_0 \rightarrow C$ to be the smallest subcategory of $C$ containing all the coherence maps.

**Corollary**

The following are equivalent:

(a) A supply $s$ of $P$ in $C$.

(b) A strong SMF $s: P \rightarrow \text{SymMonCat}_{\text{strong}}(C_0, C)$ such that

(i) $m \mapsto \text{inc}^\otimes m$

(ii) strongators are coherence maps
Preservation of supply

Let $s, t$ respectively supply $\mathbb{P}$ in $\mathcal{C}, \mathcal{D}$.

Morphisms of categories supplying $\mathbb{P}$ are defined as follows.

A strong monoidal functor $(F, \varphi): \mathcal{C} \to \mathcal{D}$ preserves supply iff for all $\mu$ in $\mathbb{P}$, $c \in \mathcal{C}$:

\[
\begin{align*}
F(c) \otimes m & \xrightarrow{t_{F(c)}(\mu)} F(c) \otimes n \\
\varphi \downarrow \cong & \quad \quad \quad \quad \quad \quad \quad \downarrow \cong \\
F(c \otimes m) & \xrightarrow{F(s_{c}(\mu))} F(c \otimes n)
\end{align*}
\]
Some fun facts

- If $A: \mathcal{P} \to \mathcal{Q}$ is a morphism of props, and $s$ supplies $\mathcal{Q}$ in $\mathcal{C}$, then $A; s$ supplies $\mathcal{P}$ in $\mathcal{C}$.

- If $F: \mathcal{C} \to \mathcal{D}$ is a symmetric, essentially surjective strict monoidal functor, then if $\mathcal{C}$ supplies $\mathcal{P}$, so does $\mathcal{D}$.

- If $\mathcal{C}$ supplies $\mathcal{P}$, so does its strictification $\tilde{\mathcal{C}}$, and the equivalence $\mathcal{C} \cong \tilde{\mathcal{C}}$ preserves supply.

- If $F$ preserves supply, it sends supply homomorphisms to supply homomorphisms.

- More in the paper...