Formal composition of hybrid systems

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Composition of Robotic Behaviors
Hybrid systems

A hybrid system $H$ consists of

- a directed graph $G = (V, E, s, t)$;
- for each mode $v \in V$,
  - an ambient smooth system $(M_v, X_v)$
  - an active set $I_v \subset M_v$
  - a flow set $F_v \subset I_v$
- for each reset $e \in E$, a guard set $Z_e \subset I_{\bar{s}(e)}$ and an associated reset map $r_e : Z_e \rightarrow I_{t(e)}$.

**Morphisms**: hybrid semiconjugacies

- “execution-preserving maps”

Cf. Lerman. “A category of hybrid systems.”
Templates and anchors

A template-anchor pair is a span $T \xleftarrow{p} S \xrightarrow{i} A$ such that

- $p$ is a hybrid subdivision;
- $i$ is a hybrid embedding;
- $i(S)$ is an isolated invariant set in $A$.

Subdivisions of hybrid systems

A **hybrid subdivision** a hybrid submersion $p: S \rightarrow H$ such that for every hybrid time execution $\chi: \tau \rightarrow H$, there exists a pullback square

\[
\begin{array}{ccc}
\sigma & \xrightarrow{\tilde{\chi}} & S \\
\downarrow{\xi} & & \downarrow{p} \\
\tau & \xrightarrow{\chi} & H
\end{array}
\]

such that $\xi$ is a **refinement** of hybrid time trajectories.

Anchoring a limit cycle in a vertical hopper

Hierarchical composition

Theorem (CGKS). Template-anchor pairs are weakly associatively composable.

Goal: define “funnel-like” sequentially composable hybrid systems
A “navigate-to-goal” funnel

Theorem 3. The piecewise continuously differentiable “move-to-projected-goal” law in (11) leaves the robot’s free space $\mathcal{F}$ (1) positively invariant; and if Assumption 2 holds, then its unique continuously differentiable flow, starting at almost¹ any configuration $x \in \mathcal{F}$, asymptotically reaches the goal location $x^*$, while strictly decreasing the squared Euclidean distance to the goal, $||x - x^*||^2$, along the way.

How to define "funnel-like" systems?

- **Problem:** the naive measure-theoretic and topologically notions of "almost all" are incompatible with fully general sequential composition

- **Example:**

  \[ H = \begin{array}{c} \bullet \\ \frac{d}{dx} \\ \downarrow \\ 0 \rightarrow 0 \\ \end{array} \begin{array}{c} \bullet \\ \{0\},0 \end{array} \quad \begin{array}{c} \bullet \\ \{1\},0 \end{array} \]

  \[ K = \begin{array}{c} \bullet \\ \frac{d}{dx} \\ \uparrow \\ 1 \rightarrow 1 \\ \end{array} \begin{array}{c} \bullet \\ \{0\},0 \end{array} \quad \begin{array}{c} \bullet \\ \{1\},0 \end{array} \]

- Is there a notion of "generalized execution" compatible with sequential composition?
Directed systems

A **directed hybrid system** $H: H_i \rightsquigarrow H_f$ is a tuple $(H, \eta_i, \eta_f)$ consisting of

- a metric hybrid system $H$,
- embeddings $\eta_i: H_i \to H$ and
- a hybrid embedding $\eta_f: H_f \to H$ such that each component $(\eta_f)_v$ is a diffeomorphism, and $G(H_f)$ is a sink in $G(H)$

such that for all $\varepsilon, T > 0$ and $x \in H$, there exists an $(\varepsilon, T)$-chain from $x$ to some $y \in H_f$.

A double category of hybrid systems

Reactive Navigation in Non-Convex Environments

Further Directions

- **Internal languages of double categories**

- **Connection to LTL/FRP**

- **Compatibility with coupled parallel compositions**

- **Triple categories**

- **Hybrid Conley theory**
Thanks!

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