Right adjoints to operadic restriction functors

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P. Hackney¹  G.C. Drummond-Cole²

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¹Department of Mathematics
University of Louisiana at Lafayette
Lafayette, Louisiana, USA

²Center for Geometry and Physics
Institute for Basic Science
Pohang, Republic of Korea
Lawvere’s Question (1963)

- Given an algebraic functor \( f : T \to T' \) between algebraic theories, there is an adjoint pair

\[
f_! : \text{Fun}^{\times}(T, \text{Set}) \rightleftarrows \text{Fun}^{\times}(T', \text{Set}) : f^*
\]

When does \( f^* \) admit a right adjoint \( f_* \)?

There is an unexpected right adjoint (Templeton 2003)

\[ \text{Opd} \xleftarrow{\phi} \text{Cyc} \]

which may be described at an operad \( P \) by

\[
(\phi_* P)(n) = \prod_{i=0}^{n} P(n) = \text{hom}_{\Sigma_n}(\Sigma_{n+1}, P(n)).
\]

When do such operadic right Kan extensions exist?
If $P$ is an operad, let $|P|$ denote the underlying monoid.

**Monoidal extension**
An operad map $P \to Q$ is a *monoidal extension* just when

$$P \circ_{|P|} |Q| \to Q \circ_{|Q|} |Q| \cong Q$$

is an isomorphism.

**Theorem (H & Drummond-Cole 2019)**
Let $\phi : P \to Q$ be a map between (monochrome) operads. The restriction functor

$$\phi^* : \text{Alg}(Q) \to \text{Alg}(P)$$

admits a right adjoint if and only if $\phi$ is a monoidal extension.
Monoidal extension
An operad map $P \to Q$ is a monoidal extension just when

$$P \circ |P| |Q| \to Q \circ |Q| |Q| \cong Q$$

is an isomorphism.

Isomorphism of underlying monoids
If $|P| \to |Q|$ is an isomorphism, then $P \to Q$ is a monoidal extension if and only if it is an isomorphism.

Standard non-example
The inclusion functor from commutative monoids to associative monoids does not admit a right adjoint.
Let \( \mathbb{D} \subseteq \mathbb{R}^2 \) be the closed unit disk.

\[
D_2(n) \subseteq D^{fr}_2(n) \subseteq \left\{ f : \bigsqcup_{k=1}^{n} \mathbb{D} \to \mathbb{D} \right\}
\]

- Each \( f_k : \mathbb{D} \to \mathbb{D} \) is an embedding.
- \( f_k(\mathbb{D}) \cap f_j(\mathbb{D}) \subseteq f_k(\partial(\mathbb{D})) \) for \( k \neq j \)
- \( D_2(n) \subseteq (\mathbb{R}_{>0} \times \mathbb{R}^2)^n \): \( f_k \) has the form \( f_k(x) = ax + b \)
- \( D^{fr}_2(n) \subseteq (SO(2) \times \mathbb{R}_{>0} \times \mathbb{R}^2)^n \) rotation–dilation–translation

**Observation**

The inclusion \( D_2 \to D^{fr}_2 \) is a monoidal extension.
The inclusion $D_2 \to D_2^{fr}$ is a monoidal extension.
If $X$ is a $D_2$-algebra, then the free loop space $LX = Map(S^1, X)$ realizes the right adjoint.

- $D^\text{fr}_2(n) \times (LX)^\times n \rightarrow LX = Map(SO(2), X)$
- The adjoint to the level $n$ action takes the form:

$$SO(2) \times D^\text{fr}_2(n) \times (LX)^\times n \rightarrow D^\text{fr}_2(n) \times (LX)^\times n \rightarrow D_2(n) \times SO(2)^\times n \times (LX)^\times n \rightarrow D_2(n) \times X^\times n \rightarrow X$$
Bicategory of colored collections

Objects: Sets named $A$, $B$, $C$, etc.

$(A, B)$ Collections:

- $S_A = \{ \sigma : a = (a_1, \ldots, a_n) \rightarrow (a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = a\sigma \}$
- $(A, B)$ collection $Y$: functor $S_A \times B \rightarrow Set$

Diagram:

```
  \begin{tikzpicture}
    \node (a1) at (0,1) {$a_1$};
    \node (a2) at (1,1) {$a_2$};
    \node (a3) at (2,1) {$a_3$};
    \node (y) at (1,0) {$y$};
    \node (b) at (1,-1) {$b$};
    \draw (a1) -- (y);
    \draw (a2) -- (y);
    \draw (a3) -- (y);
    \end{tikzpicture}
```
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Horizontal Composition

- $\circ : (B, C)\text{-Coll} \times (A, B)\text{-Coll} \to (A, C)\text{-Coll}$
- Elements of $X \circ Y$
Adjoints at level of collections

• Let $Y$ be an $(A, B)$-collection. Then

$$(-) \circ Y : (B, C)\text{-Coll} \rightarrow (A, C)\text{-Coll}$$

has a right adjoint $[Y, -]$ (Kelly, 1972)

• If $X$ is a $(B, C)$-collection, then

$$X \circ (-) : (A, B)\text{-Coll} \rightarrow (A, C)\text{-Coll}$$

only has a right adjoint, denoted by $\langle X, - \rangle$, when $X$ is concentrated in arity one

$$\langle X, Z \rangle(a; b) = \prod_{c \in C} \text{hom}(X(b; c), Z(a; c))$$

$$\text{hom}_{\text{Pl}}(1_{|\lambda|}, -) \leq \langle 1_{|\lambda|}, - \rangle$$
Colored operads

- An $A$-colored operad $P$ is a monoid in the monoidal category of $(A, A)$-collections:
  
  $\mu : P \circ P \rightarrow P \quad \eta : 1_A \rightarrow P$

- Colored operads concentrated in arity one are categories.

- A map of $A$-colored operads $\phi : P \rightarrow Q$ is a map of monoids.
Colored operads

• An $A$-colored operad $P$ is a monoid in the monoidal category of $(A, A)$-collections:

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• Colored operads concentrated in arity one are categories.
• A map of $A$-colored operads $\phi : P \rightarrow Q$ is a map of monoids.
• If $Q$ had a different color set $B$, then we also have maps $\phi : (A, P) \rightarrow (B, Q)$ lying over each function $A \rightarrow B$. 
Actions

• If $\phi : P \to Q$ is a map of $A$-colored operads, then $Q$ is a $P$-$Q$ bimodule. $P \circ Q \to Q$

• An algebra over $P$ is nothing but an $(\emptyset, A)$-collection $M$ along with a left $P$-action: $P \circ M \to M$
Definition
A map $\phi : P \to Q$ of $A$-colored operads is a categorical extension when

$$P \circ_{|P|} |Q| \to Q$$

is an isomorphism of $(A, A)$-collections.

Theorem (H & Drummond-Cole 2019)
Let $\phi : P \to Q$ be a map of $A$-colored operads. The restriction functor

$$\phi^* : \text{Alg}(Q) \to \text{Alg}(P)$$

admits a right adjoint $\phi_*$ if and only if $\phi$ is a categorical extension.
Example (Operads and Cyclic Operads)

- $R$ and $T$ are $\mathbb{N}$-colored operads
- Operations in $T$ are trees with total orderings on
  - set of vertices
  - vertex neighborhoods
  - boundaries
- $R \subseteq T$ consists of rooted trees: root of tree is first edge of boundary, root of vertex is first edge in the vertex neighborhood, and these are compatible
- $R(n; n) = \Sigma_{n-1}$ and $T(n; n) = \Sigma_n$
- $\text{Alg}(R) = \text{Opd}$ and $\text{Alg}(T) = \text{Cyc}$
- $R \subseteq T$ is a categorical extension
Example (Operads and Cyclic Operads)

Elements of $T(3, 4, 1; 4)$ and $R(3, 4, 1; 4)$
• $P \subseteq R$ are the \textit{planar} rooted trees.
• $P(n; n) = \ast$ and $R(n; n) = \Sigma_{n-1}$
• $\text{Alg}(P) = \text{nsOpd}$ and $\text{Alg}(R) = \text{Opd}$
• \textit{Not} a categorical extension:
Consider the composite $F : A \to \text{Alg}(Q)$

\[
A \xleftarrow{} (\emptyset, A)\text{-Coll} \xrightarrow{\text{free}} \text{Alg}(Q)
\]

Suppose that $\phi^* : \text{Alg}(Q) \to \text{Alg}(P)$ is a left adjoint.

If $\underline{a} = (a_1, \ldots, a_n)$ is any tuple of elements of $A$, then

\[
\bigwedge_{i=1}^{n} \phi^* F(a_i) \to \phi^* \bigwedge_{i=1}^{n} F(a_i)
\]

is an isomorphism of $P$-algebras, hence of $(\emptyset, A)$-collections.
Necessity (Idea)

\[
\prod_{i=1}^{n} \phi^* F(a_i) \equiv \left( P \circ \prod_{i=1}^{n} Q \circ a_i \right) / \sim
\]

\[
\phi^* \prod_{i=1}^{n} F(a_i) \equiv \left( Q \circ \prod_{i=1}^{n} Q \circ a_i \right) / \sim
\]

Careful analysis:

\[
\left( P \circ_{|P|} |Q| \right) \left( \frac{a}{a} \right) \subseteq \left( \prod_{i=1}^{n} \phi^* F(a_i) \right) \left( \emptyset \right)
\]

\[
\left( \prod_{i=1}^{n} F(a_i) \right) \left( \emptyset \right) \subseteq \left( \phi^* \prod_{i=1}^{n} F(a_i) \right) \left( \emptyset \right)
\]