Enriched Lawvere Theories for Operational Semantics

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Introduction

How to integrate *syntax and semantics*?

A Lawvere theory T, and a category of models $[\mathrm{T}, \mathrm{C}]$.

How to incorporate **computation**?

Enriched categories:

object	type	(carrier)
morphism	term	(operations)
2-morphism	**rewrite**	$(\Rightarrow, \vdash, \leq, \ldots)$

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Operational semantics

algebraic theories : *denotational* semantics

$$(ab)c = a(bc)$$

enriched theories : *operational* semantics

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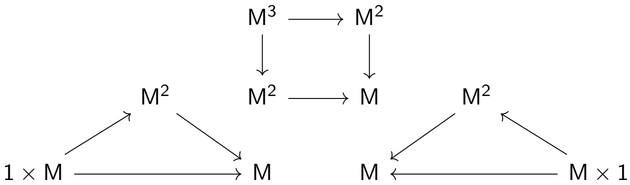
Lawvere theories

Th(Mon)

type		M		monoid
operations	$m\colon$	M^2	$\to M$	multiplication
	$e\colon$	1	$\to M$	identity

equations



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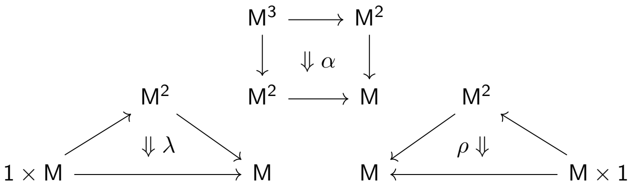
Enriched theories

Th(PsMon)

type	M		pseudomonoid
operations	$\otimes :$	$M^2 \to M$	multiplication
	$I :$	$1 \to M$	identity

rewrites



equations pentagon, triangle identities

Enriched categories

Let V be monoidal. A V-*enriched category* has hom-objects in V; composition and identity are morphisms in V, as are the components of a V-functor and a V-natural transformation:

V-**category**	$\mathrm{C}(a,b)$	$\in \mathrm{V}$
V-**functor**	$F_{ab}\colon \mathrm{C}(a,b) \to \mathrm{D}(F(a),F(b))$	$\in \mathrm{V}$
V-**transformation**	$\varphi_a\colon 1_{\mathrm{V}} \to \mathrm{D}(F(a),G(a))$	$\in \mathrm{V}.$

These form the 2-category VCat.

Our enriching category

Let V be a cartesian closed category:

$$\mathsf{V}(a \times b, c) \cong \mathsf{V}(a, [b, c]).$$

Then $\underline{\mathsf{V}} \in \mathsf{VCat}$.

Let $\mathsf{V} \in \mathsf{CCC}_{fc(1)}$, meaning assume and choose:

$$n_{\mathsf{V}} := \sum_{n} 1_{\mathsf{V}}.$$

$$\text{Let} \quad \mathsf{N}_{\mathsf{V}} := \{n_{\mathsf{V}} | n \in \mathsf{N}\} \subset_{full} \mathsf{V}$$

$$\text{and} \quad \mathsf{A}_{\mathsf{V}} := \underline{\mathsf{N}}_{\mathsf{V}}^{\mathrm{op}} \quad \text{– our “arities”}.$$

Enriched products

The V-**product** of $(a_i) \in \mathsf{C}$ is an object $\prod_i a_i \in \mathsf{C}$ equipped with a V-natural isomorphism

$$\mathsf{C}(-, \prod_i a_i) \cong \prod_i \mathsf{C}(-, a_i).$$

A V-functor $F \colon \mathsf{C} \to \mathsf{D}$ **preserves** V-products if the "projections" induce a V-natural isomorphism:

$$\mathsf{D}(-, F(\prod_i a_i)) \cong \prod_i \mathsf{D}(-, F(a_i)).$$

Let VCat_{fp} be the 2-category of V-categories with finite V-products and V-functors preserving them.

Enriched Lawvere theories

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Definition

A V-**theory** is a V-category $\mathsf{T} \in \mathsf{VCat}_{fp}$ whose objects are finite V-products of a distinguished object.

A morphism of V-theories is a V-functor $F\colon \mathsf{T} \to \mathsf{T}' \in \mathsf{VCat}_{fp}$. These and V-natural transformations form the 2-category of V-theories, VLaw.

Enriched models

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Definition

A **context** is a V-category $C \in VCat_{fp}$.
A **model** of T is a V-functor

$$\mu\colon T \to C \in VCat_{fp}.$$

The category of models is $Mod(T, C) := VCat_{fp}(T, C)$.

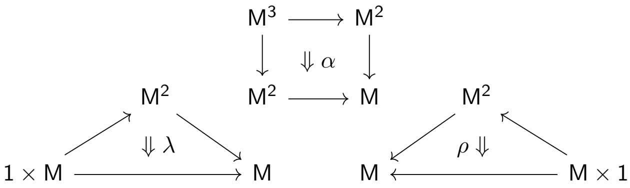
Example: monoidal categories

Let $V = \mathrm{Cat}$.

Th(PsMon)

type		M			pseudomonoid
operations	$\otimes$:	M^2	$\to$	M	multiplication
	I:	1	$\to$	M	identity

rewrites



equations pentagon, triangle identities

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Example: cartesian object

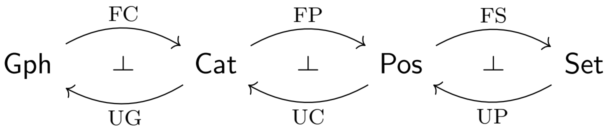
Let $V = \mathsf{Cat}$.

$$\mathrm{Th(Cart)}$$

type		X	cartesian object
operations	$m:$	$X^2 \to X$	product
	$e:$	$1 \to X$	terminal element
rewrites	$\triangle:$	$\mathrm{id}_X \Longrightarrow m \circ \Delta_X$	unit of $m \vdash \Delta_X$
	$\pi:$	$\Delta_X \circ m \Longrightarrow \mathrm{id}_{X^2}$	counit of $m \vdash \Delta_X$
	$\top:$	$\mathrm{id}_X \Longrightarrow e \circ !_X$	unit of $e \vdash !_X$
	$\epsilon:$	$!_X \circ e \Longrightarrow \mathrm{id}_1$	counit of $e \vdash !_X$
equations			triangle identities

Change of method

Idea: V is a *method of reasoning* for a theory.
There is a "spectrum" of methods of reasoning:



FC_* maps small-step to big-step operational semantics.
FP_* maps big-step to full-step operational semantics.
FS_* maps full-step to denotational semantics.

(Replace Gph with sSet_1; FC with realization.)

Change of base

Let $F\colon \mathsf{V} \to \mathsf{W}$ preserve finite products, and $\mathsf{C} \in \mathsf{VCat}$.

Then F induces a **change of base**:

$$F_*(\mathsf{C})(a, b) := F(\mathsf{C}(a, b)).$$

This gives a 2-functor

$$F_*\colon \mathsf{VCat} \to \mathsf{WCat}.$$

Enrichment provides a method of reasoning about theories, so change of base should *preserve* theories to be a *change of method*.

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Preservation of theories

Theorem

Let $F\colon \mathsf{V} \to \mathsf{W} \in \mathsf{CCC}_{fc(1)}$.
Then F *is a* **change of method***:*

F_* *preserves theories. For every* V*-theory* $\tau_{\mathsf{V}}\colon \mathsf{A}_{\mathsf{V}} \to \mathsf{T}$,

$$\tau_{\mathsf{W}} := \mathsf{A}_{\mathsf{W}} \xrightarrow{\sim} F_*(\mathsf{A}_{\mathsf{V}}) \xrightarrow{F_*(\tau_{\mathsf{V}})} F_*(\mathsf{T}) \quad \text{is a } \mathsf{W}\text{-theory.}$$

F_* *preserves models. For every model* $\mu\colon \mathsf{T} \to \mathsf{C}$,

$$F_*(\mu)\colon F_*(\mathsf{T}) \to F_*(\mathsf{C}) \quad \text{is a model of } (F_*(\mathsf{T}), \tau_{\mathsf{W}}).$$

The theory of SKI

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Th(SKI)

type t

terms

$$S : 1 \to t$$
$$K : 1 \to t$$
$$I : 1 \to t$$
$$(- \; -) : t^2 \to t$$

rewrites

$$\sigma : (((S\ a)\ b)\ c) \Rightarrow ((a\ c)\ (b\ c))$$
$$\kappa : ((K\ a)\ b) \Rightarrow a$$
$$\iota : (I\ a) \Rightarrow a$$

A model of Th(SKI)

A Gph-product preserving Gph-functor $\mu\colon \mathrm{Th(SKI)} \to \mathrm{Gph}$ yields a graph $\mu(t)$ of SKI-terms:

$$1 \cong \mu(1) \xrightarrow{\mu(S)} \mu(t) \xleftarrow{\mu((-\ -))} \mu(t^2) \cong \mu(t)^2.$$

The rewrites are transferred by the enrichment of μ:

$$\mu_{1,t}\colon \mathrm{Th(SKI)}(1,t) \to \mathrm{Gph}(1,\mu(t)).$$

The free model of SKI

The syntax and semantics of the SKI combinator calculus are given by the free model

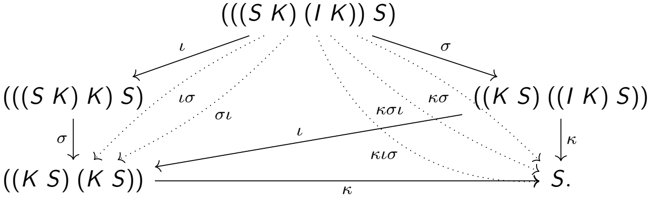
$$\mu^{\mathsf{Gph}}_{\mathsf{SKI}} := \mathrm{Th}(\mathsf{SKI})(1,-)\colon \mathrm{Th}(\mathsf{SKI}) \to \mathsf{Gph}.$$

The graph $\mu^{\mathsf{Gph}}_{\mathsf{SKI}}(t)$ is the *transition system* which represents the **small-step operational semantics** of the SKI-calculus:

$$(\mu(a) \to \mu(b) \in \mu^{\mathsf{Gph}}_{SKI}(t)) \iff (a \Rightarrow b \in \mathrm{Th}(\mathsf{SKI})(1,t)).$$

Change of method

Realization preserves products, hence gives a change of method from *small-step* to *big-step* operational semantics:



FP: Cat $\to$ Pos gives *full-step* (Hasse diagram), and
FS: Pos $\to$ Set gives *denotational* semantics, collapsing the connected component to a point.

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Enriched theories give a way to unify the structure and behavior of formal languages.

Enriching in category-like structures reifies operational semantics by incorporating rewrites between terms.

Cartesian functors between enriching categories induce change-of-method functors between categories of models.

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