THE BIG PICTURE

Each branch of mathematics is about some category:

**ALGEBRA**

- group theory - groups & group homomorphisms
  form the category $\text{Grp}$
- ring theory - rings & ring homs: the category $\text{Ring}$
- linear algebra - vector spaces & linear maps: the category $\text{Vect}$

**ANALYSIS**

- measure space & measurable functions: the category $\text{Meas}$

**TOPOLOGY**

- (topological) spaces & (continuous) maps: the category $\text{Top}$
- pointed spaces & pointed maps: the category $\text{Top}_*$

These categories are part of a single big picture, because there are things going between categories: functors!

Eg.

$$\pi_1 : \text{Top}_* \rightarrow \text{Grp}$$

HW - 58.2 no proofs needed
4,5 proofs
Classify capital letters according to
1) homeomorphism type (no proofs)
2) homotopy type
In any category, it's important to try to tell when two objects are isomorphic. (In Top, we say "homeomorphic.") Alas, it's an endless task to classify spaces and see when they're homeomorphic. This is sad, compared to, say, linear algebra: in Vect, two vector spaces V,W are isomorphic if there are linear maps \( f: V \to W \) and \( g: W \to V \) which are inverses; V and W are isomorphic if and only if they have the same dimension.

For Top, there's no simple test for when things are homeomorphic. But there are easy tests to show they aren't. This is where functors come in.

**Defn:** A category \( C \) consists of

- a collection of objects; if \( x \) is an object of \( C \) we write \( x \in C \)
- given two objects \( x, y \in C \), a set of morphisms from \( x \) to \( y \); if \( f \) is a morphism from \( x \) to \( y \), we write \( f: x \to y \) or \( x \xrightarrow{f} y \) (even though \( f \) is not necessarily a function and \( x, y \) aren't necessarily sets)
- given morphisms \( f: x \to y \) and \( g: y \to z \) there's a unique composite morphism \( g \circ f: x \to z \)
- given an object \( x \in C \), there's a unique identity morphism \( 1_x: x \to x \). (Sometimes we get lazy and write just \( 1 \).)

- The associative law holds for composition: given \( f: W \to X, g: X \to Y, h: Y \to Z \)
  \[ (h \circ g) \circ f = h \circ (g \circ f) \]
- The left and right unit laws hold: given \( f: x \to y \)
  \[ 1_y \circ f = f \circ 1_x \]
For example:

Top is a category
Top* is a category
Grp is a category
Diff (manifolds + smooth maps) is a category

Defn: Given categories $C, D$, a functor $F: C \to D$ is

- a "function" sending each object $x \in C$ to an object $F(x) \in D$
- for each $x, y \in C$ a function sending every morphism $f: x \to y$ to a morphism $F(f): F(x) \to F(y)$

such that

- composition is preserved: given

  $f: x \to y$ and $g: y \to z$

  we get

  $F(g \circ f) = F(g) \circ F(f)$

- identities are preserved:

  $F(1_x) = 1_{F(x)}$

For example:

$\Pi_1 : \text{Top}^\ast \to \text{Grp}$