So, assume we have $0 = t_0 < t_1 < \ldots < t_n = 1$ s.t. $Y(t_{i-1}, t_i)$ is contained in either $U$ or $V$, with $Y(t_i)$ in $U \cap V$.

Since $U \cap V$ is path connected, we can choose a path $\alpha_i$ from $Y(t_i)$ to $x_0$, lying in $U \cap V$. Now define paths $\delta_i$ to be the composites

$$[0, 1] \rightarrow [t_{i-1}, t_i] \xrightarrow{Y(t_{i-1}, t_i)} X$$

the unique linear 1-1 onto map.

Now we have

$$[Y] = [\delta_1] \ast [\alpha_1] \ast [\alpha_1]^{-1} \ast [\delta_2] \ast [\alpha_2] \ast [\alpha_2]^{-1} \ast \ldots \ast [\alpha_{n-1}] \ast [\delta_n]$$

each a loop in either $U$ or $V$.

Since $\delta_i$ are paths in $U$ or $V$ and $\alpha_i$ are paths in $U \cap V$.

Thus

$$[Y] = [Y_1] \ast \ldots \ast [Y_n]$$

where $Y_i$ are loops in $U$ or $V$. 

$\square$
Cor. 59.2: Under the same assumptions, if $U$ & $V$ are simply connected, then $X$ is simply connected.

Pf. Since $U$ & $V$ are simply connected, $\pi_1(U, x_0) \& \pi_1(V, x_0)$ are trivial groups, so $\text{im } \pi_1(j_1) \cup \text{im } \pi_1(j_2)$ is the trivial subgroup of $\pi_1(X, x_0).$ Baby S-v-K says $\pi_1(X, x_0)$ is generated by this, so $\pi_1(X, x_0)$ is trivial and $X$ is simply connected.

Thm 59.3: If $n \geq 2,$ the $n$-sphere is simply-connected, i.e. $\pi_1(S^n, *) \cong 1.$

Proof Sketch: Use Baby S-v-K with

- $P_+ = (0, 0, \ldots, 1) \in \mathbb{R}^{n+1}$
- $P_- = (0, 0, \ldots, -1) \in \mathbb{R}^{n+1}$

Let $U = S^n - \{P_+\} \cong \mathbb{R}^n$ (homeomorphic)

$V = S^n - \{P_-\} \cong \mathbb{R}^n$

$U \cup V = S^n - \{P_+, P_-\} \cong S^n \times (-1, 1)$

(Note $U, V$ simply connected and $U \cup V$ path connected if $n \geq 2; \ S^0$ is not connected!)

So Cor 59.2 implies the conclusion.
There's a homeomorphism $f: S^n - \mathbb{R}^n$ called **stereographic projection**:

There's a homeomorphism $g: S^n - \{p_+, p_-\} \rightarrow S^{n-1} \times (-1, 1)$

Now let's look at a more interesting bunch of spaces:

**Def**: For any $n \geq 0$ we define **real projective $n$-space** $\mathbb{R}P^n$ to be the quotient space of $S^n$ formed by identifying opposite points: $p \sim -p$ for all $p \in S^n$.

Consider the **projective line**:

The quotient map is 2-1 and onto — actually a covering map.
Next consider $\mathbb{R}P^2$, the projective plane:

$$\mathbb{R}P^2 \cong \mathbb{D}^2 / \{ p \sim -p \text{ if } p \in S^1 \}$$

$\pi \pm x$ really the same point