Proof sketch of S-vK Thm (Theorem 70).

Let \( (X,x_0) \) be a pointed space w/ open sets \( U,V \subseteq X \) such that \( U \cup V = X \), \( U \cap V \) contains \( x_0 \) and is path-connected. Then we have a pushout:

\[
\begin{array}{ccc}
\pi_1(U \cap V, x_0) & \xrightarrow{\pi_1(i_1)} & \pi_1(U, x_0) \\
\downarrow & & \downarrow \\
\pi_1(v \cap U, x_0) & \xrightarrow{\pi_1(i_2)} & \pi_1(V, x_0) \\
\downarrow & & \downarrow \\
\pi_1(X, x_0)
\end{array}
\]

where

- \( i_1 : (U \cap V, x_0) \hookrightarrow (U, x_0) \)
- \( i_2 : (U \cap V, x_0) \hookrightarrow (V, x_0) \)
- \( j_1 : (U, x_0) \hookrightarrow (X, x_0) \)
- \( j_2 : (V, x_0) \hookrightarrow (X, x_0) \)

are inclusions.

Munkres proves only the special case where \( U, V \) are path-connected, so we'll sketch that case using his notation. (Read over the proof in Munkres) The general theorem reduces to this special case.
To show that the above diamond is a pushout, i.e., given any commutative diamond

there exists a unique (hom)omorphism

such that this diagram commutes:
To show $\phi$ is unique is easy. We know $\phi$ on $\text{im}(\pi_1(j_1))$ and $\text{im}(\pi_1(j_2))$ since the triangles commute. By the baby SevK Thm, these images generate the whole group $\pi_1(X,x_0)$ so $\phi$ is uniquely determined.

The hard part is showing $\phi$ exists; we'll outline the steps.

1) We define a function $\rho$ that assigns to each loop $f$ in either $U$ or $V$ an element of $h$:

$$\rho(f) = \begin{cases} \phi_1([f]_u) & \text{if } \text{im } f \subseteq U \\ \phi_2([f]_v) & \text{if } \text{im } f \subseteq V \end{cases}$$

where $[f]_u$ means "path homotopy class of $f$ in $U$" $[f]$ with no subscript means "path homotopy class of $f$ in $X$".

To check that $\rho$ is well-defined, suppose $\text{im}(f) \subseteq U \cap V$:

$$\phi_1([f]_u) = \phi(\pi_1(j_1)[f]_u) \quad \text{since the left triangle commutes}$$

$$= \phi \circ \pi_1(j_1) \circ \pi_1(c_1)[f]_{U\cap V} \quad \text{since the top diamond commutes}$$

$$= \phi \circ \pi_1(j_2) \circ \pi_1(c_2)[f]$$

$$= \phi \circ \pi_1(j_2)[f]_v$$

$$= \phi_2([f]_v) \quad \text{since the right triangle commutes.}$$
Note $\rho$ satisfies

a) $[f]_u = [g]_u$ or $[f]_v = [g]_v \Rightarrow \rho(f) = \rho(g)$

b) if $\text{im } f, \text{im } g \subseteq U$ or $\text{im } f, \text{im } g \subseteq V$, then $\rho(f*g) = \rho(f) \rho(g)$

2) We extend $\rho$ to a function $\sigma$ that assigns to each path $f$ in either $U$ or $V$ an element of $H$.

Check that a) and b) still hold where now $f, g$ are paths.

3) Extend $\sigma$ to a function $\tau$ that assigns to every path $f \in X$ an element of $H$.

Check

$\tau(f)$ if $[f] = [g]$ for any paths $f, g$ in $X$, then $\tau(f) = \tau(g)$

$b')$ for all paths $f, g$ in $X$, $\tau(f*g) = \tau(f) \tau(g)$ if the composite $f*g$ is defined.

This is the really hard part. From this point on, the proof is easy.
If \( f \) is a based loop in \( X \), let \( \phi([f]) = \tau(f) \).

Condition \( a') \) implies that \( \phi \) is well-defined; \( b') \) implies that \( \phi \) is a homomorphism. We just need to check that \( \phi \) makes the triangles commute.

If \( f \) is a loop in \( U \), then

\[
\phi \circ \pi_1 (j_1) [f]_U = \phi[f] = \tau(f) = \rho(f)
\]

since \( \tau \) extends \( \rho \) and \( f \) is a loop in \( U \). But

\[
\rho(f) = \phi_1 [f]_U
\]

so the left-hand triangle commutes, and similarly for the right-hand triangle.

How we extend \( \rho \) (defined on loops either in \( U \) or in \( V \)) to \( \sigma \) (defined on paths either in \( U \) or in \( V \)):

![Diagram](image.png)

For every point \( x \in U \) we pick a path \( \alpha_x : x_0 \to x \), and similarly for \( V \). We can do this, because we assume \( U \) \( V \) are path-connected. Then define \( \sigma(f) = \rho(\alpha_x \ast f \ast \overline{\alpha_x}) \) if \( f \) is a path in \( U \) or \( V \) from \( x \) to \( y \).
Now we extend \( \sigma \) to \( \tau \) (defined on all paths in \( X \)):

Any path \( f \) in \( X \) can be written as \( f_1 \cdots f_n \) where each \( f_i \) lies entirely in \( U \) or \( V \), and then define

\[
\tau(f) = \sigma(f_1) \cdots \sigma(f_n)
\]

We need to check this is well-defined.