From Munkres' S-vK Thm to the Full-Fledged One

Let's assume \((X, x_0)\) a pointed space, \(U \& V\) open in \(X\), \(U \cup V\) is path connected and contains \(x_0\), \(U \cup V = X\). This is enough to imply the S-vK theorem, but Munkres also assumes \(U, V\) are path connected. How to go from his S-vK Thm to the full-fledged one?

Consider \(X_0 = \{ x \in X : \exists \text{ a path in } X \text{ from } x_0 \text{ to } x \}\) i.e. the path component of \(x_0\). The inclusion \(X_0 \hookrightarrow X\) gives a homomorphism

\[ \pi_1(X_0, x_0) \longrightarrow \pi_1(X, x_0) \]

which is actually an isomorphism, since any loop based at \(x_0\) is contained in \(X_0\). Note that \(X_0\) is path-connected.

Similarly, we can form

\[ U_0 = U \cap X_0 \]
\[ V_0 = V \cap X_0 \]

And then

\[ (U \cup V) \cap X_0 = U_0 \cap V_0 \quad \text{and} \quad U_0 \cup V_0 = (U \cap X_0) \cup (V \cap X_0) = (U \cup V) \cap X_0 = X \cap X_0 = X_0 \]
Note that the inclusions $U_0 \hookrightarrow U$, $V_0 \hookrightarrow V$, $U_0 \cap V_0 \hookrightarrow U \cap V$ all give rise to isomorphisms of fundamental groups. So to prove the full-fledged S-vK theorem, it suffices to prove the special case considered by Munkres.

![Diagram](image)

Homework: show all these are path-connected. (Hint: draw some pictures of examples first.)

Last week I told you to compute $\pi_1$ of the dunce cap:

![Dunce cap diagram](image)

formed by identifying all three edges of a solid triangle as shown. What's it like?

![Dunce cap transformation](image)

very hard to draw, but embeddable in $\mathbb{R}^3$.

(Warning! This is nothing like Munkres' dunce cap!)
The Klein bottle, on the other hand, can't be embedded in \( \mathbb{R}^3 \). This is obtained by identifying edges of a square like this:

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\[ \square \rightarrow \text{cylinder} \rightarrow K \]
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doesn't self-intersect: goes around third boundary into 4th dimension just like Möbius strip goes into third

Homework: compute fundamental group of \( K \).

For each \( n \geq 3 \) there's a group \( D_n \) called the \( n \)-th dihedral group: all symmetries of a regular \( n \)-gon, including rotations and reflections.

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\[ \text{A\hspace{1cm}rotate clockwise by } \frac{2\pi}{n} \hspace{1cm} A^n = 1 \]

\[ \text{B\hspace{1cm}reflect} \hspace{1cm} B^2 = 1 \]
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\[ BAB = A^2 \]
There's also a group $D_\infty$, $\lim_{n \to \infty} D_n$, in some intuitive sense. It's the symmetry group of

\[
\cdots \cdot 2 \cdot 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdots
\]

i.e. $\mathbb{Z}^1$.

\[
n \xrightarrow{A} n+1
\]
\[
n \xrightarrow{B} -n
\]

So $D_\infty$ is the group of bijections generated by $A, B$, i.e.

\[F(n) = \pm n + k \quad k \in \mathbb{Z}\]

Homework: Find a homomorphism from $\pi_1(K, \ast)$ to $D_\infty$. 