What's more fundamental than the fundamental group?

To avoid needing a basepoint, we should use the fundamental groupoid:

**Defn.** A groupoid is a category where every morphism \( f: x \to y \) has an inverse, i.e. some \( g: y \to x \) such that

\[
\begin{align*}
go f &= 1_x \\
fg &= 1_y
\end{align*}
\]

(If a morphism \( f \) has an inverse, it's unique, so we can call it \( f^{-1}: y \to x \).)

Any space \( X \) gives a groupoid where

- objects are points
- morphisms are homotopy classes of paths
- the inverse of any morphism \([\alpha]: x \to y\)
  is the reverse \([\alpha]: y \to x\) since \( \alpha \circ \alpha \) is contractible.

called the fundamental groupoid \( \Pi_1(X) \). It has

- \( \text{capital } \Pi \) composition \( gof = f \circ g \)
- identity \( 1_x: x \to x \)
- \( f: x \to y = f \circ 1_x = 1_y \circ f \)
Prop: Given a groupoid $G$ and any object $x \in G$, there's a group $\text{End}(x)$ of morphisms $f : x \to x$ where multiplication is composition of morphisms. Conversely, given a group $G$, there's a groupoid with one object $*$ and morphisms $f : * \to *$ for each $f \in G$, where composition is multiplication of elements.

Pf. Follows from the definitions.

\[ \square \]

Moral: "A group is a groupoid with one object."

**Group Theory $\subseteq$ Groupoid Theory**

$\pi_1(X)$ is better than $\pi_1(X,*)$ because it doesn't need a basepoint and because it contains $\pi_1(X,x_0) \forall x_0 \in X$. This simplifies life, especially the S-VK thm.

Defn - The category $\text{Gpd}$ has

- objects groupoids
- morphisms functors

\[
\begin{array}{ccc}
G & \xymatrix{\ar[r]^-{F} & } & G' \\
\end{array}
\]

a morphism $F : G \to G'$ in $\text{Gpd}$
For any space $X$, $\mathcal{P}_1(X)$ is a groupoid, but also for any map $m: X \to X'$ we get a functor $\mathcal{P}_1(m): \mathcal{P}_1(X) \to \mathcal{P}_1(X')$.

The picture explains how:

- given an object $x \in \mathcal{P}_1(X)$ i.e. a point $x \in X$,
  we define $\mathcal{P}_1(m)(x) = m(x)$

- given a morphism $g = [y] \in \mathcal{P}_1(X)$, we define $\mathcal{P}_1(m)(g) = [m \circ y]$

Prop. $\mathcal{P}_1: \text{Top} \to \text{Gpd}$ is a functor.

Pf sketch.

Check

$\mathcal{P}_1(g \circ f) = \mathcal{P}_1(g) \circ \mathcal{P}_1(f)$

$\mathcal{P}_1(1_x) = 1_{\mathcal{P}_1(x)}$

\hspace{1cm} $\Box$

Better SVK Theorem. If $X$ is a space, $U, V \subseteq X$ open, $U \cup V = X$, then

![Diagram](image_url)

is a pushout in $\text{Top}$, and
is a pushout.

Pf. Left to reader. □

This is much cleaner! There's no mention of basepoints or connectedness.