We defined the fundamental group functor

\[ \pi_i : \text{PointedSpaces} \rightarrow \text{Groups} \]

\[ (X, x_0) \mapsto \pi_i(X, x_0) \]

(pointed space) \hspace{1cm} (group of homotopy classes of loops in X with mult)

\[ [f] \circ [g] = [f \circ g] \]

\[ h : (X, x_0) \rightarrow (Y, y_0) \mapsto \pi_i(h) : \pi_i(X, x_0) \rightarrow \pi_i(Y, y_0) \]

(map of pointed spaces) \hspace{1cm} (homomorphism of groups)

\[ \pi_i(h \circ c) = \pi_i(h) \circ \pi_0(k) \]

\[ \pi_i(1_{(X, x_0)}) = 1_{\pi_i(X, x_0)} \]

This is a functor because

It was stated but not proven that \( \pi_i(h) \) is a homomorphism.

\[ \pi_i(h)([f] \circ [g]) = \pi_i(h)([f \circ g]) \]

\[ = [h \circ (f \circ g)] \]

\[ = [(h \circ f) \circ (h \circ g)] \quad \text{check this step!} \]

\[ = [h \circ f] \circ [h \circ g] \]

\[ = \pi_i(h)(f) \circ \pi_i(h)(g) \]
Next: How are $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ related?

Suppose $\alpha$ is a path with $\alpha(0) = x_0$, $\alpha(1) = x_1$.

Define a function

$\hat{\alpha}: \pi_1(X,x_0) \to \pi_1(X,x_1)$

by $\hat{\alpha}(\overline{f}) = [\overline{\alpha} * f * \alpha]$.

Thm 52.1 $\hat{\alpha}$ is a group isomorphism.

Pf.

$\hat{\alpha}(\overline{[f] * [g]}) = [\overline{\alpha} * f * g * \alpha]$

$= [\overline{\alpha} * f] * [e_x] * [g * \alpha]$

$= [\alpha * f * \alpha] * [\alpha * g * \alpha]$

$= [\overline{\alpha} * f * \alpha] * [\alpha * g * \alpha]$

$= \hat{\alpha}(\overline{[f]}) * \hat{\alpha}(\overline{[g]})$ so $\hat{\alpha}$ is a homomorphism.

$\hat{\alpha}$ has $\hat{\alpha}$ as its inverse:

$\hat{\alpha} \circ \hat{\alpha}(\overline{[f]}) = [\overline{\alpha} * (\overline{\alpha} * f * \alpha) * \alpha]$

$= [f]$ and similarly for $\hat{\alpha} \circ \hat{\alpha}$. 
Defn: If $X$ is a path-connected space, $X$ is called **simply connected** if $\pi_1(X, x_0) = 0$ for some (and hence every) $x_0 \in X$.

Examples:

- $D^2 = \bigcirc$ is simply connected
- $\mathbb{R}^n$ is simply connected
- $\mathbb{R}^2 - \{0\}$ is not simply connected: $\bigcirc$
- $\mathbb{R}^3 - \{0\}$ is simply connected
- $\mathbb{R}^3$ - line is not s.c.
- $S^1$ is not s.c.
- $S^2$ is s.c.

Lem 52.4: Let $X$ be s.c. Then any two paths $\alpha, \beta$

$x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\beta} x$, are path-homotopic, where $x_0, x_1 \in X$.

Pf. (Diagram)

or, algebraically

$$[\alpha] \neq [\alpha] *[e_{x_0}] = [\alpha] * [\beta] + [\beta] = [e_{x_0}] *[\beta] = [\beta]$$

by s.c.
Next goal: $\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$.

For this, we need the idea of

**COVERING SPACES**

**Defn.** Let $p : E \to B$ be a surjective map of spaces. An open set $U \subseteq B$ is called **evenly covered** by $p$ if

$$p^{-1}(U) = \bigcup_{\alpha} V_{\alpha}$$

where $V_{\alpha}$ are disjoint open sets in $E$, called **slices**, and

$$p|_{V_{\alpha}} : V_{\alpha} \to U$$

is a homeomorphism for each $\alpha$.

**Defn.** If every $b \in B$ has a neighborhood that is evenly covered by $p$, then $p$ is called a **covering map**; $E$ is called a **covering space** of $B$, and $B$ is called the **base space**.

**Ex (Thm 53.1)** Identify $\mathbb{S}^1$ with unit complex numbers. The map

$$p : \mathbb{R} \to \mathbb{S}^1$$

$$x \mapsto e^{\pi i x}$$

is a covering map.
$E = R$

$p$ restricted to any one of these intervals $V_x$ is a homeomorphism

$B = S'$

Other covering spaces:

$E = S'$

$B = S'$

E = $S' \sqcup S$, \hspace{1cm} \underbrace{B = S'}_{two-fold\ covers}$