Lemma 55.3 - Given a map $h: S^n \rightarrow X$ where

$$S^n = \{ x \in \mathbb{R}^n : ||x|| = 1 \}$$

$$D^{n+1} = \{ x \in \mathbb{R}^{n+1} : ||x|| \leq 1 \}$$

Then $h$ is nullhomotopic to a constant map - if and only if it extends to a map $k: D^{n+1} \rightarrow X$ such that

$$\begin{array}{c}
S^n \xrightarrow{h} X \\
\downarrow \\
D^{n+1} \xrightarrow{k} \end{array}$$

Sketch of proof (fill in details if you use this to do homework).

Suppose $h: S^n \rightarrow X$ is nullhomotopic:

If $h$ is nullhomotopic, we can extend $h$ to $k$ as shown, thinking of $D^{n+1}$ as the union of concentric $S^n$'s. Conversely, given an extension $k$, define a homotopy

$$F(x, t) = k(tx)$$

$$S^n \times [0, 1]$$
Note that $F(1,x) = k(x) = h(x)$
but $F(0,x) = k(0x) = k(0)$ is a constant!

So $F$ is a homotopy from $h$ to a constant map.

□

Cor 55.4 - The identity map $1_{S^1} : S^1 \to S^1$ isn't nullhomotopic.

Pf. By Thm 55.3, if $1_{S^1}$ were nullhomotopic, we could extend it to a map $k : D^2 \to S^1$ that is a retraction since $k|_{S^1} = 1_{S^1}$. But we've shown that no such retraction exists. □

Thm 55.5 - Given a nonvanishing vector field on $D^2$, i.e. a map

$\mathbf{\tilde{V}} : D^2 \to \mathbb{R}^2 \setminus \{0\}$

then $\exists x \in S^1 \subseteq \mathbb{R}^2$ s.t. $\mathbf{\tilde{V}}$ points directly outwards, i.e.

$\mathbf{\tilde{V}}(x) = \alpha x$ for $\alpha > 0$,

and $\exists x' \in S^1$ s.t. $\mathbf{\tilde{V}}$ points directly inwards, i.e.

$\mathbf{\tilde{V}}(x) = \alpha x$ for $\alpha < 0$.

Proof: Wlog we can assume $\|\mathbf{\tilde{V}}(x)\| = 1$ $\forall x \in D^2$ because we can normalize it: $\mathbf{\tilde{V}}$ continuous implies $\frac{\mathbf{\tilde{V}}}{\|\mathbf{\tilde{V}}\|}$ continuous.

Also, if $\mathbf{\tilde{V}}$ points in- or outwards, so does $\frac{\mathbf{\tilde{V}}}{\|\mathbf{\tilde{V}}\|}$.

So assume we have a map $\mathbf{\tilde{V}} : D^2 \to S^1$ s.t. $\mathbf{\tilde{V}}$ never points inwards; let's get a contradiction.
Let \( \tilde{w} = \tilde{v} |_{S^1} : S^1 \to S^1 \).

Then \( \tilde{w} \) extends to \( \tilde{v} \) by definition and by Lem 55.3, \( \tilde{w} \) must be nullhomotopic.

But \( \tilde{w} \) is homotopic to \( I_{S^1} \).

We have a homotopy

\[
F(x, t) = \frac{(1-t)x + t \tilde{w}(x)}{\| (1-t)x + t \tilde{w}(x) \|}
\]

Note that the only time this could fail is if \( w(x) = \alpha x \) for \( \alpha < 0 \).

Now since \( I_{S^1} \) is homotopic to \( \tilde{w} \), which by Cor 55.4 is not nullhomotopic,

So by contradiction, \( \exists x \in S^1 \) s.t. \( \tilde{v}(x) = \alpha x \) for \( \alpha < 0 \).

Since \(-v : D^2 \to S^1\) is also a vector field, it too must point directly inwards somewhere on \( S^1 \), so \( v \) must point directly outwards somewhere.

\[\square\]
Thm 55.6 (Brouwer Fixed-point Theorem) Any map $f : D^2 \to D^2$ has a fixed point, i.e. $\exists x \in D^2 \text{ s.t. } f(x) = x$.

Pf. By contradiction: suppose $\forall x \in D^2 \text{ } f(x) \neq x$. Then

$$\vec{v} = f(x) - x$$

is a vector field on $D^2$ that never vanishes. By Thm 55.5, $\vec{v} |_{S^1}$ must point directly outwards, but then $f(x)$ would not be in $D^2$. Contradiction.

□

In fact, all these results generalize if you replace $S^1$ by $S^n$ (n≥1) and $D^2$ by $D^{n+1}$. Some of these are homework problems; they all follow from the fact (which the book gives you without proof) that there’s no retraction from $D^{n+1}$ to $S^n$. Next time, I’ll discuss the proof.