Math 205B - Topology

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Exercise 55.1. Show that if A is a retract of B^2, then every continuous map $f : A \to A$ has a fixed point.

Proof. Let $r : B^2 \to A$ be a retraction, and let $i : A \to B^2$ be the inclusion map. Consider the map $g = i \circ f \circ r : B^2 \to B^2$. This is continuous, since it is the composition of continuous maps. Thus by the Brouwer Fixed Point Theorem, there exist $y \in B^2$ such that $g(y) = y$. This gives us that $y = f(r(y))$. But this means $y \in A$, and so $y = r(y)$. Therefore $f(y) = y$, and so f has a fixed point. $\square$

Exercise 55.2. Show that if $h : S^1 \to S^1$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. Since $h : S^1 \to S^1$ is nulhomotopic, then by Theorem 55.3, h can be extended to a map $k : B^2 \to S^1$. If $i : S^1 \to S^1$ is the inclusion map, then $f = i \circ k$ is a continuous map from B^2 to itself. Hence by the Brouwer Fixed Point Theorem, there exist $z \in B^2$ with $z = f(z) = i(k(z))$. This gives us that $k(z) = z$, which tells us that $z \in S^1$, and so $z = k(z) = h(z)$. Thus z is a fixed point of h.

Consider the antipode map $a : S^1 \to S^1$ given by $a(x) = -x$ and the homotopy $F(x,t) = e^{i\pi t}h(x)$. F is continuous since it is the product of continous maps, $F(x,0) = h(x)$, and $F(x,1) = -h(x) = a \circ h(x)$. Thus we have $h \simeq a \circ h$ and so $a \circ h$ is nulhomotopic. Now by the first part above $a \circ h$ has a fixed point. This means there exist $x \in S^1$ such that $x = a(h(x))$, or $x = -h(x)$. Thus $h(x) = -x$ and so h maps some point to its antipode. $\square$

Exercise 55.4. Suppose that you are given the fact that for each n, there is no retraction $r : B^{n+1} \to S^n$. Prove the following:

(a) The identity map $\iota : S^n \to S^n$ is not nulhomotopic.

(b) The inclusion map $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ is not nulhomotopic.

(c) Every non-vanishing vector field on B^{n+1} point directly outward at some point of S^n, and directly inward at some point of S^n.

(d) Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.

(e) Every $n+1 \times n+1$ matrix with positive real entries has a positive eigenvalue.

(f) If $h : S^n \to S^n$ is nulhomotopic, then h has a fixed point and h maps some point x to its antipode $-x$.

Proof. Parts (a), (b), (c), and (d) were proven previously, so we will assume they are true.

(e) Let A be a $n+1 \times n+1$ matrix with positive real entries, and let $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the linear transformation given by $T(x) = Ax$. Now consider the region $B = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1, \text{ and } x_i \geq 0 \text{ for } i = 1, ..., n+1\}$. B is homeomorphic to B^{n+1} and so all the result above about B^{n+1} hold for this new region B. Specifically we will use the generalization of the Brouwer Fixed Point Theorem; part (d).

Let $T^*(x) = \frac{T(x)}{\|T(x)\|}$. T^* is defined on B since for any $x \in B$, $\|x\| = 1$, so $\|T(x)\| \neq 0$ because A has all positive entries. Thus T^* is a continuous map from B to itself, so by part (d) T^* has a fixed point, say x_0. This gives us that $T^*(x_0) = x_0$ or $x_0 = \frac{T(x_0)}{\|T(x_0)\|}$. It then follows that $Ax_0 = \|T(x_0)\| \, x_0$, and so $\|T(x_0)\|$ is a positive eigenvalue of A.

(f) We have previously shown that we can generalize part (i) and (ii) of Theorem 55.3, so we will use that here. Since $h : S^n \to S^n$ is nulhomotopic, then by the generalization of Theorem 55.3, h can be extended to a map $k : B^{n+1} \to S^n$. Let $i : S^n \to S^n$ be the inclusion map. Then $f = i \circ k$ is a continuous map from B^{n+1} to itself. Hence by the part (d) (the generalized Brouwer Fixed Point Theroem), there exist $z \in B^{n+1}$ with $z = f(z) = i(k(z))$. This gives us that $k(z) = z$, which tells us that $z \in S^n$, and so $z = k(z) = h(z)$. Thus z is a fixed point of h.

Again, consider the antipode map $a : S^n \to S^n$ given by $a(x) = -x$. Since h is nulhomotopic so is $a \circ h$, since if $F : S^n \times I \to S^n$ is a nulhomotopy for h, $a \circ F$ is a nulhomotopy for $a \circ h$. Now by the first part above $a \circ h$ has a fixed point. This means there exist $x \in S^n$ such that $x = a(h(x))$, or $x = -h(x)$. Thus $h(x) = -x$ and so h maps some point to its antipode.

$\square$