Math 205B - Topology

Dr. Baez

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Christopher Walker

Exercise 58.2. For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine for each space which of the three alternatives holds.

Proof.

(a) The "solid torus," $B^2 \times S^1$ - Infinite Cyclic

(b) The torus T with a point removed - Figure Eight

(c) The cylinder $S^1 \times 1$ - Infinite Cyclic

(d) The infinite cylinder $S^1 \times \mathbb{R}$ - Infinite Cyclic

(e) $\mathbb{R}^3$ with the nonegative x, y, and z axes deleted - Figure Eight

(f) $\{x \mid \|x\| > 1\}$ - Infinite Cyclic

(g) $\{x \mid \|x\| \geq 1\}$ - Infinite Cyclic

(h) $\{x \mid \|x\| < 1\}$ - Trivial

(i) $S^1 \bigcup (\mathbb{R}_+ \times 0)$ - Infinite Cyclic

(j) $S^1 \bigcup (\mathbb{R}_+ \times \mathbb{R})$ - Infinite Cyclic

(k) $S^1 \bigcup (\mathbb{R} \times 0)$ - Figure Eight

(l) $\mathbb{R}^2 - (\mathbb{R}_+ \times 0)$ - Trivial

$\square$

Exercise 58.4. Let X be the figure eight and let Y be the theta space. Describe maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverse to each other.

Proof. Consider the figure eight to be two congruent, tangent circles and the theta space to be a circle with one diameter. First, the map $g : Y \to X$ can be described as contracting the circle along the diameter to the center of the circle. Similarly, the map $f : X \to Y$ can be described as stretching each tangent circle to fit into half of the theta space. A sketch of these maps is attached. $\square$

Exercise 58.5. Show that X is contractible if and only if X has the homotopy type of a one-point space.

Proof.
($\Rightarrow$)
Assume X is contractible. This means that id_X is homotopic to a constant map. Let $f : X \to X$ such that $f(x) = x_0$ be that constant map. Consider the one point $\{x_0\}$. With this we have that f is a map from X to $\{x_0\}$. Let $j : \{x_0\} \to X$ be the inclusion map. $f \circ j = id_{\{x_0\}}$, and so trivially homotopic. $j \circ f = f$ and since $f \simeq id_x$, then $j \circ f \simeq id_X$, and X has the homotopy type of a one-point space.
($\Leftarrow$)
Assume X has the homotopy type of a one-point space, say $Y = \{c\}$. This means there exist $f : X \to Y$ and $g : Y \to X$, such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Since Y is a one point set, then $g(c) = x_0$ for some $x_0 \in X$. Thus for all $x \in X$ $g \circ f(x) = g(c) = x_0$. This tells us that $g \circ f$ is a constant map. Thus id_X is homotopic to a constant map, and so X is contractible.

$\square$

Exercise. Classify the capital english letters by homeomorphism type and by homotopy type.

Proof. We will first classify by homeomorphsm type. Here is a list of sets that contain homeomorphic letter.

$$H_1 = \{\mathsf{C, G, I, J, L, M, N, S, U, V, W, Z}\}$$

$$H_2 = \{\mathsf{E, F, T, Y}\}$$

$$H_3 = \{\mathsf{H, I}\}$$

$$H_4 = \{\mathsf{K, X}\}$$

$$H_5 = \{\mathsf{D, O}\}$$

$$H_6 = \{\mathsf{B}\}$$

$$H_7 = \{\mathsf{Q}\}$$

$$H_8 = \{\mathsf{P}\}$$

$$H_9 = \{\mathsf{A, R}\}$$

The letters in H_1 are homeomorphic because the ar simply a straight line that has been curved or bent. Also for any of these if we remove any point not at the end, we get exactly 2 components. For each other set we have a similar point that when removed gives a specific number of components

We get a different result when we classify by homotopy type. This is the list of sets that contain homotopy equivalent letters.

$$F_1 = \{\mathsf{C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z}\}$$

$$F_2 = \{\mathsf{A, D, O, P, Q, R}\}$$

$$F_3 = \{\mathsf{B}\}$$

The letters in F_1 are all homotopy equivalent since each can be retracted in some way to the letter I, and I can be embedded in these letters in some way. Each letter in F_2 is homotopy equivalent since they can be retracted down to a single loop. Since B has two "holes" it is not homotopy equivalent to any other letters.

□