Math 205B - Topology

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Exercise 58.3. (Pointed version.) Show that given a collection $\mathcal{C}$ of pointed spaces, the relation of homotopy equivalence is an equivalence relation on $\mathcal{C}$.

Proof. Recall that two pointed spaces (X, x_0) and (Y, y_0) are homotopy equivalent if there exist maps $f\colon (X, x_0) \to Y$ and $g\colon (Y, y_0) \to (X, x_0)$ where $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Here the relation '$\simeq$' is the appropriate concept of 'homotopic' for pointed maps: that is, $h \simeq h'$ if that there exists a *pointed* homotopy from h to h'. We need to check that homotopy equivalence is reflexive, symmetric, and transitive.

- Reflexive - for all $X \in \mathcal{C}$, $1_X\colon (X, x_0) \to (X, x_0)$ gives us that $1_X \circ 1_X = 1_X$, so X is homotopy equivalent to X.

- Symmetric - Let $X, Y \in \mathcal{C}$. If X is homotopy equivalent to Y, then by definition of homotopy equivalence, Y is homotopy equivalent to X. This is because the definition is symmetrical.

- Transitive - Let $X, Y, Z \in \mathcal{C}$ where X is homotopy equivalent to Y, and Y is homotopy equivalent to Z. This implies there exist $f\colon X \to Y$ and $g\colon Y \to X$ with $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Also, there exist $h\colon Y \to Z$ and $k\colon Z \to Y$ with $h \circ k \simeq 1_Z$ and $k \circ h \simeq 1_Y$. Now consider $h \circ f\colon X \to Z$ and $g \circ k\colon Z \to X$. For each step, we use the fact that composition preserves the relation $\simeq$. First

  $$\begin{aligned} h \circ f \circ g \circ k &\simeq h \circ 1_Y \circ k \\ &\simeq h \circ k \\ &\simeq 1_Z \end{aligned}$$

  Similarly we have the other direction.

  $$\begin{aligned} g \circ k \circ h \circ f &\simeq g \circ 1_Y \circ f \\ &\simeq g \circ f \\ &1_X \end{aligned}$$

  Thus X is homotopy equivalent to Z.

Therefore homotopy equivalence is an equivalence relation. $\square$

Exercise 58.6. Show that a retract of a contractible space is contractible

Proof. Let A be a retract of X, and let X be contractible. This means 1_X is homotopic to a constant map, say $f(x) = x_0$. Let $H\colon X \times I \to X$ be the homotopy from 1_X to f with $H(x, 0) = x$ and $H(x, 1) = f(x) = x_0$. If $r\colon X \to A$ is the retraction of X to A, then we can consider the homotopy $r \circ H|_A\colon A \times I \to A$. This is continuous since it is the composition of continuous maps. Also $r \circ H|_A(x, 0) = r(x) = x$, since $x \in A$, and $r \circ H|_A(x, 1) = r(f(x)) = r(x_0)$. This gives us that 1_A is homotopic to the constant map $g(x) = r(x_0)$, and so A is contractible. □

Exercise 59.3. (a) Show that $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

(b) Show that $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic if $n > 2$.

Proof.

(a) We will assume that exist a map $h\colon \mathbb{R}^1 \to \mathbb{R}^n$ that is a homeomorphism. We will then consider what happens If we remove 0 in $\mathbb{R}^1$ and its image $h(0)$ from $\mathbb{R}^n$. $\mathbb{R}^1 - 0$ is a disconnected space, but $\mathbb{R}^n - h(0)$ is connected. Removal of a point and its image always preserves homeomorphism, thus $\mathbb{R}^1 - 0$ and $\mathbb{R}^n - h(0)$ are homeomorphic. But connectedness is invariant under homeomorphism, so we have a contradiction.

(b) We will use a similar technique for this part. Assume that there exist a map $h\colon \mathbb{R}^2 \to R^n$ that is a homeomorphism. We will again remove the origin from $\mathbb{R}^2$ and the image $h(0,0)$ from $\mathbb{R}^n$. Removal of a point and its image always preserves homeomorphism, thus $\mathbb{R}^1 - (0,0)$ and $\mathbb{R}^n - \{h(0,0)\}$ are homeomorphic. We now have that $\mathbb{R}^2 - \{(0,0)\}$ is homeomorphic to S^1, and $\mathbb{R}^n - \{h(0,0)\}$ is homeomorphic to S^{n-1}. S^1 is not simply connected, however S^{n-1} is simply connected for $n > 2$. Thus since simply connectedness is invariant under homeomorphism, then $\mathbb{R}^1 - (0,0)$ and $\mathbb{R}^n - \{h(0,0)\}$ are not homeomorphic, a contradiction.

$\square$