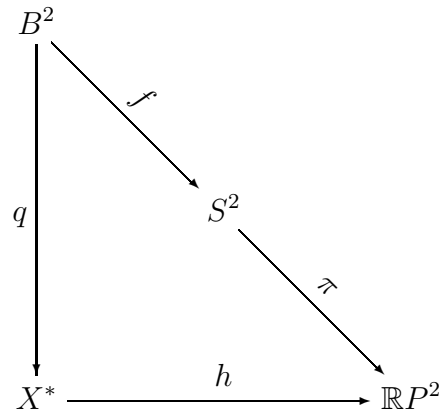


**Exercise 60.2.** Let  $X$  be the quotient space obtained from  $B^2$  by identifying each point  $x$  of  $S^1$  with its antipode  $-x$ . Show that  $X$  is homeomorphic to  $\mathbb{R}P^2$ , the real projective plane.

*Proof.* Let  $\pi : S^2 \rightarrow \mathbb{R}P^2$  be the quotient map which identifies any point  $p \in S^2$  with  $-p$ . Also, let  $q : B^2 \rightarrow X$  be the quotient map described in the question. We now define another map  $f : B^2 \rightarrow S^2$  as  $f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ . First by definition of  $f$ , we see that it is continuous and injective. Also,  $f$  is an open map since it takes  $\epsilon$ -neighborhoods of points in  $B^2$  to neighborhoods of its image in  $S^2$ . This gives us that the composition  $\pi \circ f$  is a quotient map. We also need  $\pi \circ f$  to be surjective. Let  $z \in \mathbb{R}P^2$ . First,  $\pi^{-1}(z) = \{p, -p\}$  where  $\pi(p) = z$ . At least one of  $p$  or  $-p$  has a positive value in the third coordinate. WLOG, we may assume it is  $p$ . This gives us that there exist  $b \in B^2$  with  $f(b) = p$ . Thus  $\pi \circ f(b) = \pi(p) = z$ , so  $\pi \circ f$  is surjective.

Let  $X^* = \{(\pi \circ f)^{-1}(\{z\}) \mid z \in \mathbb{R}P^2\}$ . Since  $(\pi \circ f)$  is a surjective quotient map, then by Corollary 22.3, there exist a bijective map  $h : X^* \rightarrow \mathbb{R}P^2$  that makes the following diagram commute:



and  $h$  is a homeomorphism. Next we need to show that  $X = X^*$ . For any point  $x \in B^2$  with  $x \notin S^1$ , then  $x$  is in the one point set  $\{x\}$  in  $X^*$  and  $X$ . If  $x \in S^1$  then  $x$  is in the two point set  $\{x, -x\}$  in  $X^*$  and  $X$ . Thus  $X$  and  $X^*$  are the same partition of  $B^2$ . So  $X \cong \mathbb{R}P^2$ .  $\square$

**Exercise 60.3.** Let  $p : E \rightarrow X$  be the map constructed in the proof of Lemma 60.5. Let  $E'$  be the subspace of  $E$  that is the union of the  $x$ -axis and the  $y$ -axis. Show that  $p|_{E'}$  is not a covering map.

*Proof.* Just as in the proof of Lemma 60.5, we will consider the figure eight space as two tangent circles  $A$  and  $B$ . In order to show  $p|_{E'}$  is not a covering map, we need a point that has no evenly covered neighborhoods. Consider the base point  $x_0$  which is the center of the figure eight (i.e. the point of tangency of the circles). A neighborhood  $U_0$  of  $x_0$  is the union of an open interval on  $A$  and an open interval on  $B$ , each of which contain  $x_0$ , and the intersection of these intervals is exactly  $\{x_0\}$ . The preimage of  $U_0$  under  $p|_{E'}$  is the disjoint collection of open intervals around integers on the  $x$ -axis and open intervals around integers on the  $y$ -axis. Unfortunately in either case any given interval is not homeomorphic to  $U_0$ , since removal of  $x_0$  from  $U_0$  gives 4 components, while removal of the integer (which is the preimage of  $x_0$ ) in one of these intervals only gives 2 components. Thus  $U_0$  is not evenly covered. Since,  $x_0$  does not have an evenly covered neighborhood under  $p|_{E'}$ , then  $p|_{E'}$  is not a covering map.

□

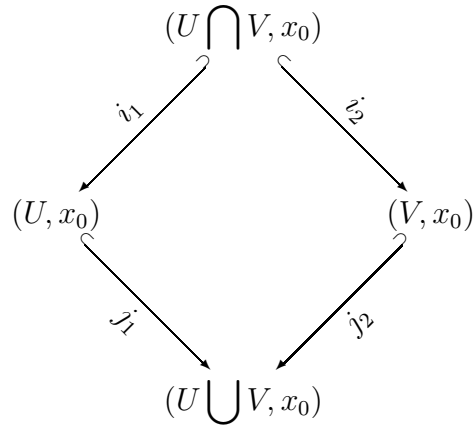
**Exercise.** Show that the quotient map  $\pi : S^n \rightarrow \mathbb{R}P^n$  is a covering map.

*Proof.* As noted on page 373, the proof of this is exactly the same as the proof of Theorem 60.3, simply replacing 2 with  $n$ .

We first show  $\pi$  is an open map (i.e. it maps open set to open sets). This along with bijection will give us later that  $\pi$  has a continuous inverse. Let  $U$  be an open set in  $S^n$ . The map  $a : S^n \rightarrow S^n$  where  $a(x) = -x$  is a homeomorphism of  $S^n$  with itself. This gives us that  $a(U)$  is also an open set in  $S^n$ . From here we get that  $U \cup a(U)$  is open (being the union of open sets), and  $\pi^{-1}(\pi(U)) = U \cup a(U)$ , so  $\pi^{-1}(\pi(U))$  is open in  $S^n$ . By definition of a quotient map, we have  $\pi(U)$  open in  $\mathbb{R}P^n$ . Thus  $\pi$  is an open map.

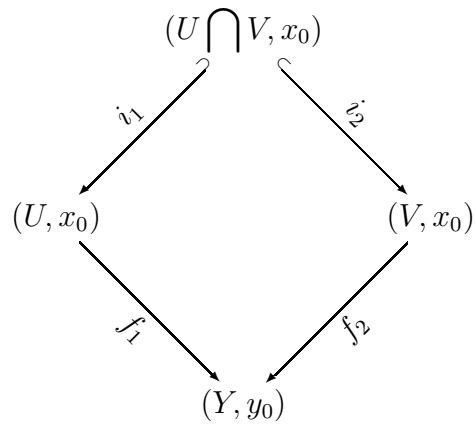
We will now show  $\pi$  is a covering map. Let  $y \in \mathbb{R}P^n$ . by definition of  $\pi$ , we know  $\pi^{-1}(y)$  is a two point set  $\{x, -x\}$ . Let  $0 < \epsilon < 1$  and let  $U$  be the  $\epsilon$  neighborhood of  $x$  in  $S^n$ , using the euclidean metric  $d$  of  $\mathbb{R}^{n+1}$ . For any point  $z \in U$  we have that  $d(z, a(z)) = 2$ , so  $a(z) \notin U$ . This tells us that  $\pi : U \rightarrow \pi(U)$  is bijective. Since  $\pi$  is also continuous and open it has a continuous inverse, so  $\pi$  is a homeomorphism. Similarly the map  $\pi : a(U) \rightarrow \pi(a(U))$  is also a homeomorphism. This tells us the set  $\pi^{-1}(\pi(U))$  is the disjoint union of the two open sets  $U$  and  $a(U)$ , and each of these is homeomorphic to the neighborhood  $\pi(U)$  of  $y$ . Thus  $\pi(U)$  is an evenly covered neighborhood of  $y$ , so  $\pi$  is a covering map.  $\square$

**Exercise.** Given a space  $X$  and open sets  $U, V \subseteq X$  with  $x_0 \in U \cap V$ , show that the following commutative diagram (call this diagram 1):

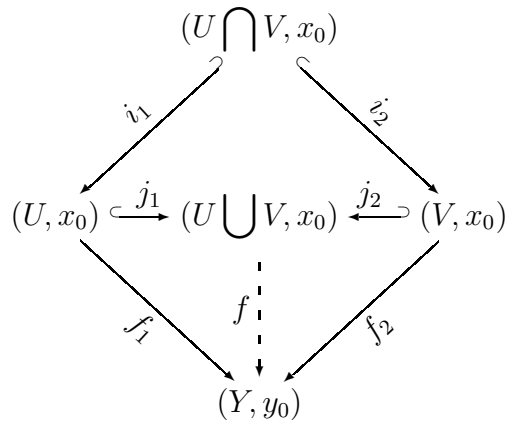


is a pushout in  $\text{Top}_*$  (the category of pointed topological spaces and point maps).

*Proof.* Given another commutative diagram (call this diagram 2):



We need to show that there exist a map  $f : (U \cup V, x_0) \rightarrow (Y, y_0)$  such that the following diagram commutes (call this diagram 3):



Define  $f : (U \cup V, x_0) \rightarrow (Y, y_0)$  as follows:

$$f(x) = \begin{cases} f_1(x) & x \in U \\ f_2(x) & x \in V \end{cases}$$

First we, need to check that this is well-defined (i.e. it agrees on  $U \cap V$ ). Let  $x \in U \cap V$ . by commutativity of diagram 2, we have that  $f_1 \circ i_1 = f_2 \circ i_2$  so  $f_1(x) = f_2(x)$ , thus  $f$  is well defined. Also since  $x_0 \in U \cap V$ , then  $f(x_0) = f_1(x_0) = y_0$ , so  $f$  is pointed. With this and the fact that each of  $f_1$  and  $f_2$  are continuous, we can use the open set version of the pasting lemma (Theorem 18.3) to see that  $f$  is also continuous. By the way we defined  $f$  we see that the entire diagram commutes. Thus diagram 1 is a pushout.  $\square$