Math 205B - Topology

Dr. Baez

February 23, 2007

Christopher Walker

Exercise 60.2. Let X be the quotient space obtained from B^2 by identifying each point x of S^1 with its antipode $-x$. Show that X is homeomorphic to $\mathbb{R}P^2$, the real projective plane.

Proof. Let $\pi : S^2 \to \mathbb{R}P^2$ be the quotient map which identifies any point $p \in S^2$ with $-p$. Also, let $q : B^2 \to X$ be the quotient map described in the question. We now define another map $f : B^2 \to S^2$ as $f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. First by definition of f, we see that its is continuous and injective. Also, f is an open map since it takes ϵ-neighborhoods of points in B^2 to neighborhoods of its image in S^2. This gives us that the composition $\pi \circ f$ is a quotient map. We also need $\pi \circ f$ to be surjective. Let $z \in \mathbb{R}P^2$. First, $\pi^{-1}(z) = \{p, -p\}$ where $\pi(p) = z$. At least one of p or $-p$ has a positive value in the third coordinate. WLOG, we may assume it is p. This gives us that there exist $b \in B^2$ with $f(b) = p$. Thus $\pi \circ f(b) = \pi(p) = z$, so $\pi \circ f$ is surjective.

Let $X^* = \{(\pi \circ f)^{-1}(\{z\}) \mid z \in \mathbb{R}P^2\}$. Since $(\pi \circ f)$ is a surjective quotient map, then by Corollary 22.3, there exist a bijective map $h : X^* \to \mathbb{R}P^2$ that makes the following diagram commute:

$$\begin{tikzcd}
B^2 \arrow[dd, "q"'] \arrow[dr, "f"] & \\
& S^2 \arrow[dr, "\pi"] & \\
X^* \arrow[rr, "h"] & & \mathbb{R}P^2
\end{tikzcd}$$

and h is a homeomorphism. Next we need to show that $X = X^*$. For any point $x \in B^2$ with $x \notin S^1$, then x is in the one point set $\{x\}$ in X^* and X. If $x \in S^1$ then x is in the two point set $\{x, -x\}$ in X^* and X. Thus X and X^* are the same partition of B^2. So $X \cong \mathbb{R}P^2$. ∎

Exercise 60.3. Let $p : E \to X$ be the map constructed in the proof of Lemma 60.5. Let $E^{'}$ be the subspace of E that is the union of the x-axis and the y-axis. Show that $p|_{E'}$ is not a covering map.

Proof. Just as in the proof of Lemma 60.5, we will consider the figure eight space as two tangent circles A and B. In order to show $p|_{E'}$ is not a covering map, we need a point that has no evenly covered neighborhoods. Consider the base point x_0 which is the center of the figure eight (i.e. the point of tangency of the circles). A neighborhood U_0 of x_0 is the union of an open interval on A and an open interval on B, each of which contain x_0, and the intersection of these intervals is exactly $\{x_0\}$. The preimage of U_0 under $p|_{E'}$ is the disjoint collection of open intervals around integers on the x-axis and open intervals around integers on the y-axis. unfortunately in either case any given interval is not homeomorphic to U_0, since removal of x_0 from U_0 gives 4 components, while removal of the integer (which is the preimage of x_0) in one of these intervals only gives 2 components. Thus U_0 is not evenly covered. Since, x_0 does not have an evenly covered neighborhood under $p|_{E'}$, then $p|_{E'}$ is not a covering map. □

Exercise. Show that the quotient map $\pi : S^n \to \mathbb{R}P^n$ is a covering map.

Proof. As noted on page 373, the proof of this is exactly the same as the proof of Theorem 60.3, simply replacing 2 with n.

We first show π is an open map (i.e. it maps open set to open sets). This along with bijection will give us later that π has a continuous inverse. Let U be an open set in S^n. The map $a : S^n \to S^n$ where $a(x) = -x$ is a homeomorphism of S^n with itself. This gives us that $a(U)$ is also an open set in S^n. From here we get that $U \bigcup a(U)$ is open (being the union of open sets), and $\pi^{-1}(\pi(U)) = U \bigcup a(U)$, so $\pi^{-1}(\pi(U))$ is open in S^n. By definition of a quotient map, we have $\pi(U)$ open in $\mathbb{R}P^n$. Thus π is an open map.

We will now show π is a covering map. Let $y \in \mathbb{R}P^n$. by definition of π, we know $\pi^{-1}(y)$ is a two point set $\{x, -x\}$. Let $0 < \epsilon < 1$ and let U be the ϵ neighborhood of x in S^n, using the euclidean metric d of $\mathbb{R}^{n+1}$. For any point $x \in U$ we have that $d(z, a(z)) = 2$, so $a(z) \notin U$. This tells us that $\pi : U \to \pi(U)$ is bijective. Since π is also continuous and open it has a continuous inverse, so π is a homeomorphism. Similarly the map $\pi : a(U) \to \pi(a(U))$ is also a homeomorphism. This tells us the set $\pi^{-1}(\pi(U))$ is the disjoint union of the two open sets U and $a(U)$, and each of these is homeomorphic to the neighborhood $\pi(U)$ of y. Thus $\pi(U)$ is an evenly covered neighborhood of y, so π is a covering map. $\square$

Exercise. Given a space X and open sets $U, V \subseteq X$ with $x_0 \in U \bigcap V$, show that the following commutative diagram (call this diagram 1):

$$\begin{tikzcd}
& (U \bigcap V, x_0) \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\
(U, x_0) \arrow[dr, hook, "j_1"'] & & (V, x_0) \arrow[dl, hook', "j_2"] \\
& (U \bigcup V, x_0) &
\end{tikzcd}$$

is a pushout in Top_* (the category of pointed topological spaces and point maps).

Proof. Given another commutative diagram (call this diagram 2):

$$\begin{tikzcd}
& (U \bigcap V, x_0) \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\
(U, x_0) \arrow[dr, "f_1"'] & & (V, x_0) \arrow[dl, "f_2"] \\
& (Y, y_0) &
\end{tikzcd}$$

We need to show that there exist a map $f : (U \bigcup V, x_0) \to (Y, y_0)$ such that the following diagram commutes (call this diagram 3):

$$\begin{tikzcd}
& (U \bigcap V, x_0) \arrow[dl, hook', "i_1"'] \arrow[dr, hook, "i_2"] & \\
(U, x_0) \arrow[r, hook, "j_1"] \arrow[dr, "f_1"'] & (U \bigcup V, x_0) \arrow[d, dashed, "f"'] & (V, x_0) \arrow[l, hook', "j_2"'] \arrow[dl, "f_2"] \\
& (Y, y_0) &
\end{tikzcd}$$

Define $f : (U \bigcup V, x_0) \to (Y, y_0)$ as follows:

$$f(x) = \begin{cases} f_1(x) & x \in U \\ f_2(x) & x \in V \end{cases}$$

First we, need to check that this is well-defined (i.e. it agrees on $U \bigcap V$). Let $x \in U \bigcap V$. by commutativity of diagram 2, we have that $f_1 \circ i_1 = f_2 \circ i_2$ so $f_1(x) = f_2(x)$, thus f is well defined. Also since $x_0 \in U \bigcap V$, then $f(x_0) = f_1(x_0) = y_0$, so f is pointed. With this and the fact that each of f_1 and f_2 are continuous, we can use the open set version of the pasting lemma (Theorem 18.3) to see that f is also continuous. By the way we defined f we see that the entire diagram commutes. Thus diagram 1 is a pushout. $\square$