Math 205B - Topology

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Question 52.1. A subset A of $\mathbb{R}^n$ is said to be ***star convex*** if for some point a_0 of A, all the line segments joining a_0 to other points in A lie in A.

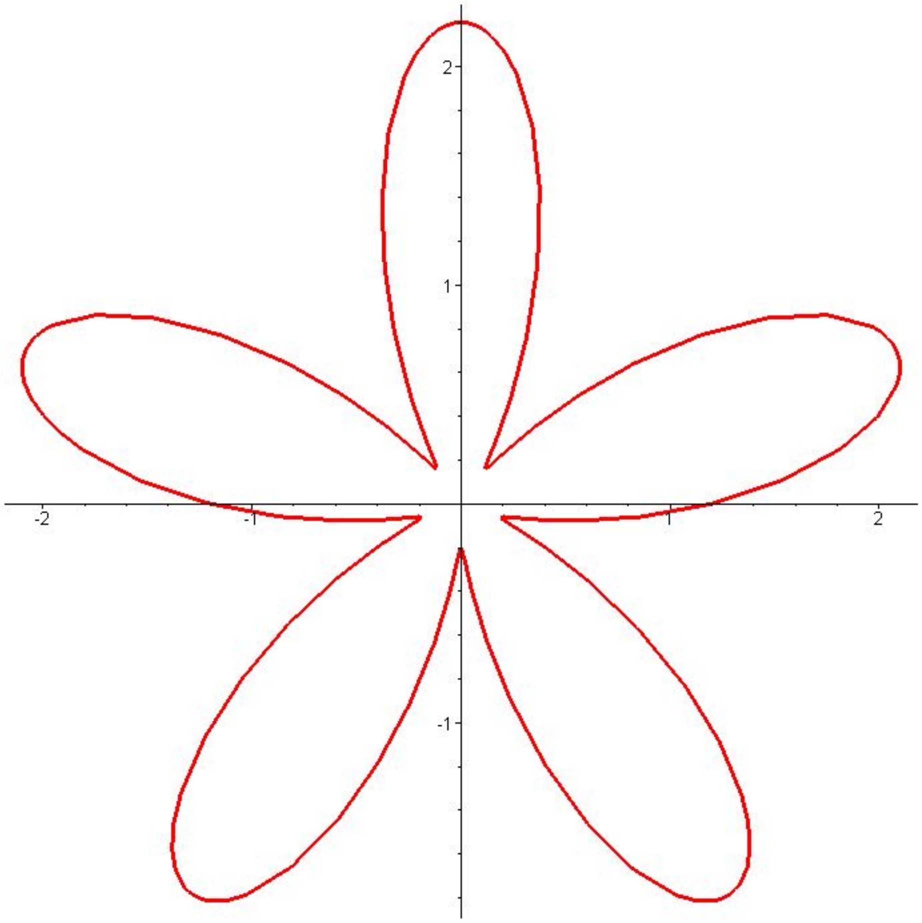
(a) Find a star convex set that is not convex.
(b) Show that if A is star convex, A is simply connected.

Proof.

(a) Consider the subset A of $\mathbb{R}^2$ bounded by the parametric polar curve
$$\{(r(t), \theta(t)) \mid r(t) = sin(5t) + 1.2, \theta(t) = t, 0 \leq t < 2\pi\}.$$
(See attached graph.) This subset is star convex if we let a_0 be the origin (and even looks like a star!). This is because the line segment from the origin to any other point in A is contained in A. This is not convex since the line segment between $(1, 0)$ and $(0, 1)$ is not contained in A.

(b) Assume A is star convex. We need to show that A is path connected and that $\pi_1(A, a_0)$ is trivial for the point a_0 in the definition of star convex. Let $a_1, a_2 \in A$. We can create a path from a_1 and a_2 by first traveling along the line segment from a_1 to a_0, then along the line segment from a_0 to a_2. This path is contained entirely in A by definition of star convex, and so A is path connected.

Next, consider a loop g at a_0. Define $H : I \times I \to A$ as the straight line path-homotopy $H(x, t) = ta_0 + (1 - t)g(x)$. We have that H is continuous, $H(x, 0) = g(x)$, and $H(x, 1) = a_0$. Thus H is a path-homotopy from g to the constant loop at a_0. This gives us that $\pi_1(A, a_0) = \{0\}$ (the trivial group), and so A is simply connected.


2
1
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□

Question 52.2. Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $\gamma = \alpha * \beta$, then $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

Proof. Let $[f] \in \pi_1(X, x_0)$. We show that $\hat{\gamma}([f]) = \hat{\beta} \circ \hat{\alpha}([f])$.

$$\begin{aligned}
\hat{\gamma}([f]) \quad &= [\bar{\gamma}] * [f] * [\gamma] \\
&= [\overline{\alpha * \beta}] * [f] * [\alpha * \beta] \\
&= [\bar{\beta} * \bar{\alpha}] * [f] * [\alpha * \beta] \\
&= [\bar{\beta}] * [\bar{\alpha}] * [f] * [\alpha] * [\beta] \\
&= [\bar{\beta}] * \hat{\alpha}([f]) * [\beta] \\
&= \hat{\beta}(\hat{\alpha}([f]) \\
&= \hat{\beta} \circ \hat{\alpha}([f])
\end{aligned}$$

Therefore $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

$\square$

Question 52.3. Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

Proof. $(\Rightarrow)$
Assume $\pi_1(X, x_0)$ is abelian. We need to show for $g \in \pi_1(X, x_0)$ that $\hat{\alpha}([g]) = \hat{\beta}([g])$. Recall that by $\bar{\beta}$ we mean the inverse path to β.

$$\begin{aligned}
\hat{\alpha}([g]) &= [\bar{\alpha}] * [g] * [\alpha] \\
&= [\bar{\alpha}] * [g] * [\alpha] * [\bar{\beta}] * [\beta] && \text{Since } [\bar{\beta}] * [\beta] \text{ is in the identity class of } \pi_1(X, x_0) \\
&= [\bar{\alpha}] * [g] * ([\alpha] * [\bar{\beta}]) * [\beta] \\
&= [\bar{\alpha}] * ([\alpha] * [\bar{\beta}]) * [g] * [\beta] && \text{Since } [\alpha] * [\bar{\beta}] \in \pi_1(X, x_0) \text{ and } \pi_1(X, x_0) \text{ is abelian.} \\
&= ([\bar{\alpha}] * [\alpha]) * [\bar{\beta}] * [g] * [\beta] \\
&= [\bar{\beta}] * [g] * [\beta] \\
&= \hat{\beta}([g])
\end{aligned}$$

Thus $\hat{\alpha} = \hat{\beta}$.

$(\Leftarrow)$
Now assume that for any two paths α and β from x_0 and x_1, We have that $\hat{\alpha} = \hat{\beta}$. Now take $f, g \in \pi_1(X, x_0)$. Let α be any path from x_0 to x_1 and define $\beta = g * \alpha$ which is a path from x_0 to x_1. Since $\hat{\alpha} = \hat{\beta}$ we get that:

$$\begin{aligned}
\hat{\alpha}([g] * [f]) &= \hat{\beta}([g] * [f]) \\
&= [\overline{g * \alpha}] * [g] * [f] * [g * \alpha] \\
&= [\bar{\alpha} * \bar{g}] * [g] * [f] * [g * \alpha] \\
&= [\bar{\alpha}] * [\bar{g}] * [g] * [f] * [g] * [\alpha] \\
&= [\bar{\alpha}] * [f] * [g] * [\alpha] \\
&= \hat{\alpha}([f] * [g])
\end{aligned}$$

This gives us that $\hat{\alpha}([g] * [f]) = \hat{\alpha}([f] * [g])$. But $\hat{\alpha}$ is an isomorphism, so $[g] * [f] = [f] * [g]$. Therefore $\pi_1(X, x_0)$ is abelian. $\square$

Question 52.4. Let $A \subset X$; suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for each point $a \in A$. (The map r is called a ***retraction*** of X onto A.) If $a_0 \in A$, show that

$$r_*; \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

Proof. To show that $r_*; \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective, we will show that for $[f] \in \pi_1(A, a_0)$ there exist $[g] \in \pi_1(X, a_0)$ such that $r_*([g]) = [f]$. This will follow from the fact that since $A \subset X$, then f is also a loop in X so consider the equivalence class $[f] \in \pi_1(X, a_0)$. With this and the fact that $r(a) = a$ for all $a \in A$, we have that $r_*([f]) = [r \circ f] = [f]$, and thus r_* is surjective. $\square$

Question. Which capital english letters are simply connected? Which ones have $\pi_1 = \mathbb{Z}$?

For simply connected we need all loops at a give point to be path-homotopic to the constant loop at that point. One way of looking at this is that the letter must not have any "holes". The set of simply connected capital english letters is

$$\{C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z\}$$

Which leaves the remaining letters as possible candidates for $\pi_1 = \mathbb{Z}$ (Since any simply connected set will have $\pi_1 = \{0\}$, the trivial group). Of these the following are the capital english letters with $\pi_1 = \mathbb{Z}$.

$$\{A, D, O, P, Q, R\}$$

And since B has two "holes" it will have a different fundamental group.