Exercise 54.7. Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Proof. Recall that a covering map of $S^1$ by $\mathbb{R}$ is the map $p : \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi ix}$. The torus is the set $T = S^1 \times S^1$. By Theorem 53.3 we have that the map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering map, since $p$ is a covering map. Let $e_0 = (0, 0) \in \mathbb{R} \times \mathbb{R}$ and let $b_0 = p(e_0)$. This gives us that $p^{-1}(b_0)$ is the set $\mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, then $\mathbb{R} \times \mathbb{R}$ is simply connected, and we have a bijective lifting correspondence

$$\phi : \pi_1(S^1 \times S^1, b_0) \to \mathbb{Z} \times \mathbb{Z}.$$  

We now prove that $\phi$ is a homomorphism. Let $[f], [g] \in \pi_1(S^1 \times S^1, b_0)$ and let $\tilde{f}$ and $\tilde{g}$ by their liftings to paths in $\mathbb{R} \times \mathbb{R}$ beginning at $(0, 0)$. If we let $(a, b) = \tilde{f}(1)$ and $(c, d) = \tilde{g}(1)$, then by definition $\phi([f]) = (a, b)$ and $\phi([g]) = (c, d)$. Now consider the path $\tilde{g}$ of $\mathbb{R} \times \mathbb{R}$ given by

$$\tilde{g}(s) = (a, b) + \tilde{g}(s).$$

The path $\tilde{g}$ is a lifting of $\tilde{g}$ since for all $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ we have:

$$(p \times p)((a, b) + (x_1, x_2)) = (p(a + x_1), p(b + x_2)) = (p(x_1), p(x_2)) = (p \times p)((x_1, x_2))$$

and $\tilde{g}$ is a path beginning at $(a, b)$. This means the product $\tilde{f} \ast \tilde{g}$ is defined, and it is a lifting of $f \ast g$ that begins at $(0, 0)$. This path ends at $\tilde{g}(1) = (a + c, b + d)$. We can then show that $\phi$ preserves the group operation:

$$\phi([f] \ast [g]) = \phi([f \ast g]) = (a + c, b + d) = (a, b) + (c, d) = \phi([f]) + \phi([g])$$

and so $\phi$ is a homomorphism. Since $\phi$ is also bijective, then

$$\pi_1(S^1 \times S^1, b_0) \cong \mathbb{Z} \times \mathbb{Z}.$$  

\[ \square \]
Exercise 55.4. Suppose that you are given the fact that for each \( n \), there is no retraction \( r : B^{n+1} \to S^n \). Prove the following:

(a) The identity map \( \iota : S^n \to S^n \) is not nulhomotopic.

(b) The inclusion map \( j : S^n \to \mathbb{R}^{n+1} - 0 \) is not nulhomotopic.

(c) Every non-vanishing vector field on \( B^{n+1} \) point directly outward at some point of \( S^n \), and directly inward at some point of \( S^n \).

(d) Every continuous map \( f : B^{n+1} \to B^{n+1} \) has a fixed point.

Proof.

(a) First, the fact that there is no retraction \( r : B^{n+1} \to S^n \) tells us that there is no extension of \( \iota \) to a map \( B^{n+1} \to S^n \). Now assume \( \iota \) is nulhomotopic. Let \( H : S^n \times I \to S^n \) be a homotopy between \( \iota \) and a constant map \( c \). Let \( \pi : S^n \times I \to B^{n+1} \) be the map

\[
\pi(x,t) = (1-t)x.
\]

\( \pi \) is a quotient map since it is continuous, surjective, and closed. Also, \( \pi \) has the property that \( S^n \times \{1\} \) goes to zero, and \( \pi \) is injective on the rest of the domain. Since \( H \) is also constant on \( S^n \times \{1\} \) then it induces, through \( \pi \), a continuous map \( f : B^{n+1} \to S^n \) that is an extension of \( \iota \) (a contradiction). Thus \( \iota \) is not nulhomotopic.

(b) Assume the inclusion map \( j \) is nulhomotopic. If we consider the map \( r : \mathbb{R}^{n+1} - 0 \to S^n \) given by the vector function \( r(x) = \frac{y}{\|y\|} \), this is a retraction since for \( y \in S^n \) we have \( \|y\| = 1 \). We thus have that the inclusion map \( \iota \) from part (a) is given by \( \iota = r \circ j \). Since \( j \) is nulhomotopic there exists a constant map \( c : S^n \to \mathbb{R}^{n+1} - 0 \) such that \( j \simeq c \). This gives us by Exercise 51.1 that \( \iota = r \circ j \simeq r \circ c \), but \( r \circ c \) is a constant map, so \( \iota \) is nulhomotopic (a contradiction to part (a)). Thus \( j \) is not nulhomotopic.

(c) We have a vector field on \( B^{n+1} \) given by ordered pairs \((x, v(x))\) where \( v \) is a continuous map \( v : B^{n+1} \to \mathbb{R}^{n+1} \). To say that this vector field is non-vanishing implies \( v(x) \neq 0 \) for all \( x \in B^{n+1} \). This tells us that we actually have \( v : B^{n+1} \to \mathbb{R}^{n+1} - 0 \). Now since the identity map on \( B^{n+1} \) is nulhomotopic and \( \mathbb{R}^{n+1} - 0 \) is path connected, then by Exercise 51.3 the set of homotopy class of maps from \( B^{n+1} \to \mathbb{R}^{n+1} - 0 \) contain a single element. This means that all of these maps are nulhomotopic, and in particular \( v \) is nulhomotopic. So we get that the restriction of \( v \) to \( S^n \) (say \( w \)) is also nulhomotopic.
We will now proceed by contradiction. Assume that $v(x)$ does not point directly inward at any point $x \in S^n$. We will show that this gives us that $w$ is homotopic to the inclusion map $j : S^n \to \mathbb{R}^{n+1}$. Consider the straight line homotopy define by:

$$F(x, t) = tx + (1 - t)w(x),$$

For all $x \in S^n$. We need that $F(x, t) \neq 0$ for any value of $t$ to verify this homotopy is continuous. $F(x, 0) \neq 0$ since $w(x) \neq 0$ for all $x$. also $F(x, 1) = x \neq 0$ since $0 \notin S^n$. Now if $F(x, t) = 0$ for some $0 < t < 1$ then $tx + (1 - t)w(x) = 0$. From here we get $w(x) = \frac{t}{1-t}x$, or $w(x)$ is a negative scalar multiple of $x$, and so it points directly inward at $x$ (a contradiction). So $F(x, t) \neq 0$ for any $t \in I$, but this implies that $j$ is homotopic to $w$, which in turn makes $j$ nullhomotopic, a contradiction to part (b) above. Thus there is a point $x \in S^n$ for which $v(x)$ points directly inward.

Now we consider the vector field $(x, -v(x))$. $-v(x)$ gives a non-vanishing vector field, and so by the proof above there is a point $x \in S^n$ where $-v(x)$ point directly inward, so there is a point $x \in S^n$ for which $v(x)$ points directly outward.

(d) We will also prove this by contradiction. Assume there is no fixed point for the given $f : B^{n+1} \to B^{n+1}$, or $f(x) \neq x$ for all $x \in B^{n+1}$. Thus we can define a map $v(x) = f(x) - x$ which is continuous since $f$ is continuous, and gives a non-vanishing vector field $(x, v(x))$. By (c) we have that there is a point $y \in S^n$ for which $v(x)$ points directly outward, or $f(y) - y = \alpha y$ for some positive scalar $\alpha \in \mathbb{R}$. But this means $f(y) = (1 + \alpha)y$, and so $f(y) \notin B^{n+1}$ since $1 + \alpha > 0$, a contradiction. Therefore there is at least one point $x \in B^{n+1}$ with $f(x) = x$. 

$\square$