Math 205B - Topology

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Exercise 54.7. Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

Proof. Recall that a covering map of S^1 by $\mathbb{R}$ is the map $p : \mathbb{R} \to S^1$ given by $p(x) = e^{2\pi ix}$. The torus is the set $T = S^1 \times S^1$. By Theorem 53.3 we have that the map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering map, since p is a covering map. Let $e_0 = (0,0) \in \mathbb{R} \times \mathbb{R}$ and let $b_0 = p(e_0)$. This gives us that $p^{-1}(b_0)$ is the set $\mathbb{Z} \times \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, then $\mathbb{R} \times \mathbb{R}$ is simply connected, and we have a bijective lifting correspondence

$$\phi : \pi_1(S^1 \times S^1, b_0) \to \mathbb{Z} \times \mathbb{Z}.$$

We now prove that ϕ is a homomorphism. Let $[f], [g] \in \pi_1(S^1 \times S^1, b_0)$ and let $\tilde{f}$ and $\tilde{g}$ by their liftings to paths in $\mathbb{R} \times \mathbb{R}$ begininning at $(0,0)$. If we let $(a,b) = \tilde{f}(1)$ and $(c,d) = \tilde{g}(1)$, then by definition $\phi([f]) = (a,b)$ and $\phi([g]) = (c,d)$. Now consider the path $\tilde{\tilde{g}}$ of $\mathbb{R} \times \mathbb{R}$ given by

$$\tilde{\tilde{g}}(s) = (a,b) + \tilde{g}(s).$$

The path $\tilde{\tilde{g}}$ is a lifting of $\tilde{g}$ since for all $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$ we have:

$$\begin{aligned} (p \times p)((a,b) + (x_1, x_2)) &= (p(a + x_1), p(b + x_2)) \\ &= (p(x_1), p(x_2)) \\ &= (p \times p)((x_1, x_2)) \end{aligned}$$

and $\tilde{\tilde{g}}$ is a path beginning at (a,b). This means the product $\tilde{f} * \tilde{\tilde{g}}$ is defined, and it is a lifting of $f * g$ that begins at $(0,0)$. This path ends at $\tilde{\tilde{g}}(1) = (a+c, b+d)$. We can then show that ϕ preserves the group operation:

$$\begin{aligned} \phi([f] * [g]) &= \phi([f * g]) \\ &= (a + c, b + d) \\ &= (a,b) + (c,d) \\ &= \phi([f]) + \phi([g]) \end{aligned}$$

and so ϕ is a homomorphism. Since ϕ is also bijective, then

$$\pi_1(S^1 \times S^1, b_0) \cong \mathbb{Z} \times \mathbb{Z}.$$

$\square$

Exercise 55.4. Suppose that you are given the fact that for each n, there is no retraction $r : B^{n+1} \to S^n$. Prove the following:

(a) The identity map $\iota : S^n \to S^n$ is not nulhomotopic.

(b) The inclusion map $j : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ is not nulhomotopic.

(c) Every non-vanishing vector field on B^{n+1} point directly outward at some point of S^n, and directly inward at some point of S^n.

(d) Every continuous map $f : B^{n+1} \to B^{n+1}$ has a fixed point.

Proof.

(a) First, the fact that there is no retraction $r : B^{n+1} \to S^n$ tells us that there is no extension of ι to a map $B^{n+1} \to S^n$. Now assume ι is nulhomotopic. Let $H : S^n \times I \to S^n$ be a homotopy between ι and a constant map c. Let $\pi : S^n \times I \to B^{n+1}$ be the map

$$\pi(x, t) = (1 - t)x.$$

π is a quotient map since it is continuous, surjective, and closed. Also, π has the property that $S^n \times \{1\}$ goes to zero, and π is injective on the rest of the domain. Since H is also constant on $S^n \times \{1\}$ then it induces, through π, a continuous map $f : B^{n+1} \to S^n$ that is an extension of ι (a contradiction). Thus ι is not nulhomotopic.

(b) Assume the inclusion map j is nulhomotopic. If we consider the map $r : \mathbb{R}^{n+1} - \mathbf{0} \to S^n$ given by the vector function $r(x) = \frac{x}{\|x\|}$, this is a retraction since for $y \in S^n$ we have $\|y\| = 1$. We thus have that the inclusion map ι from part (a) is given by $\iota = r \circ j$. Since j is nulhomotopic there exists a constant map $c : S^n \to \mathbb{R}^{n+1} - \mathbf{0}$ such that $j \simeq c$. This gives us by Exercise 51.1 that $\iota = r \circ j \simeq r \circ c$, but $r \circ c$ is a constant map, so ι is nulhomotopic (a contradiction to part (a)). Thus j is not nulhomotopic.

(c) We have a vector field on B^{n+1} given by ordered pairs $(x, v(x))$ where v is a continuous map $v : B^{n+1} \to \mathbb{R}^{n+1}$. To say that this vector field is non-vanishing implies $v(x) \neq 0$ for all $x \in B^{n+1}$. This tells us that we actually have $v : B^{n+1} \to \mathbb{R}^{n+1} - \mathbf{0}$. Now since the identity map on B^{n+1} is nulhomotopic and $R^{n+1} - \mathbf{0}$ is path connected, then by Exercise 51.3 the set of homotopy class of maps from $B^{n+1} \to R^{n+1} - \mathbf{0}$ contain a single element. This means that all of these maps are nulhomotopic, and in particular v is nulhomotopic. So we get that the restriction of v to S^n (say w) is also nulhomotopic.

We will now proceed by contradiction. Assume that $v(x)$ does not point directly inward at any point $x \in S^n$. We will show that this gives us that w is homotopic to the inclusion map $j : S^n \to \mathbb{R}^{n+1}$. Consider the straight line homotopy define by:

$$F(x,t) = tx + (1-t)w(x),$$

For all $x \in S^n$. We need that $F(x,t) \neq 0$ for any value of t to verify this homotopy is continuous. $F(x,0) \neq 0$ since $w(x) \neq 0$ for all x. also $F(x,1) = x \neq 0$ since $0 \notin S^n$. Now if $F(x,t) = 0$ for some $0 < t < 1$ then $tx + (1-t)w(x) = 0$. From here we get $w(x) = \frac{t}{t-1}x$, or $w(x)$ is a negative scalar multiple of x, and so it points directly inward at x (a contradiction). So $F(x,t) \neq 0$ for any $t \in I$, but this implies that j is homotopic to w, which in turn makes j nulhomotopic, a contradiction to part (b) above. Thus there is a point $x \in S^n$ for which $v(x)$ points directly inward.

Now we consider the vector field $(x, -v(x))$. $-v(x)$ gives a non-vanishing vector field, and so by the proof above there is a point $x \in S^n$ where $-v(x)$ point directly inward, so there is a point $x \in S^n$ for which $v(x)$ points directly outward.

(d) We will also prove this by contradiction. Assume there is no fixed point for the given $f : B^{n+1} \to B^{n+1}$, or $f(x) \neq x$ for all $x \in B^{n+1}$. Thus we can define a map $v(x) = f(x) - x$ which is continuous since f is continuous, and gives a non-vanishing vector field $(x, v(x))$. By (c) we have that there is a point $y \in S^n$ for which $v(x)$ points directly outward, or $f(y) - y = \alpha y$ for some positive scalar $\alpha \in \mathbb{R}$. But this means $f(y) = (1+\alpha)y$, and so $f(y) \notin B^{n+1}$ since $1 + \alpha > 0$, a contradiction. Therefore there is at least one point $x \in B^{n+1}$ with $f(x) = x$.

$\square$