Math 205B - Topology

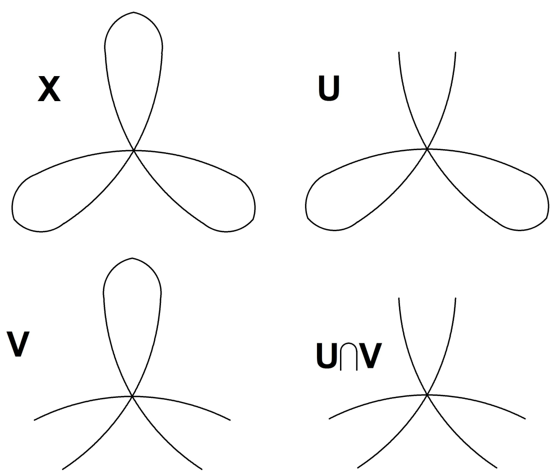
Dr. Baez

March 2, 2007

Christopher Walker

Exercise 0.1. Show that the fundamental group of the 3-bouquet of circles is $\mathbb{Z}*\mathbb{Z}*\mathbb{Z}$. Generalize this result to the n-bouquet of circles, and show that this is the free group on n generators.

Proof. We will use the Seifert-van Kampen Theorem to calculate the fundamental group. Let $U, V \subseteq X$ be as pictured (with the end points being open).



Since $U \bigcap V$ is contractible, then its fundamental group is trivial. this makes our calculation easier since we get that our normal subgroup N from the theorem is also trivial. U is homotopy equivalent to the figure eight, so $\pi_1(U, x_0) = \mathbb{Z} * \mathbb{Z}$. Also, V is homotopy equivalent to S^1, so $\pi_1(V, x_0) = \mathbb{Z}$. This tells us that the pushout of U and V is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ (since $*$ is associative). Thus π_1 of the 3-bouquet of circles is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

To generalize we simply set U and V to be the as above, where V is still homotopy equivalent to S^1, but U is homotopy equivalent to the $(n-1)$-bouquet of circles. This still gives us that $U \bigcap V$ is contractible, so by induction the fundamental group of the n-bouquet is $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$.

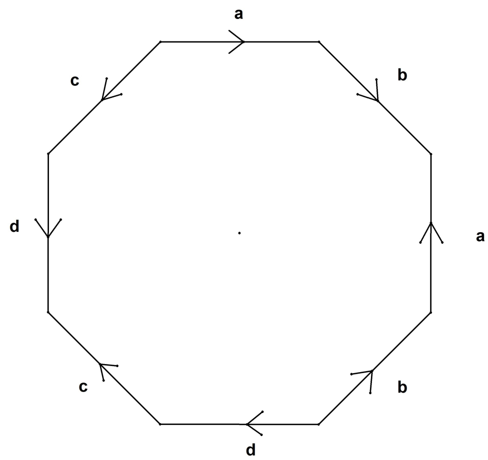
We can also show that $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$ is the free group on n generators, Denote F_n. We first use the universal property of a free group. Let X be the set $\{x_1, x_2, ..., x_n\}$ and define $f : X \to \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$ as $f(x_i) = z_i$ where z_i is the generator from the ith copy of $\mathbb{Z}$. Since F_n is free we have that there exist a homomorphism $g : F_n \to \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & F_n \arrow[d, "g"] \\ & \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}} \end{tikzcd}$$

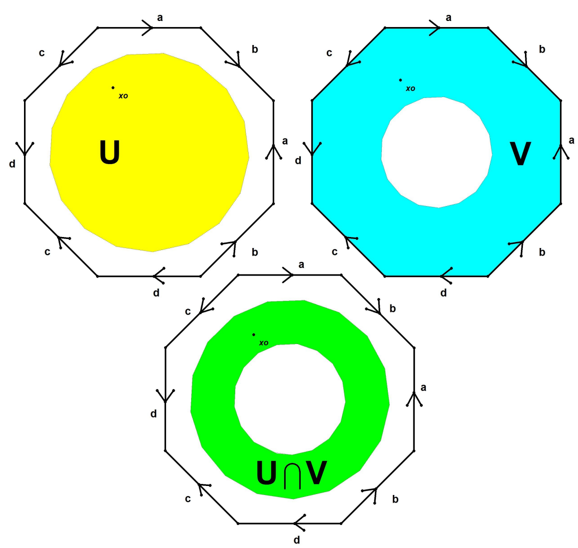
or $f = g \circ i$. Also g is bijective on the generators of each group, so it is bijective on the entire sets.

Now define $f_1 : \pi_1(U, x_0) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n-1 \text{ times}} \to F_n$ as $f_1(z_i) = x_i$ for $i = 1, ..., n-1$ where the x_i's are generators of F_n. also define $f_2 : \pi_1(V, x_0) = \mathbb{Z} \to F_n$ with $f_2(1) = x_n$. Since $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$ is a pushout then there exist a homomorphism $h : \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}} \to F_n$ such that $h(\pi_1(j_1)) = f_1$ and $h(\pi_1(j_2)) = f_2$. On the generators, $g \circ h(z_i) = g(x_i) = z_i$ and $h \circ g(x_i) = h(z_i) = x_i$, So g and h are inverses. Therefore $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}} \cong F_n$. $\square$

Exercise 0.2. Calculate the fundamental group of the two torus T^2 given as the quotient topology of the octogon by identifying edges as follows:

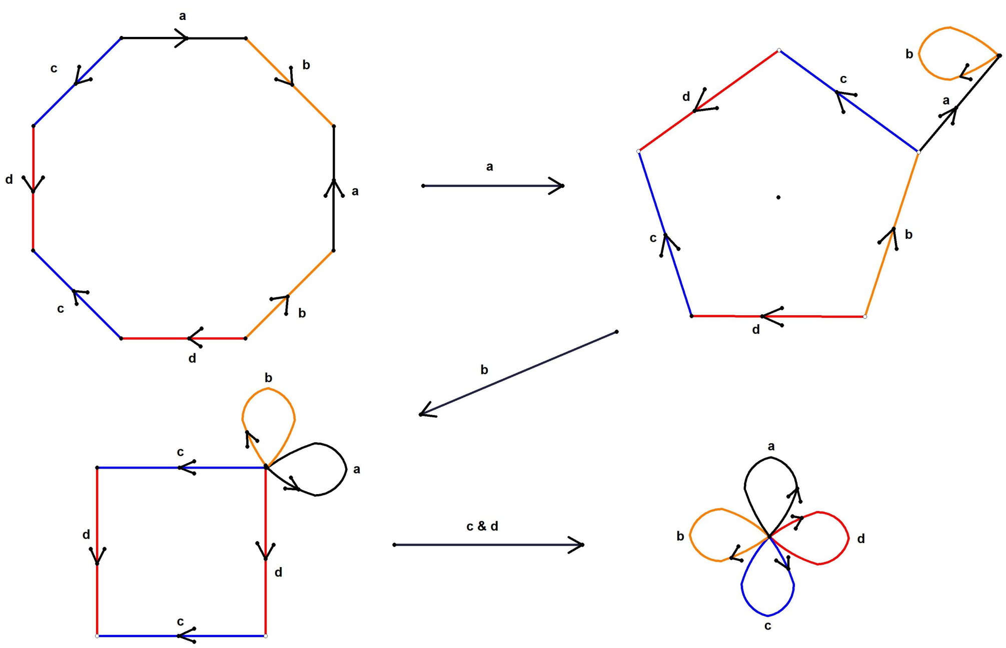


Proof. First we need to define open sets $U, V \subset T^2$ that meet the conditions of the Seifert-van Kampen Theorem. Let U, V be the subsets of T^2 as pictured here.



This choice of U and V gives what we need. The fundamental group of U is the trivial group, since U is contractible. Also, the fundamental group of $U \bigcap V$ is $\mathbb{Z}$ since it has S^1 as a deformation retract.

Now we need to consider the fundamental group of V. One way to think about this is to consider V as a "wire" (i.e. we only consider the edges of the octogon). With this we can “fold” V as follows:



So we have that V is homotopy equivalent to the bouquet of four circles. By the previous exercise, this tells us that the fundamental group of V is $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. We can now apply the Seifert-van Kampen Theorem, to get the following pushout.

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
\{0\} \arrow[dr, "\pi_1(j_1)"'] & & \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \arrow[dl, "\pi_1(j_2)"] \\
 & \pi_1(T^2, x_0) &
\end{tikzcd}$$

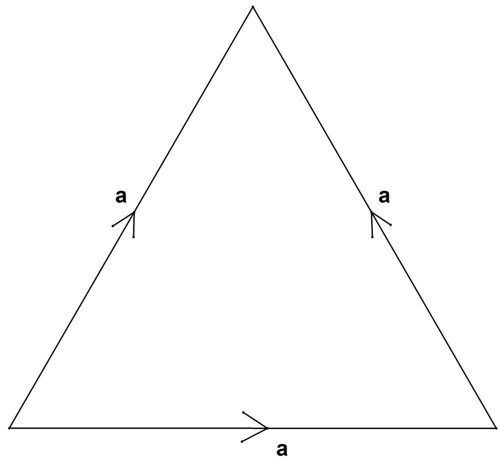
We know that for this to be a pushout, $\pi_1(T^2, x_0) = G * H/N$ where $G = \pi_1(U, x_0)$, $H = \pi_1(V, x_0)$, and N we calculate as follows. To calculate N, we need to see what the image of 1 is under the homomorphisms $\pi_1(i_1)$ and $\pi_1(i_2)$. It is clear that $\pi_1(i_1)$

is the trivial homomorphism map, so $\pi_1(i_1)(1) = 0$. As for the other direction, we need to trace the loop of $U \bigcap V$ that generates $\pi_1(U \bigcap V, x_0)$ around V to see what we get. From the picture before we see that this loop corresponds to the element $aba^{-1}b^{-1}dcd^{-1}c^{-1}$. We now take these two elements, and form the normal subgroup N of $G * H$ generated by $1(aba^{-1}b^{-1}dcd^{-1}c^{-1})^{-1} = (aba^{-1}b^{-1}dcd^{-1}c^{-1})^{-1}$.

Since $G = \{0\}$ and $H = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ we see that $G * H = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. So $\pi_1(T^2, x_0) = (\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z})/N$, where N is the normal subgroup generated by $aba^{-1}b^{-1}dcd^{-1}c^{-1}$.

$\square$

Exercise 0.3. Calculate the fundamental group of the "Dunce Cap", which is the quotient space of a triangle by associating the three side with each other as pictured below.



Proof. We will use the same argument as with the double torus (and the same U and V). Again, U is contractible, so its fundamental group is trivial. Also $U \bigcap V$ has S^1 as a deformation retract, so its fundamental group is $\mathbb{Z}$.

In order to find the fundamental group of V we again think of it as just the edges. When we do this, we identify all three sides, and then connect the ends. Thus V is homotopy equivalent to the S^1 and thus has $\mathbb{Z}$ as a fundamental group. So we get the following pushout:

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\ \{0\} \arrow[dr, "\pi_1(j_1)"'] & & \mathbb{Z} \arrow[dl, "\pi_1(j_2)"] \\ & \pi_1(X, x_0) & \end{tikzcd}$$

Now, in order for this to be a pushout we know $\pi_1(X, x_0) = G * H/N$, but we already have that $G = \{0\}$ and $H = \mathbb{Z}$. So we only need to calculate N. From the picture we see that the loop that generates $\pi_1(U \bigcap V, x_0) = \mathbb{Z}$ when considered as a loop in V gives the group element $aaa^{-1} = a$. So the normal subgroup of $G * H = \mathbb{Z}$ generated by a is $\mathbb{Z}$, so $G * H/N = \mathbb{Z}/\mathbb{Z}$. Thus $\pi_1(X, x_0)$ for the dunce cap is trivial.

$\square$