Math 205B - Topology

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Exercise 1. *Let* (X, x_0) *be a pointed space. Let* U *and* V *be open subsets of* X*, with* $U \bigcup V = X$*,* $x_0 \in U \bigcap V$*, and* $U \bigcap V$ *path connected. Let* X_0 *be the path component of the base point* x_0*. Prove that* $U_0 = U \bigcap X_0$*,* $V_0 = V \bigcap X_0$*, and* $(U \bigcap V)_0 = (U \bigcap V) \bigcap X_0$ *are path connected.*

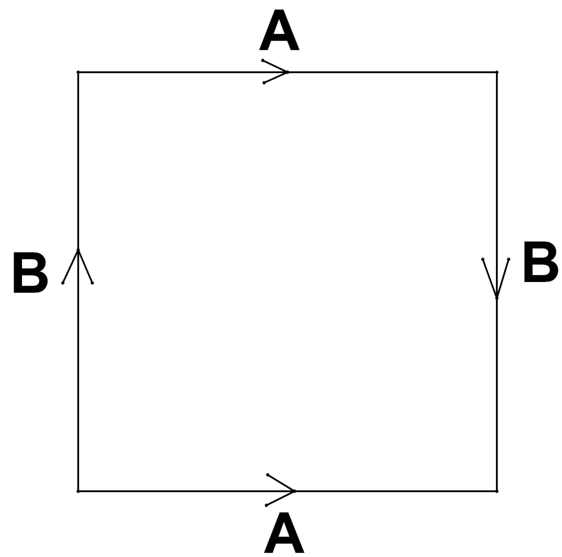
Proof. We will first look at $(U \bigcap V)_0 = (U \bigcap V) \bigcap X_0$. First $U \bigcap V$ is a path connected subset of X. By Theorem 25.2 this gives us that $U \bigcap V$ intersects exactly one path component of X. Since $x_0 \in U \bigcap V$ and $x_0 \in X_0$, then $U \bigcap V$ is contained completely in X_0, and so $(U \bigcap V)_0 = U \bigcap V$. Thus it is path connected.

Now consider $U_0 = U \bigcap X_0$. Let $y \in U_0$. If $y \in U \bigcap V$ then there is a path from x_0 to y since $U \bigcap V$ is path connected. Now assume $y \notin V$. Since $y \in X_0$ there is a path in X_0 from x_0 to y. Like in the proof of Seifert-van Kampen, there is a point on the path, say z, which is in $U \bigcap V$ and after which the rest of the path is completely contained in U. Since $z \in U \bigcap V$ which is path connected, then there is a path from x_0 to z which is in $U \bigcap V$. Now we have a path from x_0 to z in $U \bigcap X_0$, and a path from z to y in $U \bigcap X_0$, so by concatenation of paths, we get a path from x_0 to y which is in $U \bigcap X_0$. We can do this again for any other point in U_0. concatenation of these paths gives a path between any two points in U_0. Thus U_0 is path connected.

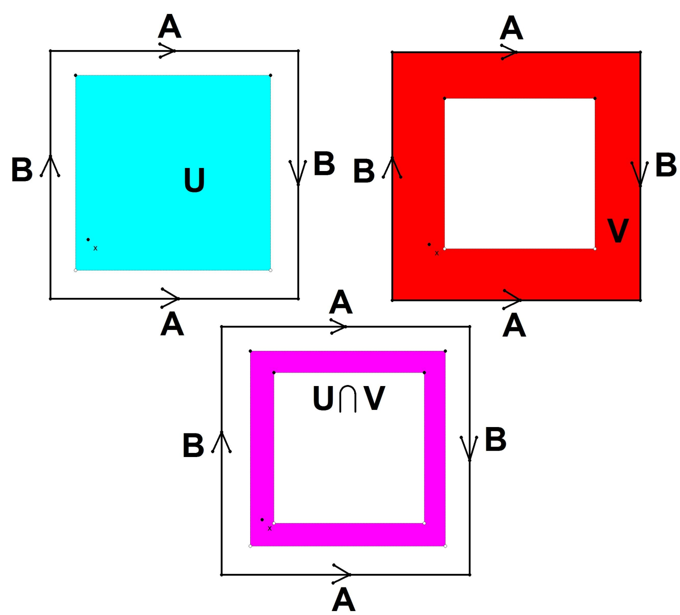
The same argument hold for V_0. $\square$

Exercise 2. *Calculate the fundamental group of the Klein Bottle.*

Proof. First the Klein Bottle can be described as the quotient space of the square by identifying the sides as denoted in the picture.



Just as with the double torus we define open sets U and V as pictured.



These U and V meet all the requirements of the Seifert-van Kampen Theorem, so we have that the following diagram is a pushout:

$$\begin{tikzcd} & \pi_1(U \bigcap V, x) \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\ \pi_1(U, x) \arrow[dr, "\pi_1(j_1)"'] & & \pi_1(V, x) \arrow[dl, "\pi_1(j_2)"] \\ & \pi_1(K, x) & \end{tikzcd}$$

We see that U is contractible, so $\pi_1(U, x) \cong \{1\}$. Also $U \bigcap V$ has S^1 as a deformation retract, so $\pi_1(U \bigcap V, x) \cong \mathbb{Z}$. For V we consider just the "wire" given by the edges. After folding we see that this gives us the figure eight space. This implies that V has the figure eight as a deformation retract, so $\pi_1(V, x) \cong \mathbb{Z} * \mathbb{Z}$. Now we can construct $\pi_1(K, x) \cong (\{1\} * (\mathbb{Z} * \mathbb{Z}))/N$ where N is the normal subgroup generated by the element $(\pi_1(i_1)(1_\mathbb{Z}) * (\pi_1(i_2)(1_\mathbb{Z}))^{-1}$. It is clear that $\pi_1(i_1)$ is a trivial homomorphism, and so $\pi_1(i_1)(1_\mathbb{Z}) = 1$. Now we know that $\pi_1(U \bigcap V, x)$ is generated by a single loop, so we can trace that loop in V to determine its image. When we do this we get that $\pi_1(i_2)(1_\mathbb{Z}) = BABA^{-1}$. we may also use the fact that if an element c generated a group, so does c^{-1}, so we may consider simply $BABA^{-1}$ instead of its inverse. Thus we have a group presentation for $\pi_1(K, x)$ given by $\{A, B \mid BABA^{-1}\}$.

$\square$

Exercise 3. *Show there is a homomorphism from* $\pi_1(K, x)$ *to* $D_\infty = \{f : \mathbb{Z} \to \mathbb{Z} \mid f(n) = \pm n + k, k \in \mathbb{Z}\}$

Proof. Recall that D_∞ is the group of symmetries on $\mathbb{Z}$ as a subset of the real line. We know that D_∞ is generated by f_a, f_b where $f_a(n) = -n$ and $f_b(n) = n + 1$. To find a homomorphism from $\pi_1(K, x)$ to D_∞, we will use the fact that $\pi_1(K, x)$ is a pushout of the U, V, and $U \bigcap V$ we defined in the previous question. We begin by defining $h_1 : \pi_1(U, x) \to D_\infty$ as the trivial homomorphism (since this is the only thing we can do). More interesting, we may define $h_2 : \pi_1(V, x) \to D_\infty$ on the generators of $\pi_1(V, x)$ as follows. Let $h_2(a) = f_a$ and $h_2(b) = f_b$. This completely determines a homomorphism since it maps generators to generators. With these maps we need to check that the following diagram commutes:

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "\pi_1(i_1)"'] \arrow[dr, "\pi_1(i_2)"] & \\
\{1\} \arrow[dr, "h_1"'] & & \mathbb{Z} * \mathbb{Z} \arrow[dl, "h_2"] \\
 & D_\infty &
\end{tikzcd}$$

On the left side, we have the composition $h_2 \circ \pi_1(i_1)$. this is the composition of the zero homomorphism with the trivial homomorphism, and so this takes $1_\mathbb{Z}$ to 1_{D_∞} which is the identity map. We now consider $h_2 \circ \pi_1(i_2)(1_\mathbb{Z})$. From the proof of the previous question we saw that $\pi_1(i_2)(1) = baba^{-1}$. We then apply h_2 to get what we need.

$$\begin{aligned}
h_2 \circ \pi_1(i_2)(1_\mathbb{Z}) \quad &= h_2(baba^{-1}) \\
&= h_2(b) \circ h_2(a) \circ h_2(b) \circ h_2(a)^{-1} \\
&= f_b \circ f_a \circ f_b \circ f_{a^{-1}}
\end{aligned}$$

So we only need to show that this is indeed the indentity map. Note that $f_{a^{-1}} = f_a$.

$$\begin{aligned}
f_b \circ f_a \circ f_b \circ f_{a^{-1}}(n) \quad &= f_b \circ f_a \circ f_b(-n) \\
&= f_b \circ f_a(-n + 1) \\
&= f_b(n - 1) \\
&= n
\end{aligned}$$

Thus the diagram commutes. So we now have a "competitor" to our pushout, which means we get a unique homomorphism, say f, from $\pi_1(K, *)$ to D_∞. $\square$