Math 205B - Topology

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Exercise 73.1. Find spaces whose fundamental group is isomorphic to the following groups.

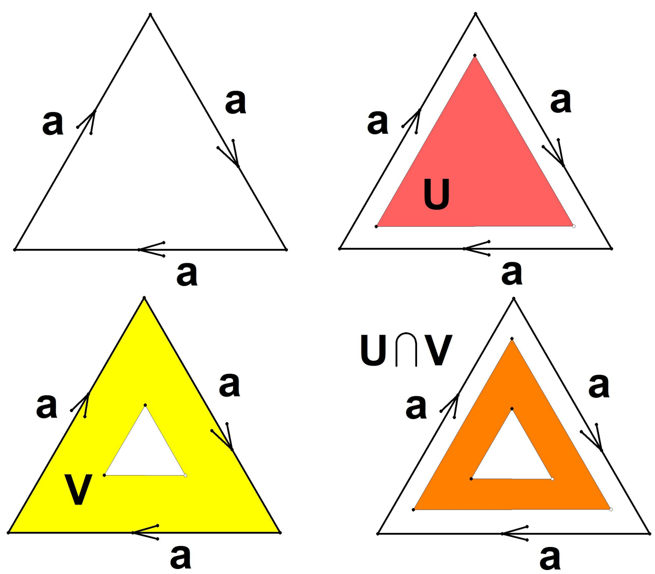
(a) $\mathbb{Z}_n \times \mathbb{Z}_m$.

(b) $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$.

(c) $\mathbb{Z}_n * \mathbb{Z}_m$

(d) $\mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \cdots * \mathbb{Z}_{n_k}$.

Proof. First we would like a space with fundamental group $\mathbb{Z}_n$ for each $n \in \mathbb{Z}^+$. We can obtain this as follows. Begin with an n-gon, and identify all the sides in the same direction. we picture this below for a triangle. also included is the break down for our open sets U and V.



From this we get that $\pi_1(U, x_0)$ is trivial, $\pi_2(U \bigcap V, x_0) \cong \mathbb{Z}$ and $\pi_1(V, x_0) \cong \mathbb{Z}$. when we track the image of $1_{\mathbb{Z}}$ around the commutative diagram, we see that our normal subgroup N from the Seifert-van Kampen Theorem is generated by a^n. Thus we have a group presentation for $\pi_1(X, x_0) = \{a \mid a^n = 1\}$, and this is isomorphic to $\mathbb{Z}_n$. Of particular interest are the cases when $n = 1$ and $n = 2$. The argument made here is true for both of these cases, but conceptually we don't usually consider what a 2-gon or 1-gon looks like. Topologically this is not a problem since we do not require the edges to be "straight". We can now construct all the requested spaces.

(a) For all $n \in \mathbb{Z}^+$, Define X_n to be the quotient space obtain from an n-gon as described above. Thus $\pi_1(X_n, *) \cong \mathbb{Z}_n$. Now by Theorem 60.1 we have that $\pi_1(X_n \times X_m, *) \cong \pi(X_n, *) \times \pi_1(X_m, *) \cong \mathbb{Z}_n \times \mathbb{Z}_m$, so we have the space we need.

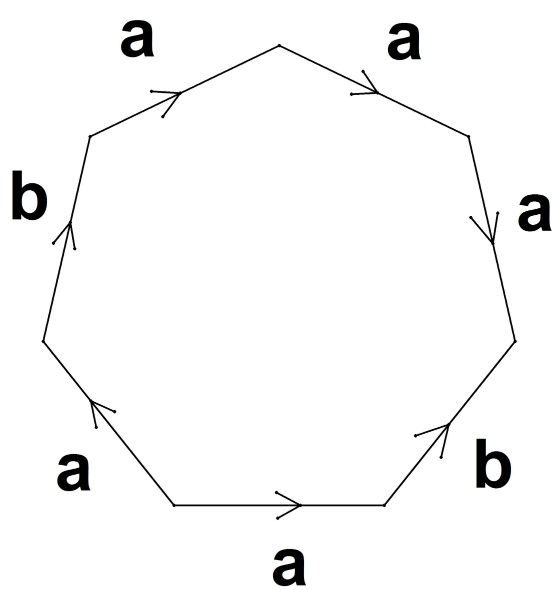
(b) Using the fact that the product operation on spaces and groups is associative, we see by induction on the number of spaces that $\pi_1(X_{n_1} \times \cdots \times X_{n_k}, *) \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$.

(c) We know that the π_1 as a functor preserves colimits of well-pointed space. In the category Top_* the colimit of two spaces is the wedge product "$\vee$". In the category *Grp* we have that the colimit of two groups is the free product "$*$". Thus for X_n as we described before $\pi_1(X_n \vee X_m, x_0) \cong \pi_1(X_n, x_0) * \pi_1(X_m, x_0) \cong \mathbb{Z}_n * \mathbb{Z}_m$.

(d) As before the wedge product in Top_* and the free product in *Grp* is associative. When can then use induction to get that $\pi_1(X_{n_1} \vee X_{n_2} \cdots \vee X_{n_k}, x_0) \cong \mathbb{Z}_{n_1} * \mathbb{Z}_{n_2} * \cdots * \mathbb{Z}_{n_k}$.

$\square$

Exercise 74.2. Consider the space X obtained from the seven sided polygon by identifying the sides as labeled in the picture.



Show that the fundamental group of X is the free product of two cyclic groups.

Proof. We approach this problem the same way as with all our quotient spaces of polygons. As usual $\pi_1(U, x_0) \cong \{1\}$ and $\pi_1(U \bigcap V, x_0) \cong \mathbb{Z}$. Since there are two different edge labels, then $\pi_1(V, x_0) \cong \mathbb{Z} * \mathbb{Z}$. To find our normal subgroup N from the Seifert-van Kampen Theorem, we check what the generating loop of $\pi_1(U \bigcap V, x_0)$ does in V. with the labelling provided we get that N is generated by the element $b^{-1}a^{-1}abaaa = a^3$. When we take the quotient group $\mathbb{Z} * \mathbb{Z}/N$ we see that N only affects the first generator a of $\mathbb{Z} * \mathbb{Z}$, and it has no bearing on the second generator b or the relationship between the two generators. Thus $\mathbb{Z} * \mathbb{Z}/N \cong \mathbb{Z}_3 * \mathbb{Z}$, and so it is the free product of two cyclic groups. $\square$