Classifying Spaces For Topological 2-Groups

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January 7, 2009

for a longer version with references, see:

http://math.ucr.edu/home/baez/barcelona/
A Famous Old Theorem

Here is the result we’d like to categorify:

**Thm.** Let $G$ be a well-pointed topological group. Let $BG$, the **classifying space** of $G$, be the geometric realization of the nerve of $G$. Then for any paracompact Hausdorff space $M$, there is a bijection

$$[M, BG] \cong H^1(M, G)$$

(A topological group $G$ is **well-pointed** if $1 \in G$ has a neighborhood of which it is a deformation retract.)
Topological 2-Groupoids

Defn. A 2-groupoid is a strict 2-category where all morphisms and 2-morphisms are strictly invertible.

Defn. A topological 2-groupoid $\mathcal{G}$ is a 2-groupoid internal to Top.

In other words, $\mathcal{G}$ has:

- a topological space of objects,
- a topological space of morphisms,
- a topological space of 2-morphisms,

and all the 2-groupoid operations are continuous.
**Topological 2-Groups**

**Defn.** A **topological 2-group** is a topological 2-groupoid with one object.

So, it has one object: \( \bullet \)

together with 1-morphisms: \( \bullet \xrightarrow{g} \bullet \)

and 2-morphisms: \( \bullet \xrightarrow{h} \bullet \)

\( g' \)}
The Čech 2-Groupoid

Let $\mathcal{U} = \{U_i\}$ be an open cover of a topological space $M$.

**Defn.** The Čech 2-groupoid $\hat{\mathcal{U}}$ is the topological 2-groupoid where:

- objects are pairs $(x, i)$ with $x \in U_i$,
- there is a single morphism from $(x, i)$ to $(x, j)$ when $x \in U_i \cap U_j$, and none otherwise,
- there are only identity 2-morphisms.

(This is just a topological groupoid promoted to a 2-groupoid by throwing in identity 2-morphisms.)
Čech Cohomology for 2-Bundles

Defn. A Čech cocycle with coefficients in a topological 2-group $G$ is a continuous weak 2-functor $g: \hat{U} \to G$.

Defn. Two Čech cocycles $g, g'$ are cohomologous if there is a continuous weak natural isomorphism $f: g \Rightarrow g'$.

Defn. Let $\check{H}^1(U, G)$ be the set of cohomology classes of Čech cocycles. We define the Čech cohomology of $M$ with coefficients in $G$ to be the limit as we refine the cover:

$$\check{H}^1(M, G) = \lim_{\rightarrow} \check{H}^1(U, G)$$
Categorifying That Famous Old Theorem

**Thm.** Suppose $G$ is a well-pointed topological 2-group and $M$ is a paracompact Hausdorff space admitting good covers. Then there is a bijection

$$\tilde{H}^1(M, G) \cong [M, B|N\mathcal{G}|]$$

where the topological group $|N\mathcal{G}|$ is the geometric realization of the nerve of $\mathcal{G}$. So, we call $B|N\mathcal{G}|$ the **classifying space** of $\mathcal{G}$.

(A topological 2-group $G$ is **well-pointed** if both the topological groups in its corresponding crossed module are well-pointed. An open cover is **good** if each nonempty finite intersection of sets in the cover is contractible.)
How to Build the Classifying Space

First we think of $G$ as a group in TopGpd and apply the nerve construction:

$$N : \text{TopGpd} \to \text{Top}^{\Delta^{\text{op}}}$$

to get a group in simplicial spaces, $NG$.

Then we use geometric realization:

$$| \cdot | : \text{Top}^{\Delta^{\text{op}}} \to \text{Top}$$

to get a topological group $|NG|$.

Then we think of $|NG|$ as a 1-object topological groupoid, and take the nerve and the geometric realization of this to get our space $B|NG|$.
A Corollary:  
Bundles vs. 2-Bundles

**Cor.** There is a 1-1 correspondence between:

- equivalence classes of principal $\mathcal{G}$-2-bundles over $M$
- elements of the Čech cohomology $\check{H}^1(M, \mathcal{G})$
- homotopy classes of maps $f : M \to B|N\mathcal{G}|$
- elements of the Čech cohomology $\check{H}^1(M, |N\mathcal{G}|)$
- isomorphism classes of principal $|N\mathcal{G}|$-bundles over $X$. 
Another Corollary

For any simply-connected compact simple Lie group $G$ there is a topological 2-group $\mathcal{G}$ called the \textbf{string 2-group} of $G$, such that $|N\mathcal{G}|$ is the 3-connected cover of $G$.

The homomorphism $|N\mathcal{G}| \xrightarrow{p} G$ gives an algebra homomorphism:

$$H^*(BG, \mathbb{R}) \xrightarrow{p^*} H^*(B|N\mathcal{G}|, \mathbb{R})$$

This is onto, with kernel generated by the ‘2nd Chern class’ $c_2 \in H^4(BG, \mathbb{R})$.

So, the real characteristic classes of String($G$)-2-bundles are just like those of $G$-bundles, but with $c_2$ set to zero!